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On extensions of mixed motives

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Abstract

In this article we give an introduction to mixed motives and sketch a couple of ways to construct examples.

0. Introduction

In this article which is addressed to a fairly general audience we give an introduction to mixed motives and sketch a couple of ways to construct examples due to Deligne, Jannsen, Scholl, Kings, Huber and myself. Our list of examples is by no means complete. For further constructions we refer e.g. to [12], [13]. Moreover we outline the relationship between periods of critical mixed motives and the values of their L–functions at integers conjectured by Deligne and Scholl building on the ideas of Bloch and Beilinson. The mixed motives we discuss are constructed from cycles, elements in algebraic K–groups, elliptic curves, modular forms and certain Laurent polynomials in several variables. In the latter case a particular Deligne period of the mixed motive attached to P has an interpretation as the logarithm of the Mahler measure of P and hence by work of Lind, Schmidt and Ward [18] as the entropy of a natural expansive $\mathbb{Z}^n$–action on a compact topological group. This connection may be of interest to mathematicians working on dynamical systems.

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Contents:

1. Mixed motives and their L–functions

The singular cohomology is a basic tool in the investigation of topological spaces. It often reflects important topological properties and can be calculated in many cases. In algebraic geometry of varieties over separably closed fields étale cohomology plays a similar role.

If the topological space is the set of $\mathbb{C}$–valued points with the analytic topology of a complex algebraic variety then its singular cohomology is functorially equipped with a mixed Hodge structure [5]. This additional structure contains important information on the analytic structure of the variety. For any two elliptic curves over $\mathbb{C}$ for example the singular cohomology groups are isomorphic. The Hodge structures on the other hand are isomorphic if and only if the elliptic curves are isomorphic as varieties.

In étale cohomology we have a similar picture. Assume that the variety $\overline{X} = X \otimes_k k^s$ is the base change to the separable closure k^s of a variety X over a field k. Then the absolute Galois group G_k of k acts on $\overline{X}$ and hence on the étale cohomology of $\overline{X}$. In many respects étale cohomology with this Galois operation behaves similar to singular cohomology with its Hodge structure. For example in both cases cohomology equipped with its additional structure can be viewed as a functor into an abelian category in which there are defined notions of a tensor product and duals.

A further example in this spirit is given by the crystalline cohomology of varieties in characteristic p with its Frobenius action.

A somewhat weaker example is the algebraic de Rham cohomology of algebraic varieties over k which takes values in the category of filtered k–vector spaces. Here the target category is exact but not abelian.

In the sixties Grothendieck developed the idea that for varieties over a field k there should exist a universal cohomology theory H^* with values in a $\mathbb{Q}$-linear abelian category $\mathcal{MM}_k$ having a notion of $\otimes$ and duals. All the aforementioned cohomologies should be obtained by specializing this universal theory. For example there should be faithful realization functors:

$$\mathcal{MM}_k \xrightarrow{(\,)_l} \text{finite dimensional continuous } \mathbb{Q}_l[G_k]\text{–modules}$$
$$\mathcal{MM}_k \xrightarrow{(\,)_B} \mathbb{Q} - \mathcal{MH} = \mathbb{Q}\text{-mixed Hodge structures} \quad (\text{if } k \subset \mathbb{C})$$
$$\mathcal{MM}_k \xrightarrow{(\,)_{dR}} \text{finite dimensional filtered } k\text{–vector spaces (if char}\, k = 0)$$

such that

$$H_l^*(X) := H^*_{\text{ét}}\big(X \otimes_k k^s, \mathbb{Q}_l\big) + G_k\text{–action} = \big(H^*(X)\big)_l$$
$$H_B^*(X) := H^*_{\text{sing}}\big(X(\mathbb{C}), \mathbb{Q}\big) + \text{Hodge structure} = \big(H^*(X)\big)_B$$

and

$$H^*_{dR}(X) := H^*_{dR}(X/k) + \text{Filtration} = (H^*(X))_{dR}\,.$$

More generally for every immersion $Y \hookrightarrow X$ of varieties there should be attached the relative motive $H^*(X \operatorname{rel} Y)$ whose cohomologies are the usual relative cohomologies. Corresponding facts should also be true for homology. As the reader may have guessed the objects of the category $\mathcal{MM}_k$ are to be called (mixed) motives over k. Like its realizations every motive M should be equipped with a functorial increasing "weight–filtration" $W_\bullet M$ indexed by the integers and satisfying $W_n M = 0$ for $n \ll 0$ and $W_n M = M$ for $n \gg 0$. For all n the associated graded object $\operatorname{Gr}_n^W M$ should be isomorphic in $\mathcal{MM}_k$ to a subquotient of $H^n(Y)$ for a suitable *smooth projective* variety Z over k. The n's for which $\operatorname{Gr}_n^W M$ is non–zero are called the weights of M. A motive is pure (of weight w) if $\operatorname{Gr}_n^W M = 0$ for $n \neq w$. For any smooth projective variety Z the motive $H^w(Z)$ is to be pure of weight w. The full subcategory of sums of pure motives should be semisimple. The extension groups $\operatorname{Ext}^i_{\mathcal{MM}_k}(N, M)$ describe to some extent how more complicated varieties are built out of simpler ones. If k is a number field these groups should be zero for $i \geq 2$ and related to higher algebraic K-groups if $i = 1$.

