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A note on p-nuclear operators

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Abstract

Let X be a Banach space and let $1 \leq r < +\infty$. We prove that X^* is isomorphic to a subspace of an $L^r(\mu)$-space if and only if the operator $(\alpha_n) \in \ell_r \to \sum \alpha_n x_n \in X$ is s-nuclear $(1/r + 1/s = 1)$ whenever $\sum \|x_n\|^s < +\infty$.

It is well known that the operator

$$(\alpha_n) \in \ell_\infty \to \sum_{n=1}^{\infty} \alpha_n x_n \in X$$

is 1-nuclear whenever $\sum \|x_n\| < +\infty$ regardless the Banach space X [1]. Then a natural question arises: Given $1 \leq r < +\infty$, which Banach spaces X have the property that the operator

$$T_{\widehat{x}} : (\alpha_n) \in \ell_r \to \sum \alpha_n x_n \in X \tag{1}$$

is s-nuclear $(r^{-1} + s^{-1} = 1)$ for every sequence $\widehat{x} = (x_n) \in \ell_a^s(X)$?. In the case $r = 1$, it seems reasonable to ask for a characterization of Banach spaces X for which the operator $(\alpha_n) \in \ell_1 \to \sum \alpha_n x_n \in X$ is ∞-nuclear for every *null sequence* (x_n) in X. Recall that an operator $T : X \to Y$ is called ∞-nuclear if T admits a representation of the form $Tx = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n$ for all $x \in X$, where (a_n) is a null sequence in X^* and (y_n) is a weakly absolutely summable sequence in Y (that is to say: $\sum |\langle y_n, y^*\rangle| < +\infty$ for all $y^* \in Y^*$). So, in an obvious way T can be written in the form $Tx = \sum\langle x, \overline{a}_n\rangle \overline{y}_n$ for all $x \in X$, with $(\overline{a}_n)$ a bounded sequence and $(\overline{y}_n)$ an unconditionally summable sequence. The following Lemma show that the converse result is true. Hence, it follows from [6] that the following statements are equivalent:

a) X^* is isomorphic to a subspace of an $L^1(\mu)$-space.

b) $(\alpha_n) \in \ell_1 \to \sum \alpha_n x_n \in X$ is ∞-nuclear for every null sequence (x_n) in X.

That is the reason why we only consider the case $1 < r < +\infty$ from now on.

Lemma 1

If (x_n) is an unconditionally summable sequence in X, then there exist an unconditionally summable sequence (y_n) in X and a scalar sequence $(\alpha_n) \in c_0$ such that $x_n = \alpha_n y_n$ for all $n \in \mathbb{N}$.

Proof. Since each unconditionally summable sequence (x_n) satisfies

$$\lim_{n\to\infty} \sup \left\{ \sum_{k=n}^{\infty} |\langle x_k, x^*\rangle| : x^* \in B_{X^*} \right\} = 0\,,$$

the proof is similar to that of the scalar case. □

We use the classical notation in Banach space theory. If X is a Banach space, X^* denotes its dual space and B_X its closed unit ball.

We refer to [5] or [7] for the definition of the r-nuclear and r-summing norm $(1 \le r \le +\infty)$ of an operator T, denoted respectively by $\nu_r(T)$ and $\pi_r(T)$. If X and Y are Banach spaces, $\mathcal{N}_r(X,Y)\,(\Pi_r(X,Y))$ will be the space of r-nuclear (r-summing) operators from X into Y. If $T : X \to Y$ is a finite rank operator, its finite r-nuclear norm is defined by

$$\nu_r^o(T) = \inf \sum_{n=1}^{m} \|x_n^*\| \|y_n\|\,,$$

where the infimum is taken over all finite representations $T = \sum_{n=1}^m x_n^* \otimes y_n$. If X or Y is finitedimensional, then the r-nuclear norm and the finite $r-$nuclear norm are equal [7].

As usual, $\ell_a^r(X)$ stands for the Banach space of all sequences (x_n) in X such that $\|(x_n)\|_r = (\sum \|x_n\|^r)^{\frac{1}{r}} < +\infty (1 \le r < +\infty)$.

DEFINITION 2. Let $1 < r < +\infty$. $P(r)$ will denote the class of all Banach spaces X such that the operator (1) is s-nuclear whenever $(x_n) \in \ell_a^s(X)\,,\, (r^{-1} + s^{-1} = 1)$.

