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An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions

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Abstract

We shall establish full asymptotic expansions for the mean squares of Lerch zeta-functions, in the form (1.1) and (1.2) given below, based on F. V. Atkinson's device (cf. [2], [4, Chapter 15] and [5, Chapter 2]). Mellin-Barnes' type integral expression for an infinite double sum (see (3.2) and (3.5)) will play a central role in the derivation of our main formulae (2.1) and (2.2).

1. Introduction

Let s be a complex variable, and α and λ fixed real parameters with $\alpha > 0$. The zeta-function

$$\phi(\lambda, \alpha, s) = \sum_{n=0}^{\infty} e^{2\pi i\lambda n}(n+\alpha)^{-s} \qquad (\mathrm{Re}\, s > 1)$$

was firstly introduced and studied by M. Lerch [14] and R. Lipschitz [15]. For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$ it is continued to an entire function over the s-plane, while for $\lambda \in \mathbb{Z}$ it reduces to the Hurwitz zeta-function $\zeta(s,\alpha)$. Note that $\zeta(s,1) = \zeta(s)$ is the Riemann zeta-function.

Extensive results have been known for the asymptotic behavior of $\zeta(s)$; however, our knowledge for more general $\phi(\lambda, \alpha, s)$ is narrowly restricted. D. Klusch [12], [13]

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proved hybrid type mean value theorems for

$$\int_0^\infty \left|\phi(\lambda, \alpha, \sigma + it)\right|^2 e^{-\delta t} dt$$

as $\delta \to +0$, while W. Zhang [22] recently obtained an asymptotic formula for the mean square

$$I(s; \lambda) = \int_0^1 \left|\phi_1(\lambda, \alpha, s)\right|^2 d\alpha, \tag{1.1}$$

where

$$\phi_1(\lambda, \alpha, s) = \phi(\lambda, \alpha, s) - \alpha^{-s}.$$

The evaluation of $I(s; \lambda)$ in the special case $\lambda \in \mathbb{Z}$ has been pursued by various authors. Recent progress has been independently made by J. Andersson [1], W. Zhang [21], [23], and K. Matsumoto and the author [9–11]. For the detailed history we refer to Section 1 of [11].

The purpose of this paper is to establish full asymptotic expansions for $I(s; \lambda)$ (Theorem 1) and its discrete analogue

$$J(s; \lambda, q) = \sum_{a=1}^{q} \left|\phi\left(\lambda, \frac{a}{q}, s\right)\right|^2, \tag{1.2}$$

where q is an arbitrary positive integer (Theorem 2). The method of proof is based on the combination of F. V. Atkinson's device (cf. [2], [4, Chapter 15], [5, Chapter 2]), which deals with the product $\zeta(u)\zeta(v)$, and a Mellin-Barnes' type integral expression for an infinite double sum (see (3.2) and (3.5)). Theorems 1 and 2 give natural generalizations of [10, Corollary 3], [11, Corollary 1] and [9, Theorem 2], respectively; however, the proofs are considerably different from our original derivation. One of the new features is a systematic application of various properties of Gauss' hypergeometric function $F(a, b; c; z)$ (see Section 4).

It should be remarked that Mellin-Barnes' type integral formulae give very powerful tools in the study of various problems related to zeta-functions. In [6] we have proved asymptotic expansions for the mean square of Dirichlet L-functions similar to (1.2), while two types of power series with the Riemann zeta-function in the coefficients have been treated in [7]. Both of these investigations are based on certain path shifting arguments for Mellin-Barnes' type integral formulae. Further applications will be given in forthcoming papers.

Our main results will be stated in the next section. Section 3 will be devoted to prepare Lemma 1, which is fundamental in deriving Theorems 1 and 2. We shall prove Theorem 1 in a more general setting (Theorem 3) in Section 4. Theorem 2 will be deduced in the final section.

2. Statement of results

Let $\Gamma(s)$ be the gamma-function, $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any $n \in \mathbb{Z}$ Pochhammer's symbol, and $\zeta_\lambda(s)$ the periodic zeta-function defined by

$$\zeta_\lambda(s) = \sum_{n=1}^{\infty} e^{2\pi i\lambda n} n^{-s} \qquad (\mathrm{Re}\, s > 1)\,,$$

and its meromorphic continuation. We set

$$E = \{n \in \mathbb{Z};\ n \geq 1\} \cup \left\{\frac{n}{2} + it;\ n \in \mathbb{Z},\ n \leq 2,\ t \in \mathbb{R}\right\}\,.$$

Theorem 1

For any integers $N \geq 1$ and $K \geq 0$, in the region $-N+1 < \sigma < N+1$ and $t > 1$ except the points of E, the formula

$$\begin{aligned} I(\sigma+it;\lambda) = {} & \frac{1}{2\sigma-1} + 2\Gamma(2\sigma-1)\mathrm{Re}\left\{\zeta_\lambda(2\sigma-1)\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right\} \\ & - 2\mathrm{Re}\sum_{n=0}^{N-1}\frac{(\sigma+it)_n}{(1-\sigma+it)_{n+1}}\left\{e^{-2\pi i\lambda}\zeta_\lambda(\sigma+it+n)-1\right\} \\ & - 2\mathrm{Re}\sum_{k=1}^{K}(-1)^{k-1}\frac{(2-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_N}\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{l^k(l+1)^{\sigma+it+N-k}} \\ & + O(t^{-K-1}) \end{aligned} \tag{2.1}$$

holds for all $\lambda \in \mathbb{R}$, where the implied constant depends only on N, K and σ.

Remark. Since

$$\frac{(2-2\sigma)_{k-1}(\sigma+it)_{N-k}}{(1-\sigma+it)_N} = O(t^{-k}),$$

$$\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{l^k(l+1)^{\sigma+it+N-k}} = O(1)$$

for $\sigma > -N+1$ and $t > 1$, (2.1) actually gives an asymptotic series for $I(\sigma+it;\lambda)$ in the descending order of t.

Theorem 2

For any integers $N \geq 1$ and $q \geq 1$, in the region $-N+1 < \sigma < N+1$ and $t \in \mathbb{R}$ except the points of E, the formula

$$\begin{aligned} J(\sigma+it;\lambda,q) = q^{2\sigma}\zeta(2\sigma) + 2q\Gamma(2\sigma-1)\mathrm{Re}\left\{\zeta_\lambda(2\sigma-1)\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)}\right\} \\ + 2\sum_{n=0}^{N-1}\frac{(-1)^n q^{\sigma-n}}{n!}\mathrm{Re}\left\{q^{-it}(\sigma+it)_n\zeta_\lambda(\sigma+it+n)\zeta(\sigma-it-n)\right\} \\ + O\left\{q^{\sigma-N}(|t|+1)^{\nu(\sigma,N,\varepsilon)}\right\} \end{aligned} \tag{2.2}$$

holds for all $\lambda \in \mathbb{R}$, where

$$\nu(\sigma,N,\varepsilon) = \begin{cases} 2N + \dfrac{1}{2} - \sigma & \text{if} \quad -N+1 < \sigma < N, \\ \dfrac{3N}{2} + \dfrac{1}{2} - \dfrac{\sigma}{2} + \varepsilon & \text{if} \quad N \leq \sigma < N+1 \end{cases}$$

with an arbitrary small $\varepsilon > 0$, and the implied constant depends at most on N, σ and ε.

Remark. The appearance of the exponent $\nu(\sigma,N,\varepsilon)$ in (2.2) is reasonable, since, from Stirling's formula (cf. [4, Appendix, p. 492, A.7(A.34)]) and an upper bound for $\mu(\sigma) = \limsup_{t\to\infty}|\zeta(\sigma+it)|/\log t$ (see [19, Chapter V, 5.1]), we have

$$(\sigma+it)_n\zeta_\lambda(\sigma+it+n)\zeta(\sigma-it-n) = O\left(|t|^{\nu(\sigma,n,\varepsilon)}\right)$$

for $-n+1 < \sigma < n+1$ $(n \geq 1)$ as $t \to \pm\infty$.

Similar asymptotic expansions for the exceptional points $\sigma + it \in E$ can be deduced as limiting cases of Theorems 1 and 2. Let $\psi(s) = (\Gamma'/\Gamma)(s)$ be the digamma function, and $\gamma = -\psi(1)$ Euler's constant. We can in particular show

Corollary 1

For any integer $K \geq 0$ and any $t > 1$, the formula

$$
\begin{aligned}
I\left(\frac{1}{2}+it;\lambda\right) = \gamma + 2\mathrm{Re}\left\{\zeta_\lambda'(0) - \zeta_\lambda(0)\psi\left(\frac{1}{2}+it\right)\right\} - 2\mathrm{Re}\frac{e^{-2\pi i\lambda}\zeta_\lambda(\frac{1}{2}+it)-1}{\frac{1}{2}+it} \\
- 2\mathrm{Re}\sum_{k=1}^{K}\frac{(-1)^{k-1}(k-1)!}{(\frac{1}{2}+it)(-\frac{1}{2}+it)\cdots(\frac{3}{2}-k+it)}\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{l^k(l+1)^{\frac{3}{2}-k+it}} \\
+ O(t^{-K-1}),
\end{aligned}
$$

holds for all $\lambda \in \mathbb{R}$.

Remark. Noting $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2)\log 2\pi$, we see that this corollary gives a generalization of [10, Theorem 1] and [11, Corollary 2].

Corollary 2

For any integer $N \geq 1$ and any real $t > 1$, the formula

$$
\begin{aligned}
I(1+it;\lambda) = 1 + \pi(1-2\lambda)t^{-1} - 2\mathrm{Re}\frac{1}{it}\sum_{n=0}^{N-1}\left\{e^{-2\pi i\lambda}\zeta_\lambda(1+it+n) - 1\right\} \\
- 2\mathrm{Re}\frac{1}{it}\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{l(l+1)^{N+it}}
\end{aligned}
$$

holds for $0 < \lambda < 1$.

Corollary 3

For any integers $N \geq 1$ and $q \geq 1$, and any $t \in \mathbb{R}$, the formula

$$
\begin{aligned}
J\left(\frac{1}{2}+it;\lambda,q\right) = q\log q + 2q\gamma + 2q\mathrm{Re}\left\{\zeta_\lambda'(0) - \zeta_\lambda(0)\psi\left(\frac{1}{2}+it\right)\right\} \\
+ 2\sum_{n=0}^{N-1}\frac{(-1)^n q^{\frac{1}{2}-n}}{n!} \\
\times \mathrm{Re}\left\{q^{-it}\left(\frac{1}{2}+it\right)_n \zeta_\lambda\left(\frac{1}{2}+it+n\right)\zeta\left(\frac{1}{2}-it-n\right)\right\} \\
+ O\{q^{\frac{1}{2}-N}(|t|+1)^{\nu(\frac{1}{2},N,\varepsilon)}\}
\end{aligned}
$$

holds for all $\lambda \in \mathbb{R}$.

Remark. This corollary gives a generalization of [9, Theorem 1].

Corollary 4

For any integers $N \geq 1$ and $q \geq 1$, and any $t \in \mathbb{R}$, the formula

$$\begin{aligned} J(1+it;\lambda,q) = q^2\zeta(2) + \pi q(1-2\lambda)t^{-1} \\ + 2\sum_{n=0}^{N-1}\frac{(-1)^n q^{1-n}}{n!}\mathrm{Re}\left\{q^{-it}(1+it)_n\zeta_\lambda(1+it+n)\zeta(1-it-n)\right\} \\ + O\{q^{1-N}(|t|+1)^{\nu(1,N,\varepsilon)}\} \end{aligned}$$

holds for $0 < \lambda < 1$.

3. Atkinson's dissection

Let u and v be independent complex variables. We suppose first that $\mathrm{Re}\, u > 1$ and $\mathrm{Re}\, v > 1$. Then

$$\phi(\lambda,\alpha,u)\phi(-\lambda,\alpha,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{2\pi i\lambda(m-n)}(m+\alpha)^{-u}(n+\alpha)^{-v}.$$

Following Atkinson [2], we classify this double sum according to the conditions $m = n$, $m > n$ and $m < n$ to get

$$\phi(\lambda,\alpha,u)\phi(-\lambda,\alpha,v) = \zeta(u+v,\alpha) + f(u,v;\lambda,\alpha) + f(v,u;-\lambda,\alpha), \tag{3.1}$$

where

$$f(u,v;\lambda,\alpha) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} e^{2\pi i\lambda m}(m+n+\alpha)^{-u}(n+\alpha)^{-v}. \tag{3.2}$$

Remark. Atkinson used the notation $f(u,v) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-u}(m+n)^{-v}$, which is equal to our $f(v,u;0,1)$. Here the reason for the use of the notation (3.2) is only for notational convenience.

One of the main difficulties in studying the mean squares of zeta-functions lies in the analysis of double sums of the type (3.2). Atkinson [2] succeeded in obtaining the analytic continuation of $f(u,v;0,1)$ (i.e. in the case of $\zeta(u)\zeta(v)$), which led him to the eventual application on taking $u = \frac{1}{2}+it$ and $v = \frac{1}{2}-it$. He treated $f(u,v;0,1)$

by Poisson's summation device (see also T. Meurman [16] for a generalization to L-functions), while K. Matsumoto and the author [9–11], based on the ideas of Y. Motohashi [17] and [8], investigated more general $f(u,v;0,\alpha)$ (i.e. in the case of $\zeta(u,\alpha)\zeta(v,\alpha)$) by using certain double loop integral expressions. In this paper we apply

$$(m+n+\alpha)^{-u}(n+\alpha)^{-v} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)} m^s (n+\alpha)^{-u-v-s} ds, \tag{3.3}$$

where c is a constant fixed with $-\operatorname{Re} u < c < -1$, and (c) denotes the vertical straight line from $c-i\infty$ to $c+i\infty$. This can be obtained by taking $-z = m/(n+\alpha)$ in

$$\Gamma(a)(1-z)^{-a} = \frac{1}{2\pi i}\int_{(\sigma)} \Gamma(a+s)\Gamma(-s)(-z)^s ds \tag{3.4}$$

for $|\arg(-z)| < \pi$ and $-\operatorname{Re} a < \sigma < 0$, which is a special case of Mellin-Barnes' formula for Gauss' hypergeometric function $F(a,b;c;z)$ (cf. [20, p. 289, 14.51, Corollary]). Integrals of the type (3.3) were firstly introduced by Motohashi [18] (see also [5, Chapter 5]) to investigate the fourth power mean of $\zeta(s)$.

We assume for brevity that all singularities appearing in the following argument are at most simple poles, since other cases can be treated by taking limits (see Corollaries 1–4 in Section 2).

Substituting (3.3) into each term on the right-hand side of (3.2), we obtain

$$f(u,v;\lambda,\alpha) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)} \zeta_\lambda(-s)\zeta(u+v+s,\alpha) ds, \tag{3.5}$$

where the interchange of the order of summation and integration can be justified, since, by virtue of the choice of c, the variables $-s$ and $u+v+s$ are both in the region of absolute convergence. As we shall see in the following, the formula (3.5) will provide the analytic continuation of $f(u,v;\lambda,\alpha)$ by modifying suitably the path of integration. Note here that (c) separates the poles at $s=-1+n$ $(n=0,1,2,\ldots)$ from the poles at $s=1-u-v$, $-u-n$ $(n=0,1,2,\ldots)$ of the integrand. If we replace (c) by a contour $\mathcal{C}$ which is suitably indented in such a manner as to separate the poles at $s=1-u-v$, $-1+n$ $(n=0,1,2,\ldots)$ from the poles at $s=-u-n$ $(n=0,1,2,\ldots)$, then we get by the theorem of residues

$$f(u,v;\lambda,\alpha) = \frac{\Gamma(u+v-1)\Gamma(1-v)}{\Gamma(u)} \zeta_\lambda(u+v-1) + g(u,v;\lambda,\alpha)\,, \tag{3.6}$$

where
$$g(u,v;\lambda,\alpha)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)}\zeta_\lambda(-s)\zeta(u+v+s,\alpha)ds. \tag{3.7}$$
Combining this with the corresponding conclusion for $f(v,u;-\lambda,\alpha)$, we see from (3.1) that

Lemma 1

In the region $\operatorname{Re} u>1$ *and* $\operatorname{Re} v>1$, *the formula*

$$\begin{aligned}&\phi(\lambda,\alpha,u)\phi(-\lambda,\alpha,v)\\&=\zeta(u+v,\alpha)+\Gamma(u+v-1)\left\{\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)}+\zeta_{-\lambda}(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right\}\\&\quad+g(u,v;\lambda,\alpha)+g(v,u;-\lambda,\alpha)\end{aligned}\tag{3.8}$$

holds, where $g(v,u;-\lambda,\alpha)$ is similarly defined as in (3.7).

Remark. The formula (3.8) gives a generalization of [11, Lemma 2, (3.4)].

This lemma is fundamental in proving Theorems 1 and 2. We note that (3.8) remains valid in a wider domain of (u,v) by modifying suitably the paths of integrations for $g(u,v;\lambda,\alpha)$ and $g(v,u;-\lambda,\alpha)$.

4. Proof of Theorem 1

The aim of this section is to prove the following Theorem 3, from which Theorem 1 follows immediately by taking $u=\sigma+it$ and $v=\sigma-it$, since

$$\frac{(2-2\sigma)_K(\sigma+it)_{N-K}}{(1-\sigma+it)_N}=O(t^{-K}),$$

$$\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{l^{2\sigma-1}}\int_l^\infty \beta^{2\sigma-K-2}(1+\beta)^{-\sigma-it-N+K}d\beta=O(t^{-1})$$

for $-N+1<\sigma<N+1$ and $t>1$.

Theorem 3

Let u and v be complex variables, and set

$$E^*=\{(u,v);\ u+v\in\mathbb{Z},\ u+v\le 2\}\cup\{(u,v);\ u\in\mathbb{Z}\ \text{ or } v\in\mathbb{Z}\}.$$

Then for any integers $N \geq 1$, in the region $-N+1 < \operatorname{Re} u < N+1$ and $-N+1 < \operatorname{Re} v < N+1$ except the points of E^*, the formula

$$\begin{aligned}
&\int_0^1 \phi_1(\lambda, \alpha, u)\phi_1(-\lambda, \alpha, v)d\alpha \\
&= \frac{1}{u+v-1} + \Gamma(u+v-1)\left\{\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} + \zeta_{-\lambda}(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\
&\quad - S_N(u,v;\lambda) - S_N(v,u;-\lambda) - T_N(u,v;\lambda) - T_N(v,u;-\lambda) \qquad (4.1)
\end{aligned}$$

holds for all $\lambda \in \mathbb{R}$, where

$$S_N(u,v;\lambda) = \sum_{n=0}^{N-1} \frac{(u)_n}{(1-v)_{n+1}} \left\{e^{-2\pi i\lambda}\zeta_\lambda(u+n) - 1\right\}, \qquad (4.2)$$

$$T_N(u,v;\lambda) = \frac{(u)_N}{(1-v)_N} \sum_{l=1}^{\infty} \frac{e^{2\pi i\lambda l}}{l^{u+v-1}} \int_l^\infty \beta^{u+v-2}(1+\beta)^{-u-N} d\beta. \qquad (4.3)$$

Furthermore, for any integer $K \geq 0$

$$\begin{aligned}
T_N(u,v;\lambda) &= \sum_{k=1}^{K}(-1)^{k-1}\frac{(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_N} \sum_{l=1}^{\infty} \frac{e^{2\pi i\lambda l}}{l^k(l+1)^{u+N-k}} \\
&\quad + (-1)^K \frac{(2-u-v)_K(u)_{N-K}}{(1-v)_N} \\
&\quad \times \sum_{l=1}^{\infty} \frac{e^{2\pi i\lambda l}}{l^{u+v-1}} \int_l^\infty \beta^{u+v-K-2}(1+\beta)^{-u-N+K} d\beta. \qquad (4.4)
\end{aligned}$$

Remark. This theorem gives a generalization of [10, Theorem 3], [11, Theorem].

Letting $N \to \infty$ in Theorem 3, since $\lim_{N\to\infty} T_N(u,v;\lambda) = 0$, we obtain

Corollary 5

For any complex u and v except the points of E^*, the formula

$$\begin{aligned}
&\int_0^1 \phi_1(\lambda, \alpha, u)\phi_1(-\lambda, \alpha, v)d\alpha \\
&= \frac{1}{u+v-1} + \Gamma(u+v-1)\left\{\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} + \zeta_{-\lambda}(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\
&\quad - \sum_{n=0}^{\infty} \frac{(u)_n}{(1-v)_{n+1}} \left\{e^{-2\pi i\lambda}\zeta_\lambda(u+n) - 1\right\} \\
&\quad - \sum_{n=0}^{\infty} \frac{(v)_n}{(1-u)_{n+1}} \left\{e^{2\pi i\lambda}\zeta_{-\lambda}(v+n) - 1\right\} \qquad (4.5)
\end{aligned}$$

holds for all $\lambda \in \mathbb{R}$.

Remark. This corollary gives a generalization of [1, (5)] and [10, Corollary 1], [11, Corollary 4].

Proof of Theorem 3. We note first that

$$\int_0^1 \phi_1(\lambda,\alpha,u)\phi_1(-\lambda,\alpha,v)d\alpha = \int_1^2 \phi(\lambda,\alpha,u)\phi(-\lambda,\alpha,v)d\alpha,$$

which is obvious from $\phi_1(\lambda,\alpha,w)=e^{2\pi i\lambda}\phi(\lambda,\alpha+1,w)$ (see (1.1) and the definition of $\phi_1(\lambda,\alpha,w)$). In order to integrate both sides of (3.8), we need

Lemma 2

For any complex $w\neq 1$, and any positive α_1 and α_2, we have

$$\int_{\alpha_1}^{\alpha_2} \zeta(w,\alpha)d\alpha = \frac{1}{w-1}\left\{\zeta(w-1,\alpha_1)-\zeta(w-1,\alpha_2)\right\},$$

and in particular

$$\int_1^2 \zeta(w,\alpha)d\alpha = \frac{1}{w-1}.$$

Proof of Lemma 2. This follows easily by integrating the expression

$$\zeta(w,\alpha) = -\frac{\Gamma(1-w)}{2\pi i}\int_\infty^{(0+)} \frac{(-z)^{w-1}e^{-\alpha z}}{1-e^{-z}}dz \tag{4.6}$$

(cf. [20, p. 266, 13.13]), which is valid for all $w\neq 1$. □

Lemma 2 immediately gives for $\operatorname{Re} u>1$ and $\operatorname{Re} v>1$ that

$$\begin{aligned}
&\int_1^2 \phi(\lambda,\alpha,u)\phi(-\lambda,\alpha,v)d\alpha \\
&= \frac{1}{u+v-1} + \Gamma(u+v-1)\left\{\zeta_\lambda(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} + \zeta_{-\lambda}(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\
&\quad + \int_1^2 g(u,v;\lambda,\alpha)d\alpha + \int_1^2 g(v,u;-\lambda,\alpha)d\alpha.
\end{aligned} \tag{4.7}$$

Moreover, from (3.7)

$$\int_1^2 g(u,v;\lambda,\alpha)d\alpha = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)}\zeta_\lambda(-s)\frac{1}{u+v+s-1}ds, \tag{4.8}$$

where the inversion of the order of integrations can be justified, since, by $\Gamma(-s)\Gamma(u+s) = O(|\mathrm{Im} s|^{\mathrm{Re}\, u-1}e^{-\pi|\mathrm{Im} s|})$ as $\mathrm{Im} s \to \pm\infty$, the right-hand side of (3.7) converges uniformly for all $\alpha \in [1,2]$.

Next we prove

Lemma 3

For any complex z and w with $0 < \mathrm{Re}\, z < \mathrm{Re}\, w$, it follows that

$$\frac{1}{w-z} = \frac{1}{2\pi i}\int_{(\rho_0)} \frac{\Gamma(w)\Gamma(z+r)\Gamma(1+r)\Gamma(-r)}{\Gamma(z)\Gamma(w+1+r)} e^{\pi i r} dr, \tag{4.9}$$

where ρ_0 is a constant fixed with $\max(-\mathrm{Re}\, z, -1) < \rho_0 < 0$.

Proof of Lemma 3. Let θ be a real number with $|\theta| < \pi$. Then Mellin-Barnes' formula for Gauss' hypergeometric function (cf. [3, p. 62, 2.1.3(15)]) implies that

$$\frac{1}{2\pi i}\int_{(\rho_0)} \frac{\Gamma(z+r)\Gamma(1+r)\Gamma(-r)}{\Gamma(w+1+r)} e^{i\theta r} dr = \frac{\Gamma(z)}{\Gamma(w+1)} F(z,1;w+1;-e^{i\theta}).$$

By continuity we may let $\theta \to \pi - 0$ in this equality, provided $\mathrm{Re}\, z < \mathrm{Re}\, w$, since the order of the integrand in (4.9) is $O(|\mathrm{Im} r|^{\mathrm{Re}\, z - \mathrm{Re}\, w - 1})$ as $\mathrm{Im} r \to \pm\infty$. This, together with the identity

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$ and $\mathrm{Re}\,(c-a-b) > 0$ (cf. [3, p. 61, 2.1.3(14)]), yield Lemma 3. $\square$

We suppose at this stage that $\mathrm{Re}\, u > 1$ and $\mathrm{Re}\, v < 1$, where $\mathcal{C}$ can be taken as a straight line (c_0) with $-\mathrm{Re}\, u < c_0 < \min(-1, 1 - \mathrm{Re}\,(u+v))$. Under this choice of c_0, it is possible to fix b_0 such as $\max(-\mathrm{Re}\, u - c_0, -1) < b_0 < 0$. Then we substitute (4.9) with $z = u+s$, $w = 1-v$ and $\rho_0 = b_0$ into the right-hand side of (4.8) to get

$$\int_1^2 g(u,v;\lambda,\alpha) d\alpha = -\frac{1}{2\pi i}\int_{(c_0)} \frac{\Gamma(-s)}{\Gamma(u)} \zeta_\lambda(-s)$$

$$\times \frac{1}{2\pi i}\int_{(b_0)} \frac{\Gamma(1-v)\Gamma(u+r+s)\Gamma(1+r)\Gamma(-r)}{\Gamma(2-v+r)} e^{\pi i r} dr ds, \tag{4.10}$$

where, by virtue of the choice of c_0, the condition $0 < \mathrm{Re}\,(u+s) < \mathrm{Re}\,(1-v)$ of Lemma 3 is fulfilled on the path $\mathrm{Re}\, s = c_0$. To invert the order of the right-hand integrals in (4.10), we temporarily restrict ourselves to the case $\operatorname{Re}(u+v) < 1$. Hence by absolute convergence

$$\int_1^2 g(u,v;\lambda,\alpha)d\alpha = -\frac{1}{2\pi i}\int_{(b_0)} \frac{\Gamma(1-v)\Gamma(u+r)\Gamma(1+r)\Gamma(-r)}{\Gamma(u)\Gamma(2-v+r)} e^{\pi i r} \times \left\{ e^{-2\pi i\lambda}\zeta_\lambda(u+r) - 1 \right\} dr, \tag{4.11}$$

where, to evaluate the resulting inner s-integral, we applied

Lemma 4

For any complex w with $\operatorname{Re} w > 1$, it follows that

$$\frac{1}{2\pi i}\int_{(\sigma_0)} \frac{\Gamma(-s)\Gamma(w+s)}{\Gamma(w)}\zeta_\lambda(-s)ds = e^{-2\pi i\lambda}\zeta_\lambda(w) - 1, \tag{4.12}$$

where σ_0 is a constant fixed with $-\operatorname{Re} w < \sigma_0 < -1$.

Proof of Lemma 4. Since $\zeta_\lambda(-s) = \sum_{l=1}^{\infty} e^{2\pi i\lambda l} l^s$ converges absolutely for $\operatorname{Re} s = \sigma_0$, the term-by-term integration is permissible on the left-hand side of (4.12). Each term in the resulting expression can be evaluated by (3.4), and this shows that the left-hand side of (4.12) is equal to

$$\sum_{l=1}^{\infty} e^{2\pi i\lambda l}(l+1)^{-w} = e^{-2\pi i\lambda}\zeta_\lambda(w) - 1, \tag{4.13}$$

by which the proof of Lemma 4 is complete. □

Let N be an arbitrary positive integer and put $b_N = b_0 + N$. We can shift the path of integration in (4.11) from (b_0) to (b_N), provided $\operatorname{Re}(u+v) < 1$, since the order of the integrand for $\operatorname{Re} r \geq b_0$ is $O(|\operatorname{Im} r|^{\operatorname{Re}(u+v)-2})$ as $\operatorname{Im} r \to \pm\infty$. Collecting the residues at the poles $r = 0, 1, 2, \ldots, N-1$, we get

$$\int_1^2 g(u,v;\lambda,\alpha)d\alpha = -S_N(u,v;\lambda) - T_N(u,v;\lambda). \tag{4.14}$$

Here

$$S_N(u,v;\lambda) = \sum_{n=0}^{N-1} \frac{(u)_n}{(1-v)_{n+1}} \left\{ e^{-2\pi i\lambda}\zeta_\lambda(u+n) - 1 \right\}, \tag{4.15}$$

which gives (4.2), and

$$T_N(u,v;\lambda) = \frac{1}{2\pi i}\int_{(b_N)} \frac{\Gamma(1-v)\Gamma(u+r)\Gamma(1+r)\Gamma(-r)}{\Gamma(u)\Gamma(2-v+r)} e^{\pi i r}\left\{e^{-2\pi i\lambda}\zeta_\lambda(u+r)-1\right\} dr. \tag{4.16}$$

To transform the last integral, we substitute (4.13) and carry out the term-by-term integration, provided $\mathrm{Re}\,(u+v)<1$. Each term in the resulting expression can be evaluated by changing the variable $r=s+N$ and using

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(b_0)} \frac{\Gamma(u+s+N)\Gamma(1+s)\Gamma(-s)(-1)^N}{\Gamma(2-v+s+N)}\left(\frac{e^{\pi i}}{l+1}\right)^{s+N} ds\\ &= (l+1)^{-N}\frac{\Gamma(u+N)}{\Gamma(2-v+N)}F\left(u+N,1;2+N-v;\frac{1}{l+1}\right),\end{aligned}$$

which can be deduced by letting $-z\to e^{\pi i}/(l+1)$ $(0<\arg(-z)<\pi)$ in Mellin-Barnes' formula

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z) = \frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds$$

for $|\arg(-z)|<\pi$ and $\max(-\mathrm{Re}\,a,-\mathrm{Re}\,b)<\sigma<0$ (cf. [3, p. 62, 2.1.3(15)]). We obtain consequently

$$T_N(u,v;\lambda) = \frac{(u)_N}{(1-v)_{N+1}}\sum_{l=1}^{\infty}\frac{e^{2\pi i\lambda l}}{(l+1)^{u+N}}F\left(u+N,1;2+N-v;\frac{1}{l+1}\right). \tag{4.17}$$

Since, by $F(u+N,1;2+N-v;\frac{1}{l+1})\sim 1$ as $l\to+\infty$, the last infinite series in (4.17) converges absolutely for $\mathrm{Re}\,u>-N+1$, $T_N(u,v;\lambda)$ is continued to a meromorphic function of (u,v) in the region $\mathrm{Re}\,u>-N+1$ and any v. We therefore conclude by analytic continuation that the relation (4.7), together with (4.14), (4.15), (4.17), and the corresponding expressions for $S_N(v,u;-\lambda)$ and $T_N(v,u;-\lambda)$, hold in the region $\mathrm{Re}\,u>-N+1$ and $\mathrm{Re}\,v>-N+1$ except the points of E^*.

If (u,v) is in the region $\mathrm{Re}\,u>-N+1$ and $\mathrm{Re}\,v<N+1$, the right-hand of side of (4.17) is further transformed by substituting

$$F\left(u+N,1;2+N-v;\frac{1}{l+1}\right) = (1+N-v)\int_0^1(1-\tau)^{-v+N}\left(1-\frac{\tau}{l+1}\right)^{-u-N}d\tau,$$

which is a special case of Euler's formula

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-\tau z)^{-a}d\tau$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$ and $|z| < 1$ (cf. [3, p. 59, 2.1.3(10)]). Changing the variable $\tau = 1 - l\beta^{-1}$, we obtain (4.3).

Finally, (4.4) can be derived by integrating (4.3) by parts K-times. It is also possible to prove (4.4) by substituting

$$\begin{aligned}
&\frac{(u)_N}{(1-v)_{N+1}}F\left(u+N,1;2+N-v;\frac{1}{l+1}\right)\\
&=\sum_{k=1}^{K}\frac{(-1)^{k-1}(2-u-v)_{k-1}(u)_{N-k}}{(1-v)_N}\left(\frac{l+1}{l}\right)^k\\
&\qquad+(-1)^K\frac{(2-u-v)_K(u)_{N-K}}{(1-v)_{N+1}}\left(\frac{l+1}{l}\right)^K F\left(u+N-K,1;2+N-v;\frac{1}{l+1}\right)
\end{aligned}$$

for any $K \geq 0$ into the right-hand side of (4.17). This in fact follows from the repeated use of a contiguity relation

$$(b-a)(1-z)F(a,b;c;z)-(c-a)F(a-1,b;c;z)+(c-b)F(a,b-1;c;z)=0$$

(cf. [3, p. 103, 2.8(37)]). The proof of Theorem 3 is now complete. □

5. Proof of Theorem 2

We first show

Lemma 5

For any complex $w \neq 1$, and any integer $q \geq 1$, we have

$$\sum_{a=1}^{q}\zeta\left(w,\frac{a}{q}\right)=q^w\zeta(w).$$

Proof of Lemma 5. The result for $\operatorname{Re} w > 1$ follows almost immediately from the definition of the Hurwitz zeta-function, and the proof for all complex $w \neq 1$ follows by analytic continuation. □

Lemma 5 immediately gives for $\mathrm{Re}\, u > 1$ and $\mathrm{Re}\, v > 1$ that

$$\begin{aligned}
&\sum_{a=1}^{q} \phi\left(\lambda, \frac{a}{q}, u\right) \phi\left(-\lambda, \frac{a}{q}, v\right) \\
&= q^{u+v}\zeta(u+v) + q\Gamma(u+v-1)\left\{\zeta_{\lambda}(u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} + \zeta_{-\lambda}(u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right\} \\
&\quad + S(u,v;\lambda,q) + S(v,u;-\lambda,q), \qquad (5.1)
\end{aligned}$$

where

$$\begin{aligned}
S(u,v;\lambda,q) &= \sum_{a=1}^{q} g\left(u,v;\lambda,\frac{a}{q}\right) \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(u+s)\Gamma(-s)}{\Gamma(u)}\zeta_{\lambda}(-s)\zeta(u+v+s)q^{u+v+s}ds. \qquad (5.2)
\end{aligned}$$

We note that $S(u,v;\lambda,q)$ can be handled almost the same manner as in [6, Section 2], so that the details of the proof will be omitted in the following.

We suppose at this stage that $\mathrm{Re}\, u > 1$ and $\mathrm{Re}\, v < 1$, where $\mathcal{C}$ can be taken as a straight line (c_0) with $-\mathrm{Re}\, u < c_0 < \min(-1, 1-\mathrm{Re}\,(u+v))$. Let N be an arbitrary positive integer and c_N a constant fixed with $-\mathrm{Re}\, u - N < c_N < -\mathrm{Re}\, u - N + 1$. Then we can shift the path of integration in (5.2) from (c_0) to (c_N), since the order of the integrand is $O(|\mathrm{Im}\, s|^{C} e^{-\pi|\mathrm{Im}\, s|})$ as $\mathrm{Im}\, s \to \pm\infty$, where $C > 0$ is a constant depending only on $\mathrm{Re}\, u$, $\mathrm{Re}\, v$ and $\mathrm{Re}\, s$. Collecting the residues at the poles $s = -u-n$ $(n = 0, 1, \ldots, N-1)$, we get

$$S(u,v;\lambda,q) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!}\frac{\Gamma(u+n)}{\Gamma(u)}\zeta_{\lambda}(u+n)\zeta(v-n)q^{v-n} + r_N(u,v;\lambda,q), \qquad (5.3)$$

where

$$r_N(u,v;\lambda,q) = \frac{1}{2\pi i}\int_{(c_N)} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)}\zeta_{\lambda}(-s)\zeta(u+v+s)q^{u+v+s}ds.$$

Here the restriction on u and v can be relaxed as

$$\mathrm{Re}\, u > -N+1 \quad \text{and} \quad \mathrm{Re}\, v < N+1. \qquad (5.4)$$

Under (5.4) we can choose c_N satisfying the condition

$$-\mathrm{Re}\, u - N < c_N < \min\big(-1, -\mathrm{Re}\, u - N + 1, 1 - \mathrm{Re}\,(u+v)\big),$$

by which (c_N) separates the poles at $s=-u-n$ $(n=N,N+1,N+2,\ldots)$ from the poles at $s=1-u-v$, $-1+n$ $(n=0,1,2,\ldots)$, $-u-n$ $(n=0,1,\ldots,N-1)$.

Theorem 2 now follows by taking $u=\sigma+it$ and $v=\sigma-it$ in (5.1), (5.3) and the corresponding expression for $S(v,u;-\lambda,q)$. Almost the same argument as in [6, Section 3] applies to deduce upper bounds for $r_N(\sigma\pm it,\sigma\mp it;\pm\lambda,q)$, which gives the error estimate in (2.2). This completes the proof of Theorem 2. □

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