In spite of much effort this program for a theory of motives has not yet been fully realized. An overview on what is known is given in [16]. At present only the constructions "via realizations" of Deligne [7], Jannsen [15] and Huber [14] yield abelian categories on which the above mentioned realization functors are defined. In their approach the étale, Hodge and de Rham cohomologies are in some sense "glued together". The explicit constructions of mixed motives in §§ 4, 5, 6 are very geometric and work in any category of mixed motives which satisfies enough of the expected formal properties e.g. in any of the categories $\mathcal{MM}_k$ defined "via realizations".

In the following we will take as ground field the field $k = \mathbb{Q}$ of rational numbers and write $\mathcal{MM}$ for $\mathcal{MM}_{\mathbb{Q}}$. Note that any variety X over a number field can be viewed as an algebraic scheme over $\mathbb{Q}$ and hence gives rise to motives $H^n(X)$ in $\mathcal{MM}$.

We next recall the definition of the L-series of a motive M in $\mathcal{MM}$ c.f. [9]. For any prime number p fix a prime $l \neq p$, let I_p be the inertia group in the Galois group G_p of $\mathbb{Q}_p$ and define the geometric Frobenius F_p to be the inverse of the canonical generator of G_p/I_p which maps x to x^p in $\overline{\mathbb{F}}_p$. If M_l denotes the l-adic realization of M define the local L-factor of M at p by the formula

$$L_p(M, s) = \det\left(1 - F_p p^{-s} \mid M_l^{I_p}\right)^{-1} .$$

It is expected that it is the inverse of a polynomial in $\mathbb{Z}[p^{-s}]$ which is independent of the chosen prime $l \neq p$. By the work of Deligne this is known for example if $M = H^n(X)$ and there is a smooth and proper scheme $\mathfrak{X}/\operatorname{spec}\mathbb{Z}_p$ such that

$$X \otimes_{\mathbb{Q}} \mathbb{Q}_p = \mathfrak{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p .$$

Assuming this the L-series of M is defined for $\operatorname{Re} s \gg 0$ as the Euler product

$$L(M, s) = \prod_p L_p(M, s) .$$

According to the conjectures $L(M, s)$ should have a meromorphic continuation to $\mathbb{C}$.

For $M = \mathbb{Q}(0) := H^0(\operatorname{spec}\mathbb{Q})$ for example we have $M_l = \mathbb{Q}_l$ with trivial Galois action and hence $L(M, s) = \zeta(s)$ is the Riemann zeta function.

If $M = H^1(A)$ where A is an abelian variety over $\mathbb{Q}$ one finds that $L(M, s) = L(A, s)$ is the L-series classically associated to A e.g. in the Birch–Swinnerton-Dyer conjecture.

Finally we need the notion of the Tate twist in $\mathcal{MM}$:

Set $\mathbb{Q}(-1) = H^1(\mathbb{G}_m)$ and $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$ and $\mathbb{Q}(n) = \mathbb{Q}(-n)^{\vee}$ for $n \geq 1$. Then the functor $M \mapsto M(n) = M \otimes \mathbb{Q}(n)$ defines an autoequivalence of $\mathcal{MM}$ for all $n \in \mathbb{Z}$ and we have:

$$L(M(n), s) = L(M, s + n)$$

since $\mathbb{Q}(1)_l = \left(\varprojlim_{\nu} \mu_{l^\nu}(\overline{\mathbb{Q}})\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ as a Galois module.

2. Extensions of mixed motives attached to algebraic cycles and to elements in K-theory

To any p-codimensional algebraic cycle given up to rational equivalence $z \in CH^p(X)$ in a smooth projective variety X over a field k one associates a cycle class $cl_l(z)$, $l \neq \operatorname{char} k$ in

$$H^{2p}\big(X \otimes_k k^s, \mathbb{Q}_l(p)\big)^{G_k} = \operatorname{Hom}_{\mathbb{Q}_l[G_k]}\big(\mathbb{Q}_l, H_l^{2p}(X)(p)\big) \,.$$

If $k \subset \mathbb{C}$ there is a cycle class $cl_\infty(z)$ in

$$H^{2p}\big(X(\mathbb{C}), \mathbb{Q}(p)\big) \cap H^{p,p}\big(X(\mathbb{C}), \mathbb{C}\big) = \operatorname{Hom}_{\mathbb{Q}-\mathcal{MH}}\big(\mathbb{Q}(0), H_B^{2p}(X)(p)\big)$$

and similarly in other cohomology theories. It is a basic requirement for a good category $\mathcal{MM}$ that these cycle classes should be induced from a homomorphism:

$$cl : CH^p(X)_{\mathbb{Q}} \longrightarrow \operatorname{Hom}_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p}(X)(p)\big) \,. \tag{2.1}$$

Here we write $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$ for any abelian group A. If $\mathcal{MM}$ is defined via realizations this property follows from the compatibility of the various cycle classes under the comparison isomorphisms.

In fact one would like to have that cl induces an isomorphism:

$$cl : CH^p(X)_{\mathbb{Q}}/CH^p(X)^0_{\mathbb{Q}} \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p}(X)(p)\big) \,,$$

where $CH^p(X)^0_{\mathbb{Q}} = \operatorname{Ker} cl_l = \operatorname{Ker} cl_\infty$ is the subspace of cycles homologically equivalent to zero. For $\mathcal{MM}$ defined via realizations this would follow from either the Hodge or the Tate conjecture. For more geometrically defined categories of motives one builds this property into the definitions.

We remark that for any $z \in CH^p(X)$ with support $Z \subset X$ the morphism $cl(z) : \mathbb{Q}(0) \to H^{2p}(X)(p)$ is induced from a morphism $cl(z) : \mathbb{Q}(0) \to H^{2p}(X \operatorname{rel} U)(p)$ by the natural map $H^{2p}(X \operatorname{rel} U)(p) \to H^{2p}(X)(p)$ where $U = X \setminus Z$.

The cycles z in $CH^p(X)^0_{\mathbb{Q}}$ give rise to extensions of motives as follows: Consider the relative exact sequence of motives in $\mathcal{MM}$

$$0 = H^{2p-1}(X \operatorname{rel} U) \to H^{2p-1}(X) \to H^{2p-1}(U) \to H^{2p}(X \operatorname{rel} U) \to H^{2p}(X) \,.$$

Since z is homologous to zero we can pull this back via $cl(z)$ after twisting by $\mathbb{Q}(p)$:

$$\begin{array}{ccc} 0 \to H^{2p-1}(X)(p) \to H^{2p-1}(U)(p) & \to & H^{2p}(X \operatorname{rel} U)(p) \\ & & \uparrow cl(z) \\ & & \mathbb{Q}(0) \end{array}$$

to get an extension of $\mathbb{Q}(0)$ by $H^{2p-1}(X)(p)$. The resulting homomorphism:

$$CH^p(X)^0_{\mathbb{Q}} \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p-1}(X)(p)\big) \tag{2.2}$$

is called the Abel–Jacobi map. For its relation to the classical Abel–Jacobi map and more details we refer to [15] § 9. In an ideal category $\mathcal{MM}$ the map (2.2) should be an isomorphism. Higher analogues of it have been described by Scholl [26]. His generalized Abel–Jacobi maps

$$CH^p(X,n)_{\mathbb{Q}} \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p-(n+1)}(X)(p)\big) \tag{2.3}$$

are defined on Bloch's higher Chow groups for $n \geq 1, p \geq 0$.

On the other hand it was pointed out by Deligne that via a Chern class map extensions of motives should come from elements of algebraic K-theory. For $\mathcal{MM}$ defined by realizations this has been made precise by A. Huber. In [14] she constructs for every simplicial variety $X_\bullet$ over $\mathbb{Q}$ such that

$$\mathrm{Hom}_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p-n}(X_\bullet)(n)\big) = 0$$

a functorial map

$$\mathrm{Gr}^{\,p}_{\gamma} K_n(X_\bullet)_{\mathbb{Q}} \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}}\big(\mathbb{Q}(0), H^{2p-(n+1)}(X_\bullet)(p)\big)\,, \tag{2.4}$$

where Gr_{γ} is the graded space associated to the γ- filtration on algebraic K-theory. If $X_\bullet$ comes from a smooth, projective variety X it is known that the left hand sides of (2.3) and (2.4) are isomorphic. However the compatibility of the two maps has not been established in the literature.

With a good category of mixed motives one would expect (2.3) and (2.4) to be isomorphisms.

3. Periods of mixed motives and L-values

For any motive M in $\mathcal{MM}$ the $\mathbb{Q}$-vector space underlying M_B carries a natural $G_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action and we write M_B^+ for the fixed part. If $M = H^n(X)$ for a variety X over $\mathbb{Q}$ then M_B^+ is the subspace of $H^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q})$ which is fixed by F_∞^* where F_∞ denotes the antiholomorphic involution of $X(\mathbb{C})$ induced by complex conjugation. There is a canonical comparison isomorphism of $\mathbb{C}$-vector spaces

$$M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes \mathbb{C} \tag{3.1}$$

which is $G_{\mathbb{R}}$-equivariant if $G_{\mathbb{R}}$ act diagonally on the left and via the second factor on the right. Taking $G_{\mathbb{R}}$-invariants one obtains the perfect $\mathbb{R}$-linear period pairing:

$$\langle , \rangle_{\mathrm{per}} : (M_B \otimes \mathbb{C})^+ \otimes_{\mathbb{R}} (\check{M}_{dR} \otimes \mathbb{R}) \longrightarrow \mathbb{R} \,. \tag{3.2}$$

For the motive $N = \check{M}(1)$ we have $F^0 N_{dR} = F^1 \check{M}_{dR}$. Following Deligne and Scholl now consider the following restriction of the pairing (3.2):

$$\langle , \rangle_{\mathrm{per}} : (M_B^+ \otimes \mathbb{R}) \otimes_{\mathbb{R}} (F^0 N_{dR} \otimes \mathbb{R}) \longrightarrow \mathbb{R} \,. \tag{3.3}$$

We will call "Deligne periods of M" the image under $\langle , \rangle_{\mathrm{per}}$ of $M_B^+ \times F^0 N_{dR}$ in $\mathbb{R}$. As an example let $M = H_1(A) = H^1(A)^\vee$ for an abelian variety A over $\mathbb{Q}$. Then

$$M_B^+ = H_1(A(\mathbb{C}), \mathbb{Q})^+$$

and

$$F^0 N_{dR} = F^1 H^1_{dR}(A/\mathbb{Q}) \cong H^0(A, \Omega^1)\,,$$

and the Deligne periods are given by integrals

$$\langle \gamma, \omega \rangle_{\mathrm{per}} = \int_\gamma \omega \,.$$

The importance of the Deligne periods stems from the following concepts and conjecture due originally to Deligne in the pure case [6] and then extended to mixed motives by Scholl [24].

The motive M in $\mathcal{MM}$ is called *critical* if the pairing (3.3) is non-degenerate. The Deligne period determinant

$$c^+(M) = \det\big(\langle \gamma_i, \omega_j \rangle_{\mathrm{per}}\big)$$

where $\{\gamma_i\}$ and $\{\omega_j\}$ are bases of M_B^+ resp. $F^0 N_{dR}$ gives a well defined element of $\mathbb{R}^*/\mathbb{Q}^*$. The motive M is called *integral* (over $\mathbb{Z}$) if for all prime numbers p and all $l \neq p$ the weight filtration on M_l splits if M_l is considered as a module under the inertia group I_p in G_p. Thus for example all pure motives are integral. Let $\mathcal{MM}_{\mathbb{Z}}$ be the full subcategory of $\mathcal{MM}$ of integral motives. Then as shown by Scholl [25] the following conjecture is equivalent under certain "standard" assumptions to the conjunction of Beilinson's conjectures [1] on special values of motivic L-series.

Conjecture 3.4

For any motive M in $\mathcal{MM}_{\mathbb{Z}}$ we have

a) $\mathrm{ord}_{s=0} L(M,s) = \dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}(M, \mathbb{Q}(1)) - \dim \mathrm{Hom}_{\mathcal{MM}_{\mathbb{Z}}}(M, \mathbb{Q}(1))$.
If in addition M is critical then:
b) $L(M,0) \equiv c^+(M) \bmod \mathbb{Q}^*$ if $L(M,0)$ is nonzero.

4. Extensions attached to CM elliptic curves

In this and the following sections we give examples of arithmetically interesting extensions of mixed motives which are directly constructed using the surrounding geometry. The constructions work in any abelian category of mixed motives over $\mathbb{Q}$ satisfying enough of the basic formal properties e.g. in the categories $\mathcal{MM}$ defined using realizations [7], [15], [14].

A much more detailed treatment of the material in this section is given in [10].

Consider a CM elliptic curve $E_0/\mathbb{Q}$ and let $k \geq 2$ be an integer. Then we have

$$\mathrm{Hom}_{\mathcal{MM}}\big(H^1(E_0)(2-k), \mathbb{Q}(1)\big) = 0$$

since $H^1(E_0)(2-k)$ has weight $2k-3 \neq -2$.

Thus by (3.4) a) the $\mathbb{Q}$-vector space

$$\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}\big(\mathbb{Q}(0), H^1(E_0)(k)\big) = \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}\big(H^1(E_0)(2-k), \mathbb{Q}(1)\big)$$

is expected to have dimension equal to the vanishing order of

$$L\big(H^1(E_0)(2-k), s\big) = L(E_0, s+2-k)$$

at $s=0$ which is known to be one. Thus we should have:

$$\dim \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}\big(\mathbb{Q}(0), H^1(E_0)(k)\big) = 1\,.$$

The following result shows that the dimension of $\mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Z}}}(\mathbb{Q}(0), H^1(E_0)(k))$ is at least one and gives evidence for conjecture (3.4) b):

Theorem 4.1

The following construction gives an extension in $\mathcal{MM}_{\mathbb{Z}}$

$$0 \longrightarrow H^1(E_0)(k) \longrightarrow M \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

which is critical and for which we have:

$$L(M,0) \equiv c^+(M) \mod \mathbb{Q}^*\,.$$

Note that

$$L(M,s) = \zeta(s) L(E_0, s+k) \text{ and hence } L(M,0) = -\frac{1}{2}\, L(E_0,k)\,.$$

Moreover $\dim M_B^+ = 2$, so that $c^+(M)$ is the determinant of a 2×2-matrix.

Remark. More generally one can treat the motives of Hecke characters of imaginary quadratic fields instead of $H^1(E_0)$.

Proof. The *construction* of M relies heavily on ideas of Beilinson and Scholl. Fix integers $n \geq 1, N \geq 3$ and an isomorphism $(\mathbb{Z}/N)^2 \xrightarrow{\sim} E_{0N}(\overline{\mathbb{Q}})$. Furthermore let $K/\mathbb{Q}$ be a finite extension such that the N-torsion of $E = E_0 \otimes K$ is K-rational and complex multiplication is defined over K. The group $(\mathbb{Z}/N)^2$ acts by translation on E. Also the group $\mu_2 = \{\pm 1\}$ acts by inversion on E. We thus get a natural action of the semidirect product

$$\Gamma_n = \left((\mathbb{Z}/N)^2 \rtimes \mu_2\right)^n \rtimes \mathfrak{S}_n$$

on E^n. Let $\varepsilon = \varepsilon_n : \Gamma_n \to \mu_2$ be the character that is trivial on $(\mathbb{Z}/N)^{2n}$, the product on μ_2^n and the sign character on $\mathfrak{S}_n$.

Let $M(N)$ be the affine modular curve over $\mathbb{Q}$ classifying elliptic curves with a level N-structure. By adding a finite scheme of cusps $M^\infty(N)$ one obtains the compactified smooth projective modular curve $\overline{M}(N) = M(N) \cup M^\infty(N)$. Let $\mathcal{E}(N) \xrightarrow{\pi} M(N)$ be the universal elliptic curve over $M(N)$ and $\overline{\mathcal{E}}(N) \xrightarrow{\pi} \overline{M}(N)$ the universal generalized elliptic curve over $\overline{M}(N)$. There exists a functorial desingularization $\overline{\overline{\mathcal{E}}}(N)^n$ of the n-fold cartesian product

$$\overline{\mathcal{E}}(N)^n = \overline{\mathcal{E}}(N) \times_{\overline{M}(N)} \ldots \times_{\overline{M}(N)} \overline{\mathcal{E}}(N)\,.$$

Note that the fibers of $\overline{\mathcal{E}}(N)$ over the cusps are singular and that $\overline{\mathcal{E}}(N)^n$ is a singular variety over $\mathbb{Q}$ for $n \geq 2$. As above the group Γ_n acts naturally on $\mathcal{E}(N)^n, \overline{\mathcal{E}}(N)^n$ and $\overline{\overline{\mathcal{E}}}(N)^n$.

For any object H in a $\mathbb{Q}$-linear abelian category on which Γ_n acts write $H(\varepsilon)$ for its ε-isotypical component:

$$H(\varepsilon) = \operatorname{Im}\left(|\Gamma_n|^{-1} \sum_{\sigma \in \Gamma_n} \varepsilon(\sigma)\sigma : H \to H\right)\,.$$

Finally we set $H^n(Z, m) = H^n(Z)(m)$ in $\mathcal{MM}$. According to a result of Scholl in [23] one has a natural exact sequence in $\mathcal{MM}$ for $n \geq 1$:

$$0 \to H^{n+1}\big(\overline{\overline{\mathcal{E}}}(N)^n, n+1\big)(\varepsilon) \to H^{n+1}\big(\mathcal{E}(N)^n, n+1\big)(\varepsilon) \underset{\overleftarrow{\ \mathrm{Eis}\ }}{\xrightarrow{\mathrm{res}}} H^0\big(M^\infty(N)\big)^{(n)} \to 0$$

which is canonically split by the Manin–Drinfeld principle: The Hecke operators act with different eigenvalues on $H^{n+1}(\overline{\overline{\mathcal{E}}}(N)^n, n+1)(\varepsilon)$ and $H^0(M^\infty(N))^{(n)}$ (which denotes the motive $H^0(M^\infty(N))$ but with a twisted action of the Hecke algebra).

Since $H^{n+1}(\overline{\overline{\mathcal{E}}}(N)^n, n+1)(\varepsilon)$ is pure of weight $-(n+1) \neq 0$ as $\overline{\overline{\mathcal{E}}}(N)^n$ is smooth projective over $\mathbb{Q}$ we get an isomorphism of $\mathbb{Q}$-vector spaces:

$$\mathbb{Q}[M^\infty(N)]$$
$$= \mathrm{Hom}_{\mathcal{MM}}\Big(\mathbb{Q}(0), H^0\big(M^\infty(N)\big)\Big) \overset{\mathrm{Eis}_*}{\cong} \mathrm{Hom}_{\mathcal{MM}}\Big(\mathbb{Q}(0), H^{n+1}\big(\mathcal{E}(N)^n, n+1\big)(\varepsilon)\Big)$$

where $\mathbb{Q}[M^\infty(N)]$ denotes the space of $\mathbb{Q}$-linear divisors on $M^\infty(N)$. The chosen level N-structure on E gives a Γ_n-equivariant embedding $i : E \hookrightarrow \mathcal{E}(N)$. It is not difficult to see that: $H^i(\mathcal{E}(N)^n)(\varepsilon) = 0$ for $i \neq n+1$ and $H^i(E^n)(\varepsilon)$ for $i \neq n$. Hence we obtain an exact sequence in $\mathcal{MM}$:

$$0 \to H^n\big(E^n, n+1\big)(\varepsilon) \to H^{n+1}\big(\mathcal{E}(N)^n \mathrm{rel}\, E^n, n+1\big)(\varepsilon)$$
$$\to H^{n+1}\big(\mathcal{E}(N)^n, n+1\big)(\varepsilon) \to 0\,.$$

To construct the extension M of theorem (4.1) one first notes that for $k \geq 2$ setting $n = 2k-3$ the motive $H^1(E_0)(k)$ is a direct summand of

$$H^n(E^n, n+1)(\varepsilon) = \mathrm{Sym}^n H^1(E)(n+1)\,.$$

If $k \geq 3$ this requires the hypothesis of complex multiplication. Now let

$$\pi : H^n(E^n, n+1)(\varepsilon) \longrightarrow H^1(E_0)(k)$$

be the projection and recall that divisors α in $\mathbb{Q}[M^\infty(N)]$ correspond to maps

$$\mathrm{Eis}_*(\alpha) : \mathbb{Q}(0) \longrightarrow H^{n+1}\big(\mathcal{E}(N)^n, n+1\big)(\varepsilon).$$

Pushing the above extension forward via π and pulling it back via $\mathrm{Eis}_*(\alpha)$ for suitable α we get the sought for extension M. For the verification that $c^+(M)$ is a non-vanishing rational multiple of $L(M,0) = -\frac{1}{2}L(E_0,k)$ one first calculates the Hodge realization of the Eisenstein splitting Eis in terms of holomorphic Eisenstein series. In the period calculations for $c^+(M)$ certain non-holomorphic Eisenstein series appear. Finally one uses the classical fact that the L-series of a CM elliptic curve can be expressed in terms of such series. For the complete proof one reformulates [10] using the considerations of [24]. Finally it can be shown that the extensions just constructed agree with those obtained via the map (2.4) from the K-theory elements constructed earlier in [3], [1] Ch. 2, [8]. □

5. Extensions attached to modular forms

The construction of extensions described in this section is due to Kings [17] building upon the results of Beilinson [2] and Scholl [27].

In this section we will work in the category $\mathcal{MM} \otimes \overline{\mathbb{Q}}$ which has the same objects as $\mathcal{MM}$ but where the morphisms are tensored with $\overline{\mathbb{Q}}$. For the following it will be useful to regard $M(N), \mathcal{E}(N)$ etc. in the limit: Consider the projective systems

$$\mathcal{E}^n = \varprojlim_N \mathcal{E}(N)^n \quad \text{and} \quad \overline{\overline{\mathcal{E}}}^n = \varprojlim_N \overline{\overline{\mathcal{E}}}(N)^n$$

of schemes with finite transition maps. For $n = 0$ set $\overline{\overline{\mathcal{E}}}^0 = M$. Their motives are defined to be

$$H^*(\mathcal{E}^n) = \varinjlim_N H^*\big(\mathcal{E}(N)^n\big) \quad \text{and} \quad H^*\big(\overline{\overline{\mathcal{E}}}^n\big) = \varinjlim_N H^*\big(\overline{\overline{\mathcal{E}}}(N)^n\big)$$

in the ind category of $\mathcal{MM} \otimes \overline{\mathbb{Q}}$. For $n \geq 1$ the group $\Gamma_n = (\mathbb{A}_f^2 \rtimes \mu_2)^n \rtimes \mathfrak{S}_n$ acts on $H^*(\mathcal{E}^n)$ and $H^*(\overline{\overline{\mathcal{E}}}^n)$ and we can consider ε-isotypic components with respect to the character $\varepsilon = \varepsilon_n : \Gamma_n \to \mu_2$ defined as before. In addition the group $\mathrm{GL}_2(\mathbb{A}_f)$ acts on these motives preserving the ε-isotypical components. According to Deligne and Scholl [23] for $k \geq 0$ the $\mathrm{GL}_2(\mathbb{A}_f)$-isotypical constituents of $H^{k+1}(\overline{\overline{\mathcal{E}}}^k)(\varepsilon)$ are precisely the motives attached to holomorphic cusp forms of weight $k+2$ i.e. the motives whose L-functions equal the L-function of the corresponding cusp forms. One shows: All the extensions of $\overline{\mathbb{Q}}(-k-l-2), l \geq 0$ by the motives of cusp forms of weight $k+2$ predicted by a $\overline{\mathbb{Q}}$-version of conjecture (3.4) a) are obtained by pushing out the extensions of $\overline{\mathbb{Q}}(-k-l-2)$ by $H^{k+1}(\overline{\overline{\mathcal{E}}}^k)(\varepsilon)$ described next.

Consider all pairs (r, s) of integers with $r + s = k + 2l$ and $r \geq s \geq l$. The natural embedding

$$i : \mathcal{E}^{k+2l} = \mathcal{E}^{r+s} \hookrightarrow \mathcal{E}^r \times_{\mathbb{Q}} \mathcal{E}^s$$

is equivariant with respect to the natural $\Gamma_r \times \Gamma_s$-action. One checks that for $s > 0$ the relative exact sequence with respect to i yields the short exact sequence

$$\begin{aligned} 0 \to H^{k+2l+1}\big(\mathcal{E}^{k+2l}\big)\big(\varepsilon_r \times \varepsilon_s\big) &\to H^{k+2l+2}\big(\mathcal{E}^r \times_{\mathbb{Q}} \mathcal{E}^s \mathrm{rel}\, \mathcal{E}^{k+2l}\big)\big(\varepsilon_r \times \varepsilon_s\big) \\ &\to H^{r+1}\big(\mathcal{E}^r\big)\big(\varepsilon_r\big) \otimes H^{s+1}\big(\mathcal{E}^s\big)\big(\varepsilon_s\big) \to 0\,. \end{aligned}$$

Moreover there is a natural projection

$$Q : H^{k+2l+1}\big(\mathcal{E}^{k+2l}\big)\big(\varepsilon_r \times \varepsilon_s\big) \longrightarrow H^{k+1}\big(\mathcal{E}^k, -l\big)\big(\varepsilon_k\big)\,.$$

Both facts are consequences of the Künneth- and Clebsch Gordan formulas. Using the Eisenstein splitting of the preceding section one gets from Q a projection

$$\overline{Q} : H^{k+2l+1}\big(\mathcal{E}^{k+2l}\big)\big(\varepsilon_r \times \varepsilon_s\big) \longrightarrow H^{k+1}\big(\overline{\overline{\mathcal{E}}}^k, -l\big)\big(\varepsilon_k\big)\,.$$

In addition one gets maps

$$\lambda : \overline{\mathbb{Q}}(0) \longrightarrow H^{r+1}(\mathcal{E}^r, r+1)(\varepsilon_r) \otimes H^{s+1}(\mathcal{E}^s, s+1)(\varepsilon_s)$$

associated to pairs of divisors on the cusps.

Pushing out the above exact sequence via $\overline{Q}$ and pulling it back via the maps λ one obtains extensions of $\overline{\mathbb{Q}}(-k-l-2)$ by $H^{k+1}(\overline{\overline{\mathcal{E}}}^k)(\varepsilon_k)$ as desired. One also needs to consider the case $s = 0$ where also $l = 0$. In this instance the construction obviously has to be modified to get non- trivial extensions. We refer to [17] II 3. for the details. In loc. cit. is is also explained that the extensions associated to modular forms just described parallel the earlier K-theory constructions by Beilinson and Scholl.

6. Extensions attached to Laurent polynomials and entropy

For a nonzero Laurent polynomial P in $\mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ set

$$\begin{aligned} m(P) &= \frac{1}{(2\pi i)^n} \int_{T^n} \log|P(z_1, \ldots, z_n)| \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n} \\ &= \int_0^1 \ldots \int_0^1 \log|P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \ldots d\theta_n \end{aligned}$$

where $T^n = (S^1)^n$ is the n-dimensional real torus. The integral exists and defines a real non-negative number whose exponential $M(P) = \exp(m(P))$ is called the Mahler measure of P. It appears in transcendence theory as a local height of the polynomial P at the infinite place [20]. The study of $M(P)$ is also important in the context of Lehmer's problem c.f. [4]. More recently $m(P)$ has been interpreted as the entropy of an associated $\mathbb{Z}^n$-action on a compact abelian topological group as follows. Let $Z(P) \subset \mathbb{G}^n_{m,\mathbb{Z}}$ denote the closed subscheme defined by P. The group $\mathbb{Z}^n$ acts by translations on the regular functions

$$\Gamma(Z(P), \mathcal{O}) = \mathbb{Z}[\mathbb{Z}^n]/(P)$$

of $Z(P)$. Viewing $\Gamma(Z(P), \mathcal{O})$ as a topological group under addition with the discrete topology we get by functoriality an action of $\mathbb{Z}^n$ on the compact Pontrjagin dual $\Gamma(Z(P), \mathcal{O})^*$.

The topological entropy of a continuous $\mathbb{Z}^n$-action on a compact metrizable space is a nonnegative real number which measures the extent to which repeated application of the action "scatters around points". The concept originated in thermodynamics and information theory where it is a measure of disorder and loss of information respectively. We refer to [19] for a lucid introduction to the concept of entropy in the one-variable case $n = 1$ and to [22] Ch. V for the general situation. There is also a notion of metric entropy for a measurable $\mathbb{Z}^n$-action on a probability space. For a continuous $\mathbb{Z}^n$-action on a compact topological group with probability measure given by the normalized Haar measure fortunately the two notions of entropy coincide [22] V. Th. 13.3.

The following result has been established by Lind, Schmidt and Ward in [18] c.f. also [22] V. Th. 18.1:

Theorem 6.1

For P as above the entropy of the associated $\mathbb{Z}^n$-action on $\Gamma(Z(P), \mathcal{O})^*$ is equal to $m(P)$.

The case $n = 1$ of this result was proved in the mid sixties by Yuzvinskii, the first results being due to Sinai by the end of the fifties. Examples by among others Smyth [28], Boyd [4] and Ray [21] show that for certain simple Laurent polynomials P the value $m(P)$ is related to special values of arithmetic L-functions e.g.:

$$m\big(1 + T_1 + T_2 + T_3\big) = \frac{7}{2\pi^2}\,\zeta(3) \tag{6.2}$$

$$m\big((T_1 + T_2)^2 \pm 3\big) = \frac{2}{3}\log 3 + \frac{\sqrt{3}}{\pi} L\big(\chi_3, 2\big) \tag{6.3}$$

where $\chi_3 : (\mathbb{Z}/3)^* \to \mu_2$ is the non-trivial character of $(\mathbb{Z}/3)^*$. See e.g. [22] VI, 19.10 and 19.11. for proofs.

In the following we will sketch basic relations of $m(P)$ with periods of mixed motives. In conjunction with conjecture 3.4 this gives some explanation why in simple cases $m(P)$ is related to special values of L-series. We will only consider the case where P does not vanish on $T^n \subset (\mathbb{C}^*)^n$ or in dynamical terms where the corresponding $\mathbb{Z}^n$-action is expansive [22] II Th. 6.5. This assumption is not fulfilled in the examples (6.2), (6.3) though. The general case is more intricate. It is considered in [11] where complete proofs for the following assertions are also given.

At first P can be any Laurent polynomial in $\mathbb{Q}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. The complement

$$X_P = \mathbb{G}_{m,\mathbb{Q}}^n \setminus Z(P)$$

of its zero locus can be viewed as a closed subvariety of $\mathbb{G}_m^{n+1}$ via the embedding (everything over $\mathbb{Q}$):

$$i : X_P \cong \mathbb{G}_m^{n+1} \cap \Gamma_P \hookrightarrow \mathbb{G}_m^{n+1}$$

where

$$\Gamma_P = \big\{ (z_0, z') \in \mathbb{A}^1 \times \mathbb{G}_m^n \,|\, z_0 = P(z') \big\}$$

is the graph of P. On $\mathbb{G}_m^{n+1}$ the group μ_2^{n+1} acts naturally. Let $\varepsilon : \mu_2^{n+1} \to \mu_2$ be the product on μ_2^{n+1}. Let X be the union in $\mathbb{G}_m^{n+1}$ of the translates of $i(X_P)$ under the automorphisms $\gamma \in \mu_2^{n+1}$:

$$X = \bigcup_{\gamma \in \mu_2^{n+1}} i(X_P)^\gamma \subset \mathbb{G}_m^{n+1} .$$

With the reduced subscheme structure X becomes a closed and hence affine subvariety of $\mathbb{G}_m^{n+1}$ of dimension n. Thus we have $H^{n+1}(X) = 0$ in $\mathcal{MM}$. Using the Künneth formula it is easy to see that $H^n(\mathbb{G}_m^{n+1})(\varepsilon) = 0$ in $\mathcal{MM}$ where as before (ε) denotes the ε-isotypical component. Hence the μ_2^{n+1}-equivariant embedding $X \subset \mathbb{G}_m^{n+1}$ gives rise to the following short exact sequence in $\mathcal{MM}$:

$$0 \to H^n(X, n+1)(\varepsilon) \to H^{n+1}\big(\mathbb{G}_m^{n+1} \operatorname{rel} X, n+1\big)(\varepsilon) \to H^{n+1}\big(\mathbb{G}_m^{n+1}, n+1\big)(\varepsilon) \to 0 .$$

Note that in $\mathcal{MM}$ we have a canonical isomorphism:

$$\mathbb{Q}(0) \xrightarrow{\sim} H^{n+1}\big(\mathbb{G}_m^{n+1}, n+1\big)(\varepsilon)$$

since $H^1(\mathbb{G}_m) = \mathbb{Q}(-1)$. Under the induced isomorphism

$$\mathbb{Q} = F^0\mathbb{Q}(0)_{dR} \xrightarrow{\sim} F^{n+1} H_{dR}^{n+1}\big(\mathbb{G}_m^{n+1}/\mathbb{Q}\big)(\varepsilon)$$

$1 \in \mathbb{Q}$ corresponds to the class of the invariant $n+1$-form $\frac{dz_0}{z_0} \wedge \ldots \wedge \frac{dz_n}{z_n}$ where $z_0, \dots, z_n$ are the coordinates of $\mathbb{G}_m^{n+1}$.

Setting

$$N = H^{n+1}\big(\mathbb{G}_m^{n+1} \operatorname{rel} X, n+1\big)(\varepsilon)$$

the above short exact sequence gives an identification

$$F^0 N_{dR} \cong F^{n+1} H^{n+1}_{dR}\big(\mathbb{G}_m^{n+1}/\mathbb{Q}\big)(\varepsilon)$$

since $F^{n+1} H^n_{dR}(X/\mathbb{Q}) = 0$. Let $\omega_{\mathcal{H}} \in F^0 N_{dR}$ be the form corresponding to the class of $\frac{dz_0}{z_0} \wedge \ldots \wedge \frac{dz_n}{z_n}$. On the other hand for the Betti realization of

$$M = \check{N}(1) = H_{n+1}\big(\mathbb{G}_m^{n+1} \operatorname{rel} X, -n\big)(\varepsilon)$$

we find, again by using the exact sequence that

$$M_B^+ \cong H_n^B\big(X, \mathbb{Q}(-n)\big)^+(\varepsilon)$$

since $\mathbb{Q}(1)_B^+ = 0$.

Now assume that P does not vanish on T^n i.e. that $T^n \subset X_P(\mathbb{C})$. Then T^n defines a homology class $[T^n]$ in $H_n(X_P(\mathbb{C}), \mathbb{Q})$ and we let $i_*[T^n](\varepsilon)$ be the ε-isotypical component of $i_*[T^n]$ in $H_n(X(\mathbb{C}), \mathbb{Q})$. Let $c \in M_B^+$ be the element corresponding to the cycle

$$i_*[T^n](\varepsilon) \otimes (2\pi i)^{-n} \quad \text{in} \quad H_n^B\big(X, \mathbb{Q}(-n)\big)^+(\varepsilon)\,.$$

Then our observation is this:

Theorem 6.4

For $P \in \mathbb{Q}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ without zeroes on T^n we have under the period pairing

$$\langle , \rangle_{\mathrm{per}} : M_B^+ \times F^0 N_{dR} \longrightarrow \mathbb{R}$$

of section 2 that

$$\langle c, \omega_{\mathcal{H}} \rangle_{\mathrm{per}} = m(P)\,.$$

If P has integer coefficients we thus get an interpretation of the Deligne period $\langle c, \omega_{\mathcal{H}} \rangle$ as an entropy of a natural $\mathbb{Z}^n$-action.

The proof is not difficult [11] § 2.

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