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Spherical spaces for which the Hasse principle and weak approximation fail

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Abstract

We construct spherical homogeneous spaces X of semisimple simply connected groups with connected stabilizers such that the Hasse principle or weak approximation fail for X.

Introduction

A homogeneous space $\bar{X} = \bar{H}\backslash\bar{G}$ of a connected semisimple simply connected group $\bar{G}$ defined over an algebraically closed field $\bar{k}$ of characteristic 0 is called *spherical* if a Borel subgroup $\bar{B} \subset \bar{G}$ acts on $\bar{X}$ with an open orbit.

This class of varieties includes many classical ones (cf. [7]):

(1) flag varieties ($\bar{H}$ is a parabolic subgroup);
(2) symmetric spaces ($\bar{H}$ is the set of fixed points of an involution of $\bar{G}$);
(3) horospherical spaces ($\bar{H}$ contains a maximal unipotent subgroup $\bar{U}$ of $\bar{G}$).

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Now let X be a homogeneous space defined over a number field k. This means that there is a transitive action $X \times G \to X$ defined over k. We call X spherical if $\bar{X} = X \times_k \bar{k}$ is spherical in the above sense.

Let Σ be a finite set of places v of k, and let k_v denote the completion of k at v.

We are interested in the existence of rational points of X and their density. To be more precise, we ask whether the two following properties hold:

(1) *The Hasse principle*: the existence of k_v-points in X for all the places v of k implies the existence of a k-point of X;

(2) *Weak approximation*: $X(k)$ is dense in the product $\prod_{v \in \Sigma} X(k_v)$ for any finite set Σ of places.

We are interested in homogeneous spaces of semisimple simply connected groups with connected stabilizers. For this class of varieties there are some positive results. Namely, the Hasse principle and weak approximation are known to hold for projective homogeneous spaces (that is, flag varieties) [6]. It is also true for symmetric spaces [9], [2] and, more generally, for all affine spherical spaces [8].

In this paper we show that for a general spherical homogeneous space of a semisimple simply connected group, the Hasse principle or weak approximation may fail. We find counter-examples among horospherical spaces.

Our general approach is that of [1], [2]. One may interpret our counter-examples in terms of the Brauer–Manin obstruction which is known to be the only one for homogeneous spaces with connected stabilizer [4].

We describe the construction of counter-examples in Section 1. In Section 2 we prove that they are indeed counter-examples.

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1. Construction of counter-examples

1.1. We need some notation.

Let H be a connected group. We denote by H^{u} the unipotent radical of H. Set $H^{\mathrm{red}} = H/H^{\mathrm{u}}$, the reductive part of H; $H^{\mathrm{ss}} = (H^{\mathrm{red}})^{\mathrm{derived}}$, the semisimple part of H; $H^{\mathrm{tor}} = H^{\mathrm{red}}/H^{\mathrm{ss}}$, the toric part of H.

For any algebraic torus T denote

$$Ш^2(k,T) = \ker\left[H^2(k,T) \to \bigoplus_v H^2(k_v,T)\right];$$

for any finite set Σ of places of k let

$$Ч^1_\Sigma(k,T) = \operatorname{coker}\left[H^1(k,T) \to \bigoplus_{v\in\Sigma} H^1(k_v,T)\right].$$

1.2. We wish to construct a counter-example to weak approximation in the form $X = H\backslash G$, where G is semisimple simply connected. By [1], if $Ч^1_\Sigma(k,H^{\mathrm{tor}}) = 0$, then weak approximation with respect to Σ holds for X. Therefore we should look for X with $Ч^1_\Sigma(k,H^{\mathrm{tor}}) \neq 0$.

We wish to construct a counter-example to the Hasse principle in the form $(\tilde{G},\tilde{X}) = {}_\psi(G,X)$, as a twisted form of (G,X), where $X = H\backslash G$ and $\psi \in Z^1(k,\mathrm{Aut}\,(G,X))$ is a cocycle (cf. [12], Ch. I, 5.3, [2], 1.3, and Subsection 2.1 below). Having such a ψ, one can define the twisted toric part ${}_\psi H^{\mathrm{tor}}$ (cf. [12], Ch. I, 5.6, [2], 1 .5, 1.7, and Subsection 2.3 below). By [2], 3.3, if $Ш^2(k,{}_\psi H^{\mathrm{tor}}) = 0$, then the Hasse principle holds for $\tilde{X}$. Therefore we should look for examples with $Ш^2(k,{}_\psi H^{\mathrm{tor}}) \neq 0$.

1.3. Let L be the splitting field of a torus T, i.e. the minimal Galois extension of k such that $T\times_k L \simeq \mathbb{G}^d_{m,L}$, and let $\mathfrak{g} = \mathrm{Gal(L/k)}$. Denote by $\hat{T} = \mathrm{Hom(T_L,\mathbb{G}_{m,L})}$ the $\mathfrak{g}$-module of characters of T, and set

$$Ш^1_\omega(\mathfrak{g},\hat{T}) = \ker\left[H^1(\mathfrak{g},\hat{T}) \to \prod_C H^1(C,\hat{T})\right],$$

where C runs over all the cyclic subgroups of $\mathfrak{g}$.

One can show that if $Ш^1_\omega(\mathfrak{g},\hat{T}) = 0$, then

(i) $Ш^2(k,T) = 0$, and

(ii) $Ч^1_\Sigma(k,T) = 0$ for any finite Σ.

Hence we should look for a $\mathfrak{g}$-module M with $Ш^1_\omega(\mathfrak{g},M) \neq 0$.

1.4. A simple example of such a module can be constructed as follows. Let $L = k(\sqrt{a},\sqrt{b})$ be a biquadratic extension, $\mathfrak{g} = \mathrm{Gal(L/k)} \simeq (\mathbf{Z}/2)^2$, and let M be the augmentation ideal of the group ring $\mathbf{Z}[\mathfrak{g}]$, i.e.

$$M = \ker[\varepsilon\colon \mathbf{Z}[\mathfrak{g}] \to \mathbf{Z}],$$

where

$$\varepsilon\left(\sum_{g\in\mathfrak{g}} a_g g\right) = \sum_{g\in\mathfrak{g}} a_g\,.$$

In the exact sequence

$$0 \to M \to \mathbf{Z}[\mathfrak{g}] \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

the middle term is a free $\mathfrak{g}$-module, so its cohomology is trivial. Hence $H^1(\mathfrak{g}, M) = \hat{H}^0(\mathfrak{g}, \mathbf{Z})$, where $\hat{H}^0$ stands for the modified Tate cohomology. We have $\hat{H}^0(\mathfrak{g}, \mathbf{Z}) = \mathbf{Z}/4$. On the other hand, for all the cyclic subgroups $C \subset \mathfrak{g}$ we have $H^1(C, M) = \hat{H}^0(C, \mathbf{Z}) = \mathbf{Z}/2$. Hence

$$Ш^1_\omega(\mathfrak{g}, M) = \ker\left[\mathbf{Z}/4 \to (\mathbf{Z}/2)^3\right] \neq 0.$$

In fact, one can show that $Ш^1_\omega(\mathfrak{g}, M) = \mathbf{Z}/2$. Indeed, for any subgroup $C \subset \mathfrak{g}$ the map $\hat{H}^0(\mathfrak{g}, \mathbf{Z}) \to \hat{H}^0(C, \mathbf{Z})$ is just the natural projection $\mathbf{Z}/n_\mathfrak{g} \to \mathbf{Z}/n_C$, where $n_\mathfrak{g}$ denotes the order of $\mathfrak{g}$, similarly for C.

1.5. Let $L = k(\sqrt{a}, \sqrt{b})$. Set $T = R_{L/k}\mathbb{G}_m/\mathbb{G}_{m,k}$, where $R_{L/k}$ denotes Weil's restriction of the ground field. Then from the exact sequence of tori

$$1 \to \mathbb{G}_{m,k} \to R_{L/k}\mathbb{G}_m \to T \to 1$$

one can get the dual exact sequence of character modules

$$0 \to \hat{T} \to \mathbf{Z}[\mathfrak{g}] \to \mathbf{Z} \to 0.$$

We see that $\hat{T}$ is isomorphic to M constructed in 1.4.

1.6. Let us show that one can choose k and L in such a way that $Ш^2(k, T) \neq 0$.

Lemma 1.6.1

Let k, L, and T be as in 1.5. *If all the decomposition groups are cyclic, then* $Ш^2(k, T) \neq 0$.

Proof. For any k-torus T, the group $Ш^2(k,T)$ is dual to $Ш^1(k,\hat{T})$. Under the assumptions of the lemma, $Ш^1(k,\hat{T}) = Ш^1_\omega(k,\hat{T})$ and $Ш^1_\omega(k,\hat{T}) = \mathbf{Z}/2$ as shown in 1.4. □

Remark 1.6.2. The hypotheses of Lemma 1.6.1 are satisfied if $k = \mathbf{Q}$, $a = 13$, $b = 17$.

1.7. Let us show that one can choose k, L in 1.5, and Σ in such a way that $Ч^1_\Sigma(k,T) \neq 0$.

For any k-torus T denote

$$Ш^1_\Sigma(k,\hat{T}) = \ker\left[H^1(k,\hat{T}) \to \prod_{v\notin\Sigma} H^1(k_v,\hat{T})\right],$$

$$Ш^1(k,\hat{T}) = \ker\left[H^1(k,\hat{T}) \to \prod_{v} H^1(k_v,\hat{T})\right].$$

For any finite abelian group A let $A^{\sim}$ denote the dual group, $A^{\sim} = \mathrm{Hom}(\mathrm{A}, \mathbf{Q}/\mathbf{Z})$.

Lemma 1.7.1

For any k-torus T there exists an exact sequence

$$0 \to Ш^1(k,\hat{T}) \to Ш^1_\Sigma(k,\hat{T}) \to Ч^1_\Sigma(k,T)^{\sim} \to 0.$$

Proof. In [10], Lemma 1.4, it is proved that for any finite abelian k-group B there exists an exact sequence

$$0 \to Ш^1(k,\hat{B}) \to Ш^1_\Sigma(k,\hat{B}) \to Ч^1_\Sigma(k,B)^{\sim} \to 0.$$

To obtain the desired exact sequence, one has to follow, word by word, Sansuc's proof replacing the reference to the Poitou–Tate duality for finite abelian groups by a reference to the Nakayama–Tate duality for tori (cf. [5], 3.0.7). □

Lemma 1.7.2

Let k, L, and T be as in 1.5. Denote by Σ_0 the set of places v such that the decomposition group $\mathfrak{g}_v = \mathrm{Gal}(L_w/k_v)$ is noncyclic. If $\Sigma_0 \neq \emptyset$, then for any finite $\Sigma \supseteq \Sigma_0$ we have $Ч^1_\Sigma(k,T) \neq 0$.

Proof. Under the hypotheses of the Lemma, $Ш^1_\Sigma(k,\hat{T}) = Ш^1_\omega(k,\hat{T})$. By 1.4, $Ш^1_\omega(k,\hat{T}) = \mathbf{Z}/2$. By the Nakayama–Tate duality, $Ш^1(k,\hat{T}) = Ш^2(k,T)^{\sim}$. By [10], 1.9.4, we have $Ш^2(k,T) = 0$. By Lemma 1.7.1 we conclude that $Ч^1_\Sigma(k,T) = \mathbf{Z}/2$. □

Remark 1.7.3. The hypotheses of Lemma 1.7.2 are satisfied if $k = \mathbf{Q}, a = 2,\ b = -1, \Sigma_0 = \{2\}$.

1.8. Let us embed T into a semisimple simply connected group G.

From the definition of $\hat{T}$ we see that as $\mathbf{Z}$-module it is generated by the elements of the form $(g-1)$, g running over $\mathfrak{g}$, so $\hat{T} =< s-1, t-1, st-1 >$, where s and t denote two generators of $\mathfrak{g}$. Let us consider the homomorphism of $\mathfrak{g}$-modules

$$\mu\colon \mathbf{Z}[\mathfrak{g}] \oplus \mathbf{Z}[\mathfrak{g}] \to \hat{T},$$

mapping two generators of the free module to $(s-1)$ and $(t-1)$. As $\mu(1,s) = (s-1)+s(t-1) = st-1$, we conclude that μ is surjective. Thus one may consider the dual inclusion of tori

$$\nu\colon T \hookrightarrow R_{L/k}\mathbb{G}_m \times R_{L/k}\mathbb{G}_m.$$

Composing ν with the diagonal embedding $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2, \mathrm{x} \mapsto \mathrm{diag}(\mathrm{x}, \mathrm{x}^{-1})$, we obtain an embedding

$$T \hookrightarrow (R_{L/k}\mathbb{G}_m)^2 \hookrightarrow (R_{L/k}\mathrm{SL}_2)^2 = \mathrm{G}.$$

Let $U_0 \subset \mathrm{SL}_{2,\mathrm{L}}$ be the unipotent upper-triangular subgroup, and set $U = (R_{L/k}U_0)^2 \subset G$. Then $T \subset G$ normalizes U, hence TU is a subgroup of G.

Now we can state the main result.

Theorem 1.9

Let G, T, and U be as in 1.8, *and let $H = TU$, $X = H\backslash G$.*
(i) If k and L are chosen as in 1.6, *then there exists a k-form $(\tilde{G},\tilde{X}) = {}_\psi(G,X)$ of the pair (G,X) twisted by a cocycle $\psi \in Z^1(k, \mathrm{Aut}\,(G,X))$ such that for $\tilde{X}$ the Hasse principle fails.*
(ii) If k, L, and Σ are chosen as in 1.7, *then weak approximation with respect to Σ fails for X.*

Remark 1.9.1. From the construction it is clear that X is horospherical, hence spherical.

2. Proofs

In this section we prove Theorem 1.9.

2.1. Let us recall (cf. Subsection 1.2) that we wish to construct a counter-example to the Hasse principle, twisting (G, X) by a cocycle $\psi \in Z^1(k, \mathrm{Aut}\,(G, X)$. By technical reasons we are looking for a cocycle with values in $\mathrm{Aut}\,(G, X)^\circ$, the identity component of $\mathrm{Aut}\,(G, X)$. Let us explicitly describe $\mathrm{Aut}\,(G, X)^\circ$.

We need some additional notation. For a connected group H set $H^{\mathrm{cf}} = \ker[H \to H^{\mathrm{tor}}]$, the character-free part of H. We denote by $\mathrm{Int}\, G$ the group of inner automorphisms of a group G, and by $\mathrm{int}(g)$ the inner automorphism $g' \mapsto gg'g^{-1}$, defined by $g \in G$. Let $Z(G)$ denote the center of G.

As in [2], we introduce the group $\tilde{A}(G, H)$ of pairs (s, a), where $s \in G,\ a \in \mathrm{Aut}\, G$ are such that

$$s \cdot a(H) \cdot s^{-1} = H$$

with the composition

$$(s_1, a_1) \cdot (s_2, a_2) = (s_1 \cdot a_1(s_2), a_1 a_2).$$

Let $N(H)$ denote the normalizer of H in G. We have embeddings

$$i\colon N(H) \to \tilde{A}(G, H), \quad s \mapsto (s, 1) \text{ for } s \in N(H);$$
$$j\colon G \to \tilde{A}(G, H), \quad t \mapsto (t, \mathrm{int}(t^{-1})) \text{ for } t \in G.$$

One checks immediately that $i(N(H))$ and $j(G)$ commute in $\tilde{A}(G, H)$. We have a canonical exact sequence

$$1 \to H \to \tilde{A}(G, H) \to \mathrm{Aut}\,(G, H\backslash G) \to 1 \tag{2.1.1}$$

cf. [2], 1.2.

Lemma 2.2

Let G be a connected group, and let H be a connected subgroup of G.
(i) The connected component $\tilde{A}(G, H)^\circ$ of $\tilde{A}(G, H)$ equals $i(N(H)^\circ) \cdot j(G)$.
(ii) $i(N(H)) \cap j(G) = j(Z(G))$.

Proof. (i) Let $(s,a) \in \tilde{A}(G,H)^{\circ}$. Then $a \in \operatorname{Int} G$. Write $a = \operatorname{int}(t)$, then

$$(s,a) = (st,1) \cdot (t^{-1}, \operatorname{int}(t)) = i(st) \cdot j(t^{-1}),$$

where $st \in N(H)$. Since $(s,a) \in \tilde{A}(G,H)^{\circ}$, we have $st \in N(H)^{\circ}$.

(ii) If $(s,a) \in i(N(H)) \cap j(G)$, then $(s,a) \in i(N(H))$, hence $a = 1$. Thus $(s,1) \in j(G)$, hence $\operatorname{int}(s) = 1$, and we see that $s \in Z(G)$. Conversely, $j(Z(G)) \subset i(N(H)) \cap j(G)$. □

2.3. Define $\tilde{A}'(G,H) = \tilde{A}(G,H)/H^{\mathrm{cf}}$ (we write H^{cf} for $i(H^{\mathrm{cf}})$). From Lemma 2.2

$$\tilde{A}'(G,H)^{\circ} = (N(H)/H^{\mathrm{cf}})^{\circ} \cdot j(G).$$

The canonical exact sequence (2.1.1) gives rise to the exact sequence

$$1 \to H^{\mathrm{tor}} \to \tilde{A}'(G,H) \to \operatorname{Aut}(G, H\backslash G) \to 1$$

and to the exact sequence

$$1 \to H^{\mathrm{tor}} \to \tilde{A}'(G,H)^{\circ} \to \operatorname{Aut}(G, H\backslash G)^{\circ} \to 1.$$

Since H^{tor} is abelian, for any cocycle $\psi \in Z^1(k, \operatorname{Aut}(G,X))$ one can define the twisted group ${}_{\psi}H^{\mathrm{tor}}$ and the cohomology class $\Delta(\psi) \in H^2(k, {}_{\psi}H^{\mathrm{tor}})$ (cf. [12], Ch. I, 5.6).

The group $\operatorname{Aut}(G, H\backslash G)^{\circ}$ is connected; the automorphism group of the torus H^{tor} is discrete; thus the action of $\operatorname{Aut}(G, H\backslash G)^{\circ}$ on H^{tor} is trivial, and for $\psi \in Z^1(k, \operatorname{Aut}(G, H\backslash G)^{\circ})$ we have ${}_{\psi}H^{\mathrm{tor}} = H^{\mathrm{tor}}$.

Since H^{tor} is central in $\tilde{A}'(G,H)^{\circ}$, one can define the connecting map

$$\Delta \colon H^1(k, \operatorname{Aut}(G, H\backslash G)^{\circ}) \to H^2(k, H^{\mathrm{tor}}),$$

cf. [12], 1.5.7. Abusing the notation we will write $\Delta(\psi)$ for $\Delta(\mathrm{Cl}(\psi))$, where $\mathrm{Cl}(\psi)$ is the cohomology class of a cocycle ψ.

2.4 We will need some results from [2], [3]. For any reductive group G over a field K of characteristic 0, let $H^i_{\mathrm{ab}}(K,G)$ be the abelian Galois cohomology of G (cf. [3], Section 2). We need the following facts concerning H^i_{ab}.

Lemma 2.4.1 ([3], 5.8, 5.10).

Let K be a local or a number field, and let

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of connected reductive groups. Then the sequence

$$H^1(K,G_2) \to H^1(K,G_3) \xrightarrow{\Delta'} H^2_{\mathrm{ab}}(K,G_1) \xrightarrow{\varepsilon} H^2_{\mathrm{ab}}(K,G_2)$$

is exact.

Lemma 2.4.2 ([3], 2.11(2)).

Suppose that the semisimple part H^{ss} *of* H *is simply connected. Then* $H^i_{\mathrm{ab}}(K, H) = H^i(K, H^{\mathrm{tor}})$.

Remark 2.4.3. If H is abelian, then by the definition of abelian cohomology $H^i_{\mathrm{ab}}(K, H) = H^i(K, H)$.

Remark 2.4.4. One can show that if G_1 is a torus, then the connecting map Δ' defined in 2.4.1 coincides with the connecting map Δ defined in [12], I.5.7, see Subsection 2.3.

The following fact will be used in the proof of Theorem 1.9(ii).

Lemma 2.4.5 ([2], 1.4, 1.5).

Let $\psi \in Z^1(k, \mathrm{Aut}\,(G, H\backslash G))$. *If* ${}_\psi(G, H\backslash G)$ *has a* k*-point, then* $\Delta(\psi) = 0$.

Proposition 2.5

Let G, H *be as in the statement of Theorem* 1.9, *and let* $\eta \in Ш^2(k, H^{\mathrm{tor}}) \subset H^2(k, H^{\mathrm{tor}})$. *Then there exists* $\psi \in Z^1(k, \mathrm{Aut}\,(G, H\backslash G)^\circ)$ *such that* $\eta = \Delta(\psi)$.

Proof. By Lemma 2.4.1, we have the exact sequence

$$H^1\big(k, \mathrm{Aut}\,\big(G, H\backslash G)^\circ\big) \xrightarrow{\Delta'} H^2(k, H^{\mathrm{tor}}) \xrightarrow{\varepsilon} H^2_{\mathrm{ab}}\big(k, \tilde{A}'(G, H)^\circ\big). \tag{2.5.1}$$

According to Remark 2.4.4, there is no difference between Δ and Δ'. We have $H = TU$, $N(H) = SU$ where $S = (R_{L/k}\mathbb{G}_m)^2$ is a maximal torus in $G = (R_{L/k}\mathrm{SL}_2)^2$, cf. 1.8. Since S normalizes U, so does T. Hence $H^{\mathrm{tor}} = T$, $H^{\mathrm{cf}} = U$, $N(H)/H^{\mathrm{cf}} = S$, and $\tilde{A}'(G, H)^\circ = i(S) \cdot j(G)$. The semisimple part of $\tilde{A}'(G, H)^\circ$ is $j(G) \simeq G$, hence it is simply connected. By Lemma 2.4.2,

$$H^2_{\mathrm{ab}}\big(k, \tilde{A}'(G, H)^\circ\big) = H^2\big(k, (\tilde{A}'(G, H)^\circ)^{\mathrm{tor}}\big).$$

But $(\tilde{A}'(G, H)^\circ)^{\mathrm{tor}} = i(S) \cdot j(G)/j(G) = S/Z(G)$, hence $H^2_{\mathrm{ab}}(k, \tilde{A}'(G, H)^\circ) = H^2(k, S/Z(G))$.

We have $Z(G) = (R_{L/k}\mu_2)^2$, $S/Z(G) \simeq (R_{L/k}\mathbb{G}_m)^2$, hence $Ш^2(k, S/Z(G)) = 0$, cf. [10], 1.9. By hypothesis $\eta \in Ш^2(k, H^{\mathrm{tor}})$, hence $\varepsilon(\eta) \in Ш^2(k, S/Z(G)) = 0$, with the notation of (2.5.1). It follows that $\eta \in \mathrm{im}\,\Delta'$. Thus there exists $\psi \in Z^1(k, \mathrm{Aut}\,(G, H\backslash G)^\circ)$ such that $\eta = \Delta'(\psi) = \Delta(\psi)$. □

2.6. *Proof of Theorem* 1.9 (i).

Let $\eta \in Ш^2(k, H^{\mathrm{tor}})$, $\eta \neq 0$. Such an η exists by Lemma 1.6.1. By Proposition 2.5 there exists $\psi \in Z^1(k, \mathrm{Aut}\,(G, H\backslash G))$ such that $\Delta(\psi) = \eta$. Let $(\tilde{G}, \tilde{X}) = {}_\psi(G, X)$ be the twisted form of (G, X).

We claim that $\tilde{X}$ is a counter-example to the Hasse principle. Since $\Delta(\psi) \neq 0$, we have $X(k) = \emptyset$, cf. Lemma 2.4.5. It remains to show that $X(k_v) \neq 0$ for any place v of k.

Consider the commutative diagram

$$\begin{array}{ccccc} H^1(k, \tilde{A}'(G,H)^\circ) & \longrightarrow & H^1(k, \mathrm{Aut}\,(G, H\backslash G)^\circ) & \xrightarrow{\Delta} & H^2(k, H^{\mathrm{tor}}) \\ \downarrow & & \downarrow \mathrm{loc}_v & & \downarrow \mathrm{loc}_v \\ H^1(k_v, \tilde{A}'(G,H)^\circ) & \longrightarrow & H^1(k_v, \mathrm{Aut}\,(G, H\backslash G)^\circ) & \xrightarrow{\Delta} & H^2(k_v, H^{\mathrm{tor}}) \end{array}$$

where loc_v denotes localization at v.

Lemma 2.6.1

$H^1(K, \tilde{A}'(G,H)^\circ) = 1$ *for any field* $K \supset k$.

Proof. We have an exact sequence

$$1 \to G \to \tilde{A}'(G,H) \to S/Z(G) \to 1.$$

Here $H^1(K, G) = 1$ because $G = (R_{L/k}\mathrm{SL}_2)^2$ (cf. [11], Ch. X, §1), and $H^1(K, S/Z(G)) = 1$ because $S/Z(G) \simeq (R_{L/k}\mathbb{G}_m)^2$. It follows that $H^1\big(K, \tilde{A}'(G,H)^\circ\big) = 1$. □

Let $\xi \in H^1(k, \mathrm{Aut}\,(G, H\backslash G)^\circ)$ be the cohomology class of ψ. We have

$$\Delta(\psi) = \eta \in Ш^2\big(k, H^{\mathrm{tor}}\big)\,.$$

It follows that for the localization $\mathrm{loc}_v\xi$ we have

$$\Delta_v\big(\mathrm{loc}_v\xi\big) = 0.$$

Hence $\mathrm{loc}_v\xi$ comes from $H^1(k_v, \tilde{A}'(G,H)^\circ)$. But by Lemma 2.6.1, $H^1(k_v, \tilde{A}'(G,H)^\circ) = 1$. Thus $\mathrm{loc}_v\xi = 1$. We see that $\mathrm{loc}_v\xi$ is a coboundary. It follows that $\tilde{X}_{k_v} \simeq X_{k_v}$, hence $\tilde{X}_{k_v}(k_v) \neq 0$. This proves Theorem 1.9 (i).

2.7. *Proof of Theorem* 1.9 (ii).

The proof immediately follows from [1]. Indeed, Theorem 1.1 of [1] states that $Ч^1_\Sigma(k,H)=0$ if and only if weak approximation with respect to Σ holds for $X=H\backslash G$. Theorem 1.4 of [1] gives $Ч^1_\Sigma(k,H)=Ч^1_\Sigma(k,H^{\mathrm{tor}})$. By Lemma 1.7.2 we have $Ч^1_\Sigma(k,H^{\mathrm{tor}})\neq 0$. □

2.8. For the reader's convenience we give below a self-contained proof of Theorem 1.9(ii) including an easier part of [1].

Let $k_\Sigma=\prod_{v\in\Sigma}k_v$. Let $\mathcal{O}(X,G,k)$ be the set of orbits of $G(k)$ in $X(k)$, and let $\mathcal{O}(X,G,k_\Sigma)$ be the set of orbits of $G(k_\Sigma)$ in $X(k_\Sigma)$. Since G is simply connected, by a theorem of Kneser–Harder it satisfies weak approximation with respect to any finite Σ. Hence the closure of an orbit $(x\cdot G(k))$ in $X(k_\Sigma)$ equals $x\cdot G(k_\Sigma)$ for any $x\in X(k)$. Thus weak approximation for X with respect to Σ holds if and only if the map

$$i_\Sigma\colon \mathcal{O}\big(X,G,k\big)\to\mathcal{O}\big(X,G,k_\Sigma\big),$$

induced by the embedding $X(k)\hookrightarrow X(k_\Sigma)$, is surjective. But one can describe the above orbit spaces in cohomological terms:

$$\mathcal{O}\big(X,G,k\big)=\ker\big[H^1(k,H)\to H^1(k,G)\big],$$

$$\mathcal{O}\big(X,G,k_\Sigma\big)=\ker\big[H^1(k_\Sigma,H)\to H^1(k_\Sigma,G)\big]$$

(cf. [12], Ch. I, 5.4, Cor. 1 of Prop. 36). Thus nonsurjectivity of the map

$$\ker\big[H^1(k,H)\to H^1(k,G)\big]\to\bigoplus_{v\in\Sigma}\ker\big[H^1(k_v,H)\to H^1(k_v,G)\big]$$

implies lack of weak approximation for X with respect to Σ.

In our case $G=(R_{L/k}\mathrm{SL}_2)^2$, hence $H^1(K,G)=1$ for any $K\supset k$ (cf. [11], Ch. X, §1). Since $H^{\mathrm{cf}}=U$, we have $H^1(K,H)=H^1(K,H^{\mathrm{tor}})$ for any $K\supset k$ (cf. [10], 1.13). So we see that weak approximation with respect to Σ holds for X if and only if the map

$$H^1\big(k,H^{\mathrm{tor}}\big)\to\bigoplus_{v\in\Sigma}H^1\big(k_v,H^{\mathrm{tor}}\big)$$

is surjective, that is $Ч^1_\Sigma(k,H^{\mathrm{tor}})=0$. But we have $Ч^1_\Sigma(k,H^{\mathrm{tor}})\neq 0$, thus weak approximation with respect to Σ does not hold for X. This proves Theorem 1.9 (ii). □

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