Remark 3. If $X \in P(r)$, we can consider the natural linear map

$$\widehat{x} = (x_n) \in \ell_a^s(X) \to T_{\widehat{x}} \in \mathcal{N}_s(\ell_r, X).$$

Since this map has closed graph, it follows that $X \in P(r)$ if and only if there exists a constant $c > 0$ such that

$$\nu_s\left(\sum_{n=1}^{m} e_n^* \otimes x_n : \ell_r \to X\right) \leq c\left(\sum_{n=1}^{m} \|x_n\|^s\right)^{1/s} \tag{2}$$

for every finite subset $\{x_1, .., x_m\}$ of X ($\{e_n^* : n \in \mathbb{N}\}$ denotes the canonical basis of ℓ_s).

The next Proposition contains some properties of the class $P(r)$.

Proposition 4
i) $L^s(\mu) \in P(r)$ for every extended positive measure μ.
ii) If $X^{**} \in P(r)$, then $X \in P(r)$.
iii) If $X \in P(r)$, then $Z \in P(r)$ for every quotient Z of X.

Proof. i) Since $L^s(\mu)$ is an $\mathcal{L}^s-$space it suffices to prove that ℓ_s belongs to $P(r)$. So, let (x_n) be a sequence in ℓ_s such that $\sum \|x_n\|_s^s < +\infty$. Define an operator $T : \ell_r \to \ell_s$ by

$$T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n x_n \text{ for any } (\alpha_n) \in \ell_r .$$

By [7, Proposition 9.8], we have

$$\nu_s(T) \leq \left(\sum_{n=1}^{\infty} \|T^*(e_n)\|_s^s\right)^{1/s} ,$$

where $\{e_n\}$ is the unit vector basis of ℓ_r. But the last sum is equal to $\sum \|x_n\|_s^s$. In fact, we have

$$\sum_{n=1}^{\infty} \|T^*(e_n)\|_s^s = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle e_m, T^*(e_n)\rangle|^s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |\langle T(e_m), e_n\rangle|^s$$
$$= \sum_{m=1}^{\infty} \|T(e_m)\|_s^s = \sum_{m=1}^{\infty} \|x_m\|_s^s .$$

ii) If $X^{**} \in P(r)$, then there exists a constant $c > 0$ such that

$$\nu_s \left(\sum_{n=1}^{m} e_n^* \otimes x_n^{**} : \ell_r \to X^{**} \right) \leq c \left(\sum_{n=1}^{m} \|x_n^{**}\|^s \right)^{1/s}$$

for every finite subset $\{x_1^{**}, .., x_m^{**}\}$ of X^{**}. Then we have

$$\nu_s \left(\sum_{n=1}^{m} e_n^* \otimes x_n : \ell_r \to X^{**} \right) \leq c \left(\sum_{n=1}^{m} \|x_n\|^s \right)^{1/s}$$

for every finite set $\{x_n\}_1^m \subset X$. If T denotes the operator $\sum_{n=1}^m e_n^* \otimes x_n : \ell_r \to X$, we have to prove that

$$\nu_s(T) = \nu_s(T^{**}). \tag{3}$$

For this, we consider the operator $T_m : \ell_r^m \to X$ defined by $T_m(\alpha_n) = \sum_{n=1}^m \alpha_n x_n$ for all $(\alpha_n)_1^m \in \ell_r^m$. It is easy to prove that $\nu_s(T) = \nu_s(T_m)$ and $\nu_s((T_m)^{**}) = \nu_s(T^{**})$. Hence, it suffices to prove that (3) also holds changing T by T_m. But, as mentioned earlier, the norm s-nuclear and the norm finite s-nuclear are equal if the operator is defined on a finitedimensional space. So, (3) follows from the equality $\nu_s^o(T_m) = \nu_s^o((T_m)^{**})$ [1, Proposition 17.3].

iii) If $X \in P(r)$, there exists a constant $c > 0$ such that (2) holds for every finite subset $\{x_1, .., x_m\}$ of X. Given $\epsilon > 0$ and $\{\overline{x}_n\}_{n=1}^m \subset Z$, choose $x_n \in \overline{x}_n$ such that $\|x_n\| \leq \epsilon + \|\overline{x}_n\|$ for all $n \leq m$. If $\phi : X \to Z$ is the canonical surjection, we have

$$\begin{aligned} \nu_s \left(\sum_{n=1}^{m} e_n^* \otimes \overline{x}_n : \ell_r \to Z \right) &\leq \|\phi\| \, \nu_s \left(\sum_{n=1}^{m} e_n^* \otimes x_n : \ell_r \to X \right) \\ &\leq c \left[\sum_{n=1}^{m} (\|\overline{x}_n\| + \epsilon)^s \right]^{1/s} \end{aligned}$$

The statement follows letting ϵ tend to zero. □

Now we are ready for the main Theorem.

Theorem 5

Let X be a Banach space and let $1 < r < +\infty$. $X \in P(r)$ if and only if X^* is isomorphic to a subspace of an $L^r(\mu)$-space.

Proof. If $X \in P(r)$, we consider the continuous linear map

$$U : \widehat{x} = (x_n) \in \ell_a^s(X) \to T_{\widehat{x}} \in \mathcal{N}_s(\ell_r, X)\,.$$

Since with the trace duality $\Pi_r(X, \ell_r)$ (resp. $\ell_a^r(X^*)$) can be isometrically identified to the dual space of $\mathcal{N}_s(\ell_r, X)$ (resp. $\ell_a^s(X)$) [2], in a standard way we obtain

$$U^* : T \in \Pi_r(X, \ell_r) \to (T^\star(e_n^*)) \in \ell_a^r(X^*)\,.$$

So, there is a constant $c > 0$ such that

$$\left(\sum_{n=1}^m \|x_n^*\|^r\right)^{1/r} \leq c\ \pi_r\left(\sum_{n=1}^m x_n^* \otimes e_n : X \to \ell_r\right) \tag{4}$$

for every finite subset $\{x_1^*, .., x_n^*\}$ of X^*. To prove that X^* is isometric to a subspace of an $L^r(\mu)$-space, we will use the well known characterization of J. Lindenstrauss and A. Pelczynski [3]: A Banach space X is isomorphic to a subspace of an $L^r(\mu)$-space if and only if there exists a constant $\rho > 0$ such that

$$\left(\sum_{i=1}^m \|x_i\|^r\right)^{1/r} \leq \rho\ \left(\sum_{j=1}^n \|y_j\|^r\right)^{1/r}$$

whenever

$$\sum_{i=1}^m |\langle x_i, x^*\rangle|^r \leq \sum_{j=1}^n |\langle y_j, x^*\rangle|^r$$

for all $x^* \in X^*$. If $\{x_i^*\}_{i=1}^m$ and $\{y_j^*\}_{j=1}^n$ are finite subsets of X^* so that

$$\sum_{i=1}^m |\langle x, x_i^*\rangle|^r \leq \sum_{j=1}^n |\langle x, y_j^*\rangle|^r \text{ for all}\, x \in X \tag{5}$$

we can define two linear operators T and S from X into ℓ_r by

$$Tx = \sum_{i=1}^m \langle x, x_i^*\rangle e_i \ \text{ and }\ Sx = \sum_{j=1}^n \langle x, y_j^*\rangle e_j$$

for all $x \in X$. From (5) it follows that $\pi_r(T) \leq \pi_r(S)$. This and (4) yields

$$\left(\sum_{i=1}^m \|x_i^*\|^r\right)^{1/r} \leq c\,\pi_r(T) \leq c\,\pi_r(S) \leq c\,\nu_r(S)\,.$$

As mentioned earlier, $\nu_r(S) \leq \left(\sum_{j=1}^{n} \|y_j^*\|^r\right)^{\frac{1}{r}}$. Hence, X^* is isomorphic to a subspace of an $L^r(\mu)$-space.

Conversely, if X^* is isomorphic to a subspace of an L^r-space, then X^{**} is isomorphic to a quotient of an L^s-space. Proposition 4 assures us that $X \in P(r)$. □

Remark 6. a) It is clear that, for every sequence (x_n) belonging to $\ell_a^r(X^*)$, the operator $T : x \in X \to (\langle x, x_n^* \rangle) \in \ell_r$ is r-nuclear with

$$\nu_r(T) \leq \left(\sum_{n=1}^{\infty} \|x_n^*\|^r\right)^{1/r} .$$

Hence, the linear map $(x_n^*) \in \ell_a^r(X^*) \to T \in \mathcal{N}_r(X, \ell_r)$ is continuous and injective. On the other hand, if X^* is isomorphic to a subspace of an L^r-space, it follows from the proof of Theorem 5 that the map

$$T \in \Pi_r(X, \ell_r) \to (T^\star(e_n^*)) \in \ell_a^r(X^*)$$

is continuous. