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Galois representations, embedding problems and modular forms

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Abstract

To an odd irreducible 2-dimensional complex linear representation of the absolute Galois group of the field $\mathbb{Q}$ of rational numbers, a modular form of weight 1 is associated (modulo Artin's conjecture on the L-series of the representation in the icosahedral case). In addition, linear liftings of 2-dimensional projective Galois representations are related to solutions of certain Galois embedding problems.

In this paper we present some recent results on the existence of liftings of projective representations and on the explicit resolution of embedding problems associated to orthogonal Galois representations, and explain how these results can be used to construct modular forms.

0. Introduction

In this paper, we survey results on the explicit resolution of Galois embedding problems, liftings of projective Galois representations, and a method of construction of modular forms of weight one.

Looking first at embedding problems, we present an explicit resolution, obtained by the author, of certain embedding problems whose obstruction to solvability can be computed by means of the formulae of Serre and Fröhlich. More precisely, the embedding problems considered by Serre are those given by double covers of symmetric and alternating groups; Fröhlich generalizes the result of Serre to embedding problems associated to orthogonal representations. The method for the computation of the solutions exploits the connection of these embedding problems with Clifford algebras. It can also be applied in the case of extensions of symmetric and alternating groups with a kernel of order a power of 2.

For 2-dimensional projective representations of the absolute Galois group of a field K, we present the results of Quer on the existence of liftings to linear representations. By relating these liftings to solutions of certain Galois embedding problems, a criterion is obtained for the existence of a lifting of index 2. In the case when K is the field $\mathbb{Q}$ of rational numbers, further results are obtained, namely, a criterion for the existence of a lifting with any given index.

The method of construction of modular forms of weight one is due to Bayer and Frey. It is worth stressing that the spaces of modular forms of higher weight can be computed from the trace formulae for Hecke operators or by viewing these acting on modular symbols. These ideas do not work for weight one. In this case, the method is based on the connection of modular forms of weight one with odd irreducible 2-dimensional complex linear representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ of the field $\mathbb{Q}$ of rational numbers, and exploits the previously mentioned results.

More precisely, a Galois realization over $\mathbb{Q}$ of a group G equal to the alternating groups A_4, A_5 or the symmetric group S_4, gives a projective representation of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. The linear liftings of this projective representation are associated to the resolution of embedding problems given by extensions of the group G with a cyclic kernel. If the linear representation is odd, by a theorem of Weil-Langlands (modulo Artin's conjecture), it gives rise to a modular form of weight one. The explicit resolution of the embedding problem makes possible the determination of the Fourier coefficients of the modular form. Some numerical examples are provided of resolution of embedding problems on A_4-, S_4- and A_5-extensions of $\mathbb{Q}$ and construction of the corresponding modular form.

1. Explicit resolution of Galois embedding problems

We will now present a method to obtain the solutions to embedding problems given by double covers and to embedding problems given by central extensions of alternating and symmetric groups with a kernel of order a power of 2. For the general ideas concerning Galois embedding problems, we refer to [17]. The resolution of Galois embedding problems is an aspect of the inverse Galois problem. We address the reader interested in an overview on the present state of this subject to [21], [30] and [39].

Let us fix the following notation. We shall denote by $E \to G \simeq \mathrm{Gal}(L|K)$ the Galois embedding problem given by a Galois field extension $L|K$ with Galois group isomorphic to G (corresponding to an epimorphism $\varphi : G_K \to G$, for G_K the absolute Galois group of K) and a group extension $E \to G$ with abelian kernel A (corresponding to an element ε in the cohomology group $H^2(G, A)$).

We note that the element $\varphi^*\varepsilon \in H^2(G_K, A)$ giving the obstruction to the solvability of the embedding problem is in general not directly computable. For embedding problems given by double covers of alternating and symmetric groups, Serre [28 Th.1] gives a formula relating the element $\varphi^*\varepsilon$ with Hasse-Witt invariants and allowing its computation. This result has been used by several authors (see [22], [33], [37], [38]) to deduce the Galois realizability over $\mathbb{Q}$ and $\mathbb{Q}(T)$ of the double covers of alternating and symmetric groups. This formula of Serre has been generalized by Fröhlich to the wider context of embedding problems associated to orthogonal representations.

Our method of computation of the solutions to embedding problems given by the nonsplit double cover of an alternating group is based on the fact that the solvability of this kind of embedding problems is equivalent to the existence of a certain isomorphism between Clifford algebras ([4]). For embedding problems given by double covers of symmetric groups, we obtained two different methods of reduction to the alternating case ([5]). Inside Fröhlich's general framework, we obtained a generalization of our previous methods which can be applied to an embedding problem given by an orthogonal representation if this orthogonal representation satisfies certain conditions which can be proved to be nonrestrictive ([6]).

1.1. Embedding problems associated to orthogonal representations

We consider Galois embedding problems

$$2G \to G \simeq \mathrm{Gal}(L|K) \tag{1}$$

given by a central extension

$$1 \to C_2 \to 2G \to G \to 1\ . \tag{2}$$

First we note that if (2) is non-split, the solutions to (1) are all proper and are the fields $L(\sqrt{r\gamma})$, for $L(\sqrt{\gamma})$ a particular solution and r running over K^*/K^{*2}. The fact that $L(\sqrt{\gamma})$ is a solution to (1) is equivalent to γ being an element in L^* satisfying

$$\gamma^s\gamma^{-1} \in L^{*2} \text{ for all } \ s \in G$$

with b_s defined by $b_s^2 := \gamma^s\gamma^{-1}$ such that

$$a_{s,t} = b_s b_t^s b_{st}^{-1} \tag{3}$$

is a 2-cocycle representing the element $\varepsilon \in H^2(G, C_2)$ corresponding to (2).

Thus, giving the solutions to an embedding problem of this kind means finding an element γ in L^* satisfying the above conditions but, in general, such an element γ cannot be found just by operating in the field L.

Let us consider the case when the group G is a subgroup of the symmetric group S_n and $2G$ is the preimage of G in the double cover $2S_n^+$ of S_n, characterized by the property that transpositions lift to involutions. We denote by s_n^+ the element in the cohomology group $H^2(S_n, C_2)$ corresponding to $2S_n^+$. Then the element $\varphi^* s_n^+$ giving the obstruction to the solvability of the corresponding embedding problem can be computed by the following formula of Serre.

Theorem ([28], Th.1)

Let Q_E be the quadratic trace form of the extension $E|K$, for E the fixed field of the isotropy group of one letter, $\mathrm{w}(Q_E)$ its Hasse-Witt invariant, and d_E its discriminant. We have the following equality in $H^2(G_K, C_2)$

$$\varphi^*(s_n^+) = \mathrm{w}(Q_E)(2, \mathrm{d}_E)$$

where $(\cdot, \cdot)$ denotes the Hilbert symbol.

Let us observe that the group G is contained in the alternating group A_n if and only if $d_E = 1$ (modulo squares) and denote by a_n the element in the cohomology group $H^2(A_n, C_2)$ corresponding to the nontrivial double cover $2A_n$ of A_n. We obtain

$$\varphi^*(a_n) = \mathrm{w}(Q_E) \in H^2(G_K, C_2)\,.$$

Let us denote now by $2S_n^-$ the second double cover of the symmetric group reducing to the nontrivial double cover of the alternating group, and the corresponding element in $H^2(S_n, C_2)$ by s_n^- . Then, by observing that s_n^+ and s_n^- differ in $(-1, d_E)$, we obtain

$$\varphi^*(s_n^-) = \mathrm{w}(Q_E)(-2, \mathrm{d}_E) \in H^2(G_K, C_2)\,.$$

We assume now that the element ε is the second Stiefel-Whitney class of the linear representation χ_ρ associated to an orthogonal representation

$$\rho : G \to \mathrm{O}(Q)$$

of the group G in a K-quadratic space (V, Q), i.e. that we are in Fröhlich's situation (cf. [16]). We note that this includes the double covers of the alternating and symmetric groups considered by Serre. Indeed, if we consider the orthogonal representation of the symmetric group S_n obtained by embedding it in the orthogonal space of the identity quadratic form in n variables, then the corresponding second Stiefel-Whitney class gives the double cover $2S_n^+$.

We shall see now how we can then use Clifford algebras to compute the element γ. For the general definitions and results concerning Clifford algebras, we refer to [18] and [24].

We note that the orthogonal representation ρ can be seen as a 1-cocycle $G \to \mathrm{O}_L(Q)$ and denote by Q_ρ the twisted quadratic form of Q by ρ, i.e. the quadratic form corresponding to ρ by the isomorphism between $H^1(G, \mathrm{O}_L(Q))$ and the set of classes of K-quadratic forms which are L-isomorphic to Q. The quadratic spaces (V, Q) and (V_ρ, Q_ρ) are related by an isomorphism:

$$f : V \otimes_K L \to V_\rho \otimes_K L \tag{4}$$

satisfying $f^{-1} f^s = \rho(s)$, for all $s \in G$ ([27] X 2). We also denote by f the extension to the Clifford algebras of Q and Q_ρ over L:

$$f : C_L(Q) \to C_L(Q_\rho). \tag{5}$$

Now, the obstruction to the solvability of the embedding problem is given by the following result obtained by Fröhlich.

Theorem ([16], Th.3)

Let ε be the second Stiefel-Whitney class of the linear representation χ_ρ associated to an orthogonal representation ρ of the group G. We have the following equality in $H^2(G_K, C_2)$

$$\varphi^*(\varepsilon) = \mathrm{w}(Q)\mathrm{w}(Q_\rho)\mathrm{sp}_2(\rho)\big(\mathrm{d}(Q), -\mathrm{d}(Q_\rho)\big)$$

where w *denotes the Hasse-Witt invariant of a quadratic form,* d *its discriminant,* sp_2 *is an invariant associated to the spinor norm* sp *defined on the orthogonal group and* $(\cdot, \cdot)$ *denotes the Hilbert symbol.*

In the case of the symmetric group, if we consider the orthogonal representation given by embedding the group S_n in the orthogonal group $\mathrm{O}(Q)$ of the identity quadratic form Q in n variables over K, the twisted form of Q by this representation is the trace form Q_E of the extension $E|\mathbb{Q}$ and the invariant sp_2 equals the Hilbert symbol $(2, d_E)$, for d_E the discriminant of the quadratic form Q_E or, equivalently, of the extension $E|\mathbb{Q}$. We then recover the formula of Serre from the formula of Fröhlich.

Coming back to the general case, we shall force the orthogonal representation to fulfill the conditions that the application of our method requires: namely, we assume that the embedding problem considered is associated to an orthogonal representation ρ of G satisfying:

- ρ is special, i.e. $\operatorname{Im}\rho \subset \mathrm{SO}(Q)$,
- $\mathrm{sp} \circ \rho = 1$.

We note that the first condition implies equality between the discriminants of the quadratic forms Q and Q_ρ and the second the vanishing of the invariant sp_2 (cf. [16] (7.8)). So, under these conditions, Fröhlich's formula above gives that an equivalent condition for the solvability of the considered embedding problem is

$$\mathrm{w}(Q)\mathrm{w}(Q_\rho) = 1 \in H^2(G_K, C_2). \tag{6}$$

We obtain then the following.

Theorem [6]

If the embedding problem $2G \to G \simeq \mathrm{Gal}(L|K)$ is solvable, there exists a $\mathbb{Z}/2\mathbb{Z}$-graded algebras isomorphism $g : C(Q) \to C(Q_\rho)$ such that the even element in $C_L(Q_\rho)$:

$$z := \sum_{\epsilon_i=0,1} f(e_1)^{-\epsilon_1} f(e_2)^{-\epsilon_2} \dots f(e_n)^{-\epsilon_n} g(e_n)^{\epsilon_n} \dots g(e_2)^{\epsilon_2} g(e_1)^{\epsilon_1},$$

where $(e_1, e_2, \dots, e_n)$ is an orthogonal basis in (V, Q), is invertible.

A solution to the embedding problem is then provided by any non-zero coordinate γ of the spinor norm $\mathrm{N}(z)$ of z in the basis $\{g(e_1)^{\epsilon_1} g(e_2)^{\epsilon_2} \dots g(e_n)^{\epsilon_n}\}_{\epsilon_i=0,1}$ of $C_L(Q_\rho)$.

An important step in the proof of the theorem is to obtain that, for z the element in the statement, the element b_s defined for each $s \in G$ by

$$b_s := f(x_s)^{-1} {}^s z z^{-1} \tag{7}$$

for $\{x_s\}_{s\in G}$ a system of representatives of G in $2G$, lies in L^* and satisfies equality (3). Now, the explicit expression of the element z makes possible the computation of $\mathrm{N}(z)$ whenever we can explicitly write an isomorphism g between the two Clifford algebras over K. In the cases given below, we get an expression of the element γ in terms of matrices.

Corollary

Let M be the matrix in $\mathrm{GL}(n, L)$ *associated to the inverse isomorphism* f^{-1} *of (4).*

1) If Q and Q_ρ are K-equivalent, we can choose P in $\mathrm{GL}(n, K)$ *such that*

$$P^t[Q_\rho]P = [Q] \quad \textit{and} \quad \det(MP + I) \neq 0\,,$$

where I denotes the $n \times n$-identity matrix. Then $\gamma = \det(MP + I)$ provides the solutions to the embedding problem.

2) If Q is the identity form and Q_ρ is K-equivalent to a quadratic form $Q_q = -I_q \oplus I_{n-q}$, then for a matrix P in $\mathrm{GL}(n, K)$ *such that*

$$P^t[Q_\rho]P = [Q_q]\,,$$

an element γ providing the solutions to the embedding problem is obtained as a sum of minors of the matrix MP.

Remarks. In case 1), g is the isomorphism associated to the matrix P. In case 2), P provides an isomorphism between $C(Q_\rho)$ and $C(Q_q)$ and we can write down an isomorphism between $C(Q)$ and $C(Q_q)$, not arising from an isometry. The formula giving γ is explicit but too long to be specified here (cf. [4], [6]).

In the case of alternating groups, the orthogonal representation given by embedding the group A_n in the special orthogonal group $\mathrm{SO}(Q)$ of the identity quadratic form Q in n variables over K satisfies the above conditions. Moreover, if $K = \mathbb{Q}$, we are always under the conditions of the corollary and so an element γ providing the solutions to the embedding problem can be computed in terms of matrices (cf. [4]).

In the case of an embedding problem $2S_n \to S_n \simeq \mathrm{Gal}(L|K)$, in order to apply the above method, we can either construct an orthogonal representation of the group S_n in $n+2$ variables satisfying the above conditions or obtain the solutions to the embedding problem considered from the solutions to the embedding problem $2A_n \to A_n \simeq \mathrm{Gal}(L|K(\sqrt{d_E}))$ (cf. [5]).

In the case of symmetric and alternating groups, the matrix M is given by

$$M = \left(x_i^j\right)_{\substack{1\le i\le n\\ 0\le j\le n-1}}$$

for $\{x_i\}$ the roots of a polynomial realizing G over K and so the element γ is obtained in terms of the x_i.

We can see now that in the general case, the assumptions made on ρ are not restrictive. We note that a non-special orthogonal representation $r : G \to \mathrm{O}(Q)$ determines two double covers of the group G, namely the ones associated to $\mathrm{sw}(\chi_r)$ and $\mathrm{sw}(\chi_r \perp \det \chi_r)$.

Proposition ([6], Prop.3)

Let $r : G \to \mathrm{O}(Q)$ be a non special orthogonal representation of G. There exist two orthogonal representations ρ_1 and ρ_2 of G such that:

1) ρ_1 and ρ_2 are special,
2) $\mathrm{sp} \circ \rho_1 = \mathrm{sp} \circ \rho_2 = 1$,
3) $\mathrm{sw}(\chi_{\rho_1}) = \mathrm{sw}(\chi_r)$ and $\mathrm{sw}(\chi_{\rho_2}) = \mathrm{sw}(\chi_r \perp \det \chi_r)$.

We get ρ_1 and ρ_2 by adding variables to Q and using the explicit expression of the invariant sp_2 in terms of Hilbert symbols given in [16] (7.8).

We can apply these results to the embedding problems given by the dihedral group D_{4n} and the quaternion group H_{4n} as double covers of the dihedral group D_{2n} and to the case G abelian of exponent 2 (cf. [6] Props.5-7).

Recently Swallow has noted that the formula of Fröhlich and the above results on computation of solutions remain valid if we allow scalar extension to L in the orthogonal space under the condition that the elements in the image of ρ have spinor norm in $K^*.\overline{K}^{*2}$ ([35]). With this result, he is able to compute the obstruction to the solvability of embedding problems given by the extension $C_{16} \to C_8$. To this end, he uses a parametric family of C_8-extensions obtained by Schneps giving all C_8-extensions for fields belonging to a certain class containing $\mathbb{Q}$ ([32]).

1.2. Embedding problems with kernel of order 2^r

We now consider embedding problems given by central extensions of alternating or symmetric groups with a cyclic kernel. First we note that if the kernel has odd order, all such extensions are trivial. We then consider extensions with a kernel of even order m. In the case of the alternating group, there is a unique non-split extension mA_n of A_n with a cyclic kernel of order m. In the case of the symmetric group, we denote by mS_n^- the central extension of the symmetric group S_n corresponding to the image of s_n^- by the map $j_* : H^2(S_n, C_2) \to H^2(S_n, C_m)$ induced by the embedding $j : C_2 \to C_m$. We note that $j_* s_n^+ = j_* s_n^-$. Now, if we consider mA_n (respectively mS_n^-) with $m = 2^r m'$, m' odd, we have $mA_n = 2^r A_n \times C_{m'}$ (respectively $mS_n^- = 2^r S_n^- \times C_{m'}$) and so, the solutions to an embedding problem $mA_n \to A_n \simeq \mathrm{Gal}(L|K)$ (respectively $mS_n^- \to S_n \simeq \mathrm{Gal}(L|K)$) are the fields $\widehat{L}.K'$, for $\widehat{L}$ a solution to $2^r A_n \to A_n \simeq \mathrm{Gal}(L|K)$ (respectively $2^r S_n^- \to S_n \simeq \mathrm{Gal}(L|K)$) and K' a Galois realization of $C_{m'}$ over K.

We consider then the case when the kernel has order a power of 2. We recall that $H^2(S_n, C_{2^r}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and denote by $2^r S_n^+$ the second extension of S_n with kernel C_{2^r} reducing to $2^r A_n$.

Let now G be a transitive subgroup of S_n, $n \geq 4$, $L|K$ a Galois extension with Galois group G. We consider the embedding problem

$$2^r G \to G \simeq \mathrm{Gal}(L|K) \tag{8}$$

for $2^r G$ the preimage of G in $2^r S_n^-$. We shall see that it is possible to compute the solutions to an embedding problem of this kind by reducing to the case of an embedding problem with kernel of order 2.

First we note that if the embedding problem (8) is solvable, so is any embedding problem $2^s G \to G \simeq \mathrm{Gal}(L|K)$, with $s \geq r$.

Now, if the field $\widehat{L}$ is a solution to the embedding problem (8), for L_1 the subfield of $\widehat{L}$ fixed by the subgroup of order 2 of C_{2^r}, we have $\mathrm{Gal}(L_1|K) \simeq G \times C_{2^{r-1}}$. By taking $K_1 := L_1^G$, we get $\mathrm{Gal}(K_1|K) \simeq C_{2^{r-1}}$ and $L \cap K_1 = K$. Moreover, $\hat{L}$ is a solution to the embedding problem with kernel C_2:

$$2^r G \to G \times C_{2^{r-1}} \simeq \mathrm{Gal}(L_1|K) \tag{9}$$

and all solutions to (8) are obtained in this way from a Galois extension $K_1|K$ with Galois group $C_{2^{r-1}}$ and $K_1 \cap L = K$ and such that the corresponding embedding problem (9) is solvable.

From the above discussion we get the following criterion for the solvability:

Proposition [11]

The embedding problem

$$2^r G \to G \simeq \mathrm{Gal}(L|K)$$

is solvable if and only if there exists a Galois extension $K_1|K$ with Galois group $C_{2^{r-1}}$ and $K_1 \cap L = K$ and such that the obstructions to the solvability of the embedding problems $2G \to G \simeq \mathrm{Gal}(L|K)$ and $C_{2^r} \to C_{2^{r-1}} \simeq \mathrm{Gal}(K_1|K)$ agree in $H^2(G_K, C_2)$.

Now the preceding method of computation of the solutions can be applied to (9) whenever the element $b \in H^2(C_{2^{r-1}}, C_2)$ corresponding to the exact sequence

$$1 \to \{\pm 1\} \to C_{2^r} \to C_{2^{r-1}} \to 1$$

is the second Stiefel-Whitney class of an orthogonal representation of $C_{2^{r-1}}$. This is so in the case when $K \supset \mu_{2^{r-1}}$ for any r and in the cases $r = 2, 3$ for any field K and $r = 4$ for fields K for which all C_8-extensions are admissible (cf. [35]). In these cases, the condition for the solvability is that $\varphi^*(s_n^-)$ can be expressed as a certain product of Hilbert symbols (cf. [7], [9], [10], [11]). In the case $r = 4$, we use the result of Swallow mentioned above (cf. [12]).

In general, to obtain results for $r > 4$, it would be necessary to have a parametrization of $C_{2^{r-1}}$-extensions of the field K and an orthogonal representation of $C_{2^{r-1}}$ whose second Stiefel-Whitney class corresponds to its cyclic double cover.

2. Linear and projective representations

I shall now report Quer's results on liftings of projective representations [25].

We consider continuous 2-dimensional linear representations $\rho : G_K \to \mathrm{GL}(2, F)$, for $G_K = \mathrm{Gal}(\overline{K}|K)$ the absolute Galois group of a field K and F an algebraically closed field.

The continuity of the representation ρ, relative to the profinite topology of $\mathrm{Gal}(\overline{K}|K)$ and the discrete topology of F, implies that its image is a finite subgroup of $\mathrm{PGL}(2, F)$.

If $\overline{\rho}$ is the projective representation determined by ρ, i.e. $\overline{\rho} = \pi \circ \rho$, for π the projection from $\mathrm{GL}(2, F)$ onto $\mathrm{PGL}(2, F)$, we say that ρ is a lifting of $\overline{\rho}$.

The obstruction to the existence of a lifting for a given $\overline{\rho}$ is the element $\overline{\rho}^* c \in H^2(G_K, F^*)$, for $c \in H^2(\mathrm{PGL}(2, F), F^*)$ the element corresponding to the exact sequence

$$1 \to F^* \to \mathrm{GL}(2, F) \to \mathrm{PGL}(2, F) \to 1$$

and $\overline{\rho}^* : H^2(\mathrm{PGL}(2, F), F^*) \to H^2(G_K, F^*)$ the morphism induced by $\overline{\rho}$ on cohomology.

The index of a lifting ρ is defined as the degree of the cyclic extension $\widetilde{L}|L$, for $\widetilde{L} = \overline{K}^{\mathrm{Ker}\,\rho}$, $L = \overline{K}^{\mathrm{Ker}\,\overline{\rho}}$. A minimal lifting is a lifting with minimum index and a trivial lifting is a lifting with index 1.

It can be seen that a minimal lifting has index a power of 2 and that there are such liftings with determinant of order a power of 2. If $\mathrm{char}F = 2$, every representation has a trivial lifting and so, from now on, we shall assume $\mathrm{char}F \neq 2$.

Here we shall deal with representations of types PSL and PGL, that is, representations whose image is isomorphic to one of the groups $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$, occurring for q a power of the characteristic p of F. This includes the groups $A_4 \simeq \mathrm{PSL}(2,3)$, $S_4 \simeq \mathrm{PGL}(2,3)$, $A_5 \simeq \mathrm{PSL}(2,5)$, occurring for any characteristic.

We see first the relation between the index of the lifting and the order of the determinant.

Proposition ([25], 1.2)

Let $\overline{\rho}$ be a projective representation of type PSL *or* PGL.

i) $\overline{\rho}$ has no trivial liftings.

ii) Let ρ be a lifting with determinant ϵ of order 2^s. The index of ρ is

**) 2^{s+1} for type* PSL,

**) 2^s or 2^{s+1} for type* PGL *depending on whether the maximal abelian subextension M of $L|K$ is contained or not in the fixed field N of* $\mathrm{Ker}(\det\rho)$.

If $M \subset N$ (resp. $M \not\subset N$), we say that ρ is a lifting of class II (resp class I).

2.1 Relations with embedding problems

We recall that $H^2(\mathrm{PSL}(2,q), C_{2^r}) \simeq \mathbb{Z}/2\mathbb{Z}$ and denote by $2^r\mathrm{PSL}(2,q)$ the extension corresponding to the nonzero element.

Proposition [25]

Let $L|K$ be a Galois extension with Galois group $\mathrm{PSL}(2,q)$ and let $\overline{\rho} : G_K \to \mathrm{PGL}(2,F)$ be a projective representation obtained from an embedding of $\mathrm{Gal}(L|K)$ into $\mathrm{PGL}(2,F)$, by identifying $\mathbb{F}_q$ with a subfield of F. Then the Galois embedding problem $2^r\mathrm{PSL}(2,q) \to \mathrm{PSL}(2,q) \simeq \mathrm{Gal}(L|K)$ is solvable if and only if ρ has a lifting of index 2^r.

We recall that $H^2(\mathrm{PGL}(2,q), C_{2^r}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and let $2\mathrm{PGL}(2,q)^-$ be the pull-back of the diagram

$$\begin{array}{ccc} & & \mathrm{PGL}(2,q) \\ & & \downarrow \\ \mathrm{SL}(2,F) & \longrightarrow & \mathrm{PGL}(2,F) = \mathrm{PSL}(2,F). \end{array}$$

Now, identifying the extensions of $\mathrm{PGL}(2,q)$ with the corresponding elements in the cohomology groups, we denote by $2^r\mathrm{PGL}(2,q)^-$ the image of $2\mathrm{PGL}(2,q)^-$ by the morphism

$$j_* : H^2\big(\mathrm{PGL}(2,q), C_2\big) \to H^2\big(\mathrm{PGL}(2,q), C_{2^r}\big)$$

induced by the embedding $j : C_2 \to C_{2^r}$. Let $2^r\mathrm{PGL}(2,q)^+$ be the second extension of $\mathrm{PGL}(2,q)$ with kernel C_{2^r} reducing to $2^r\mathrm{PSL}(2,q)$.

Proposition [25]

Let $L|K$ be a Galois extension with Galois group $\mathrm{PGL}(2,q)$ and let $\overline{\rho} : G_K \to \mathrm{PGL}(2,F)$ be a projective representation obtained from an embedding of $\mathrm{Gal}(L|K)$ into $\mathrm{PGL}(2,F)$. Then the Galois embedding problem $2^r\mathrm{PGL}(2,q)^- \to \mathrm{PGL}(2,q) \simeq \mathrm{Gal}(L|K)$ (resp. $2^r\mathrm{PGL}(2,q)^+ \to \mathrm{PGL}(2,q) \simeq \mathrm{Gal}(L|K)$) is solvable if and only if $\overline{\rho}$ has a lifting of index 2^r and class I (resp. class II).

To see when the embedding problems given by extensions of groups PSL and PGL are solvable, we can use the formula of Serre that we referred to above. To this end, we consider the permutation representation $\pi_q : \mathrm{PGL}(2,q) \to S_{q+1}$ induced by the action of $\mathrm{PGL}(2,q)$ on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. Let $\pi_q^* : H^2(S_{q+1}, C_2) \to H^2(\mathrm{PGL}(2,q), C_2)$ be induced by π on cohomology.

Let $\beta^+, \beta^- \in H^2(\mathrm{PGL}(2,q), C_2)$ correspond to $2\mathrm{PGL}(2,q)^+$ and $2\mathrm{PGL}(2,q)^-$, respectively. By determining the graph of π_q^*, we obtain the following result (cf. [25], Section 3).

Proposition

Let $q \equiv \pm 3 \pmod 8$, $\overline{\rho}$ of type PGL. Then:

$$\overline{\rho}^*\beta^- = \mathrm{w}(Q_E).(\mp 2, d_E)\ , \quad \overline{\rho}^*\beta^+ = \mathrm{w}(Q_E).(\pm 2, d_E)\,,$$

for $E|K$ the fixed field of the isotropy group of some element of $\{1, \dots, q+1\}$.

Let $q \equiv \pm 3 \pmod 8$, $\overline{\rho}$ of type PSL. Then $\beta^- = \beta^+$, the discriminant d_E is a square and

$$\overline{\rho}^*\beta^- = \mathrm{w}(Q_E)\,.$$

Now, for $2^r S_n^-$, we have $\pi_q^* 2^r S_{q+1}^- = 2^r \mathrm{PGL}(2,q)^-$ for $q \equiv \pm 3 \pmod 8$ and so we can translate the results obtained on the solvability for embedding problems given by $2^r S_n^-$ (cf. [10], [11]) to embedding problems given by $2^r\mathrm{PSL}(2,q)$ and $2^r\mathrm{PGL}(2,q)^-$, with $q \equiv \pm 3 \pmod 8$ (cf. [25]).

2.2 The case $K = \mathbb{Q}$

We consider now the case when K is the field $\mathbb{Q}$ of rational numbers. We note first that, in this case, the existence of a lifting is always guaranteed by the following theorem of Tate.

Theorem ([29])

Let K be a local or global field. Then $H^2(G_K, F^*)$ is trivial.

We shall see now when a projective representation has a lifting with a given determinant. Consider the exact sequence

$$1 \to C_2 \to F^* \xrightarrow{(.)^2} F^* \to 1 . \tag{10}$$

Given a character $\epsilon \in \mathrm{Hom}(G_{\mathbb{Q}}, F^*)$, let $\epsilon^* c_1 \in H^2(G_{\mathbb{Q}}, C_2)$ be the image by the morphism $\epsilon^* : H^2(F^*, C_2) \to H^2(G_{\mathbb{Q}}, C_2)$, induced by ϵ on cohomology, of the element $c_1 \in H^2(F^*, C_2)$, corresponding to (10).

Consider now the exact sequence

$$1 \to C_2 \to \mathrm{SL}(2,F) \to \mathrm{PSL}(2,F) \to 1 . \tag{11}$$

Given a projective representation $\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{PGL}(2,F)$, let $\overline{\rho}^* c_2 \in H^2(G_{\mathbb{Q}}, C_2)$ be the image by the morphism $\overline{\rho}^* : H^2(\mathrm{PSL}(2,F), C_2) \to H^2(G_{\mathbb{Q}}, C_2)$, induced by $\overline{\rho}$ on cohomology, of the element $c_2 \in H^2(\mathrm{PSL}(2,F), C_2)$ corresponding to (11).

Theorem (Tate) ([29])

The projective representation $\overline{\rho}$ has a lifting with determinant ϵ if and only if

$$\epsilon^* c_1 = \overline{\rho}^* c_2 \quad \textit{in} \;\; \mathrm{Br}_2(\mathbb{Q}) .$$

Now, with this theorem of Tate, together with the previous results connecting the index of a lifting with the order of its determinant, the index of minimal liftings is obtained.

Let us recall that equality in $\mathrm{Br}_2(\mathbb{Q})$ is equivalent to equality for every local component. For the local components $(\epsilon^* c_1)_p \in \mathrm{Br}_2(\mathbb{Q}_p) = \{\pm 1\}$, viewing the local character ϵ_p as a character of $\mathbb{Q}_p^*$, we have $(\epsilon^* c_1)_p = \epsilon_p(-1)$.

On the other hand, $\overline{\rho}^* c_2$ is just the element denoted above by $\overline{\rho}^*(\beta^-)$ and so can be computed from the preceding results.

If we define, for a projective representation $\overline{\rho}$, $P(\overline{\rho}) = \{p \text{ prime: } (\overline{\rho}^* c_2)_p = -1\}$, the problem of determining the minimal index of linear liftings of $\overline{\rho}$ can be solved by determining the order of a global character ϵ with local conditions $\epsilon_p(-1) = -1$ for all p in $P(\overline{\rho})$.

For an odd prime p, let $\mu(p)$ be the maximum power of 2 dividing $p-1$. Let $\mu(2) = 1$. For a projective representation $\overline{\rho}$, let

$$\mu(\overline{\rho}) = \begin{cases} \max\{\mu(p) : p \in P(\overline{\rho})\} & \text{if} \quad P(\overline{\rho}) \neq \emptyset \\ 0 & \text{if} \quad P(\overline{\rho}) = \emptyset\,. \end{cases}$$

Let χ be the quadratic character corresponding to the maximal abelian subextension M of $L = \overline{K}^{\text{Ker } \overline{\rho}}$.

Proposition ([25], 4.3)

Let $\overline{\rho}$ be a projective representation

i) If $\overline{\rho}$ is of type PSL*, the index of a minimal lifting is* $2^{\mu(\overline{\rho})+1}$.

ii) If $\overline{\rho}$ is of type PGL*, the index of a minimal lifting is*

**) $2^{\mu(\overline{\rho})}$ when $\overline{\rho}$ has some lifting of class II and $P(\overline{\rho}) \neq \emptyset$ (resp. $P(\overline{\rho}) \setminus \{2\} \neq \emptyset$) if $\chi_2(-1) = -1$ (resp. $\chi_2(-1) = 1$).*

**) $2^{\mu(\overline{\rho})+1}$, otherwise.*

Remark. For odd representations, the component at ∞ of the element $\overline{\rho}^* c_2$ is -1. Hence, by the product formula, $P(\overline{\rho})$ must be nonempty and $\mu(\overline{\rho}) \geq 1$. In particular, odd projective representations of type PSL never have liftings of index 2.

Now, the preceding result has a consequence for the embedding problem $2^r G \to G \simeq \text{Gal}(L|\mathbb{Q})$. As before, G is a transitive subgroup of the symmetric group on n letters, $L|\mathbb{Q}$ a Galois extension with Galois group G, $2^r G$ the preimage of G in $2^r S_n^-$. For E the fixed field of the isotropy group of one letter, let $P(E) = \{p \text{ prime: } \text{w}(Q_E)_p.(-2, d_E)_p = -1\}$,

$$\mu(E) = \begin{cases} \max\{\mu(p) : p \in P(E)\} & \text{if} \quad P(E) \neq \emptyset \\ 0 & \text{if} \quad P(E) = \emptyset\,. \end{cases}$$

Corollary

The embedding problem $2^r G \to G \simeq \mathrm{Gal}(L|\mathbb{Q})$ *is solvable if and only if* $r > \mu(E)$.

3. Modular forms of weight 1

Let us recall that a theorem of Deligne-Serre and a theorem of Weil-Langlands give a 1-1 correspondence between normalized new forms of weight 1 and isomorphism classes of odd irreducible two-dimensional complex linear representations ρ of $G_{\mathbb{Q}}$ such that ρ and all its twists by characters of $G_{\mathbb{Q}}$ satisfy Artin's conjecture, that is, their Artin L-series are holomorphic in $\mathbb{C} \setminus \{0,1\}$ (cf. [29]). Through this correspondence, the Artin L-series of the representation agrees with the Dirichlet L-series of the modular form and, moreover, the Artin conductor and the determinant of the representation are given by the level and the character of the modular form.

When dealing with complex irreducible representations, $\mathrm{Im}\overline{\rho}$ can be either a dihedral group or one of the groups A_4, S_4, A_5. In the first case, the modular form attached to ρ can be easily obtained as a linear combination of θ-series of binary quadratic forms. In the others, the corresponding modular forms are called exotic. These cases are classically referred to as tetrahedral, octahedral and icosahedral, due to the representation of these groups as groups of rotations of the corresponding polyhedra. We note that Artin's conjecture has been proved by Langlands and Tunnell for types A_4, S_4 using methods of base change in GL(2), but the icosahedral case remains open.

For type A_5, Artin's conjecture has been verified by Frey and his collaborators ([14]) in a finite number of examples corresponding to the case when the determinant is a quadratic character and the conductor N has the shape $N = 2^{\alpha}.n$, with n odd and square free and $\mathrm{lcm}(N, 4n) \leq 10^4$, by systematizing the work of Buhler ([3]) in the case $N = 800$. The verification of the conjecture is made by checking that the number of irreducible odd two-dimensional linear representations of given conductor and determinant is equal to the dimension of the corresponding space of new forms. The dimension of spaces of forms of weight one are computed by determining a subspace isomorphic to them in a space of cusp forms of higher weight.

3.1 Method of construction

We start with a Galois realization $L|\mathbb{Q}$ of one of the groups $G = A_4, S_4, A_5$ and denote by E the fixed field of the isotropy group of one letter. We compute the minimal r such that the embedding problem given by $2^r G$ is solvable. If $r \leq 4$, we can compute the element γ providing the solutions to $2^rG \to G \simeq \mathrm{Gal}(L|\mathbb{Q})$ and consider the representation ρ obtained from the Galois realization of 2^rG and a lifting of a projective representation of G to a linear representation of 2^rG. The prime factors of the conductor N of the representation are the primes ramifying in the extension $E|\mathbb{Q}$ and their exponents in N are determined by the higher ramification groups. The explicit computation has been carried out by Rio in the octahedral case (cf. [26]).

For the modular form f_ρ corresponding to ρ, the character ϵ is the Dirichlet character (modulo N) corresponding to $\det\rho$ by class field theory. As f_ρ is an eigenvector of the Hecke operators, its Fourier coefficients a_n are determined by the coefficients a_p, for p prime. Now, the equality of the L-series gives $a_p = \mathrm{Tr}(\rho \mathrm{Frob}_p)$, for $p \not| N$, where Frob_p denotes the Frobenius at p, and a_p is computed by determining the conjugacy class in 2^rG of the corresponding Frobenius substitution. Let us make this more precise in the case of the double cover $2S_4^+$ of S_4 (cf. [2]).

When ρ is given by one of the two (conjugate) irreducible representations of $2S_4^+$ in $\mathrm{GL}(2,\mathbb{C})$, $\det\rho$ is the quadratic character of $\mathbb{Q}(\sqrt{d_E})$ and so we get an odd ρ by taking $d_E < 0$. For $p \not| N$, let us denote by $\widetilde{\pi}_p$, π_p the conjugation class of the Frobenius at p in $2S_4^+$ and S_4, respectively. Now, π_p is determined by the reduction modulo p of the polynomial realizing S_4 over $\mathbb{Q}$ and the five conjugation classes of S_4 lift to eight conjugation classes in $2S_4^+$. On the other hand, the character table of $2S_4^+$ gives the value of a_p for each $\widetilde{\pi}_p$. These data are gathered in the table. In the first column, we give the prime decomposition type of the extension of the prime p to the ring of integers O_E of E. The subindex of each prime ideal denotes its residue degree.

pO_E	π_p	$\widetilde{\pi_p}$	a_p
$P_1P_1'P_1''P_1'''$	1A	1A	2
		2A	-2
P_2P_2'	2A	4A	0
$P_1P_1'P_2$	2B	2B	0
P_1P_3	3A	3A	-1
		6A	1
P_4	4A	8A	$i\sqrt{2}$
		8B	$-i\sqrt{2}$

Let us choose now one of the two conjugate representations of $2S_4^+$ in $\mathrm{GL}(2,\mathbb{C})$. We see that $\widetilde{\pi}_p$ and so a_p is determined if $\pi_p = 2A$ or $2B$. For $\pi_p = 1A, 3A$, it is enough to know if γ is a square in the completion $L_{\mathfrak{P}}$ at a prime $\mathfrak{P}$ in L over p.

For $\pi_p = 4A$, $\widetilde{\pi}_p$ is determined by the action of the two liftings of s on $\sqrt{\gamma}$, for s an element in the conjugation class $4A$. The exact condition is expressed in terms of the element b_s defined by $b_s^2 := \gamma^s \gamma^{-1}$.

Proposition [2]

i) If $\pi_p = 1\mathrm{A}$ *(resp.* $3\mathrm{A}$*), we have*
a_p = *2 (resp -1) if* γ *is a square in* $L_{\mathfrak{P}}$*, for* $\mathfrak{P}|p$*,*
a_p = *-2 (resp 1) , otherwise.*
ii) If p *is an odd prime and* $\pi_p = 4\mathrm{A}$*, we have*
$a_p = i\sqrt{2}$ *if* $\gamma^{\frac{p-1}{2}} \equiv b_s$ *(mod* $\mathfrak{P}$*),*
$a_p = -i\sqrt{2}$ *otherwise.*

Let us note that the condition in i) can be checked from the explicit expression of the element γ. For the case ii), we take $s = (1234) \in 4A$ and in order to check the condition, together with the explicit expression of the element γ, we need an explicit expression of the element b_s, corresponding to $s = (1234)$. By following the steps of construction of the element γ, which is obtained as a generator of the solutions to an embedding problem given by the double cover $2A_4$ of the alternating group A_4 over $\mathbb{Q}(\sqrt{d_E})$, we see that $b_s = b_\sigma$ for $\sigma = (123)$ (cf. [5]). Then, by using equality (7), a computation in the Clifford algebra leads to an explicit expression for the element b_s which can be given in terms of the element γ as

$$b_s = -\frac{1}{2} + \frac{\gamma^{\sigma^{-1}} - \gamma^{\sigma}}{2\gamma}\,.$$

The preceding results have been generalised by Quer to extensions of the groups A_4, S_4, A_5 with kernel up to order 8. The character tables of the group considered and its extension lead to results similar to the preceding ones and to the corresponding criteria determining the value of each a_p. In each case, to check these criteria, the explicit expression of the element γ and some element b_s is needed. In general, the element b_s is obtained as a sum of minors of a base change matrix between quadratic forms and cannot be given by such a compact formula as in the case S_4. We note that in the case of the alternating groups the double covers do not give odd representations.

By methods similar to the preceding ones it is possible to compute the coefficients of the L-series of an odd irreducible representation of $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ over the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$. These computations provide numerical examples for the verification of the conjecture of Serre predicting the modularity modulo p of this kind of representations (cf. [31], [20]).

3.2 Examples

From the examples computed by Quer, we exhibit the sample below. We note that in the third example, given by an A_5-extension, the obtained coefficients do belong to a modular form, since it is a twist of the one obtained by Buhler.

In each example, $f(X)$ is a polynomial realizing the group G over $\mathbb{Q}$ and L its decomposition field. We give the element γ providing the solution $\widetilde{L}$ of the embedding problem $\widetilde{G} \to G \simeq \mathrm{Gal}(L|\mathbb{Q})$ in terms of distinct roots x_i of the polynomial f. For the corresponding modular form F of weight one, we give the first coefficients, the level and the character (we denote by $S_1(N, \epsilon)$ the space of cusp forms of weight one, level N and character ϵ).

EXAMPLE 1:

$$G = A_4 \ ; \ f(X) = X^4 - 2X^3 - 2X + 3 \ ; \ \widetilde{G} = 4A_4$$

$$\gamma = 108 - 105i - 241x_1 + 34ix_1 + 65x_1^2 + 45ix_1^2 - 42ix_1^3 - 241x_2 - 118ix_2 + 70x_1x_2 + 140ix_1x_2 - 35x_1^2x_2 - 126ix_1^2x_2 + 98ix_1^3x_2 + 65x_2^2 + 39ix_2^2 - 35x_1x_2^2 - 112ix_1x_2^2 + 49ix_1^2x_2^2$$

$$\widetilde{L} = L(i, \sqrt{\gamma})$$

$$F = q - iq^3 - q^5 - iq^{11} + iq^{15} - q^{17} - iq^{19} - iq^{23} - iq^{27} + iq^{31} - q^{33} - q^{37} - iq^{47} + iq^{51} - q^{53} + iq^{55} - q^{57} - iq^{59} + q^{61} + iq^{67} - q^{69} + q^{73} - iq^{79} - q^{81} + q^{85} + q^{89} + q^{93} + iq^{95} - q^{101} \dots$$

$$F \in S_1(2^5 7^4, \chi_{\mathbb{Q}(i)})$$

EXAMPLE 2:

$$G = S_4 \ ; \ f(X) = X^4 + X - 1 \ ; \ \widetilde{G} = 2S_4^+$$

$$\gamma = 30 + 7x_1 - 20x_1^2 + 4x_1^3 + 27x_2 - 12x_1x_2 - 12x_1^2x_2 + 12x_1^3x_2 + 24x_2^2 + 6x_1^2x_2^2 + 24x_1^3x_2^2$$

$$\widetilde{L} = L(\sqrt{\gamma})$$

$$F = q + i\sqrt{2}q^2 - i\sqrt{2}q^3 - q^4 - i\sqrt{2}q^5 + 2q^6 - q^7 - q^9 + 2q^{10} + q^{11} + i\sqrt{2}q^{12} + q^{13} - i\sqrt{2}q^{14} - 2q^{15} - q^{16} - i\sqrt{2}q^{18} + i\sqrt{2}q^{19} + i\sqrt{2}q^{20} + i\sqrt{2}q^{21} + i\sqrt{2}q^{22} - q^{23} - q^{25} +$$

$i\sqrt{2}q^{26} + q^{28} - q^{29} - 2i\sqrt{2}q^{30} + i\sqrt{2}q^{31} - i\sqrt{2}q^{32} - i\sqrt{2}q^{33} + i\sqrt{2}q^{35} + q^{36} - 2q^{38} - i\sqrt{2}q^{39} + q^{41} - 2q^{42} + i\sqrt{2}q^{43} - q^{44} + i\sqrt{2}q^{45} - i\sqrt{2}q^{46} - i\sqrt{2}q^{47} + i\sqrt{2}q^{48} - i\sqrt{2}q^{50} - q^{52} - i\sqrt{2}q^{55} + 2q^{57} - i\sqrt{2}q^{58} + q^{59} + 2q^{60} + q^{61} - 2q^{62} + q^{63} + q^{64} - i\sqrt{2}q^{65} + 2q^{66} + i\sqrt{2}q^{69} - 2q^{70} + i\sqrt{2}q^{75} - i\sqrt{2}q^{76} - q^{77} + 2q^{78} + i\sqrt{2}q^{80} - q^{81} + i\sqrt{2}q^{82} - 2q^{83} - i\sqrt{2}q^{84} - 2q^{86} + i\sqrt{2}q^{87} - q^{89} - 2q^{90} - q^{91} + q^{92} + 2q^{93} + 2q^{94} + 2q^{95} - 2q^{96} + q^{97} - q^{99} + q^{100} \dots$

$$F \in S_1(283, \chi_{\mathbb{Q}(\sqrt{283})})$$

EXAMPLE 3:

$$G = A_5 \ ; \ f(X) = X^5 + 10X^3 - 10X^2 + 35X - 18 \ ; \ \widetilde{G} = 4A_5$$

$\gamma = -216701 - 531333i - 180828x_1 - 164249ix_1 - 78682x_1^2 - 164682ix_1^2 - 30566x_1^3 - 25908ix_1^3 + 3792x_1^4 - 9363ix_1^4 - 399567x_2 + 141716ix_2 + 83532x_1x_2 - 207984ix_1x_2 + 237x_1^2x_2 + 126635ix_1^2x_2 + 6465x_1^3x_2 - 3995ix_1^3x_2 + 1071x_1^4x_2 + 9123ix_1^4x_2 - 125404x_2^2 - 53646ix_2^2 + 25533x_1x_2^2 - 14645ix_1x_2^2 + 2031x_1^2x_2^2 - 12714ix_1^2x_2^2 - 645x_1^3x_2^2 + 2592ix_1^3x_2^2 + 144ix_1^4x_2^2 - 41024x_2^3 - 14442ix_2^3 + 22917x_1x_2^3 - 22513ix_1x_2^3 - 249x_1^2x_2^3 + 2916ix_1^2x_2^3 + 1320x_1^3x_2^3 + 72ix_1^3x_2^3 + 108x_1^4x_2^3 - 36ix_1^4x_2^3 - 7017x_3 - 12840ix_3 - 45276x_1x_3 - 97176ix_1x_3 - 12072x_1^2x_3 + 4335ix_1^2x_3 - 4578x_1^3x_3 - 13188ix_1^3x_3 - 459x_1^4x_3 - 2463ix_1^4x_3 - 96588x_2x_3 - 68802ix_2x_3 + 70374x_1x_2x_3 + 58542ix_1x_2x_3 - 47715x_1^2x_2x_3 + 5430ix_1^2x_2x_3 + 3801x_1^3x_2x_3 + 2139ix_1^3x_2x_3 - 3468x_1^4x_2x_3 + 2880ix_1^4x_2x_3 + 18915x_2^2x_3 + 5034ix_2^2x_3 - 9300x_1x_2^2x_3 - 12135ix_1x_2^2x_3 - 42x_1^2x_2^2x_3 - 12999ix_1^2x_2^2x_3 + 1530x_1^3x_2^2x_3 - 78ix_1^3x_2^2x_3 + 162x_1^4x_2^2x_3 - 1686ix_1^4x_2^2x_3 - 12162x_2^3x_3 - 4542ix_2^3x_3 + 4350x_1x_2^3x_3 + 708ix_1x_2^3x_3 - 4998x_1^2x_2^3x_3 + 2958ix_1^2x_2^3x_3 + 246x_1^3x_2^3x_3 - 348ix_1^3x_2^3x_3 - 459x_1^4x_2^3x_3 + 153ix_1^4x_2^3x_3 - 15162x_3^2 - 79362ix_3^2 - 11442x_1x_3^2 - 20607ix_1x_3^2 - 2289x_1^2x_3^2 - 10914ix_1^2x_3^2 - 5244x_1^3x_3^2 - 7296ix_1^3x_3^2 - 432ix_1^4x_3^2 - 40431x_2x_3^2 - 132ix_2x_3^2 + 11745x_1x_2x_3^2 - 22830ix_1x_2x_3^2 + 1128x_1^2x_2x_3^2 + 6273ix_1^2x_2x_3^2 + 702x_1^3x_2x_3^2 + 1938ix_1^3x_2x_3^2 + 162x_1^4x_2x_3^2 + 762ix_1^4x_2x_3^2 - 6081x_2^2x_3^2 - 2271ix_2^2x_3^2 - 1473x_1x_2^2x_3^2 - 2268ix_1x_2^2x_3^2 - 210x_1^2x_2^2x_3^2 + 294ix_1^2x_2^2x_3^2 + 246x_1^3x_2^2x_3^2 + 468ix_1^3x_2^2x_3^2 - 306x_1^4x_2^2x_3^2 + 102ix_1^4x_2^2x_3^2 - 7296x_2^3x_3^2 - 5244ix_2^3x_3^2 + 4578x_1x_2^3x_3^2 - 2370ix_1x_2^3x_3^2 + 246x_1^2x_2^3x_3^2 + 1284ix_1^2x_2^3x_3^2 - 153x_1^3x_2^3x_3^2 + 51ix_1^3x_2^3x_3^2$

$$\widetilde{L} = L(i, \sqrt{\gamma})$$

$$F = q - iq^3 + \left(\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^7 + \left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}\right)q^{13} + \left(-\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^{19} + \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}\right)q^{21} - iq^{23} - iq^{27} + \left(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}\right)q^{29} - iq^{31} - q^{37} + \left(\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^{39} + \left(\tfrac{i}{2} - \tfrac{i}{2}\sqrt{5}\right)q^{43} + \left(-\tfrac{i}{2} - \tfrac{i}{2}\sqrt{5}\right)q^{47} + \left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}\right)q^{49} - q^{53} + \left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}\right)q^{57} + \left(-\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^{59} + q^{61} - q^{69} + \left(\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^{71} + q^{73} + \left(\tfrac{i}{2} - \tfrac{i}{2}\sqrt{5}\right)q^{79} - q^{81} + iq^{83} + \left(-\tfrac{i}{2} + \tfrac{i}{2}\sqrt{5}\right)q^{87} + \left(-\tfrac{3i}{2} - \tfrac{i}{2}\sqrt{5}\right)q^{91} - q^{93} + \left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}\right)q^{97} - q^{101} \dots$$

$$F \in S_1\left(2^5 5^4, \chi_{\mathbb{Q}(i)}\right)$$

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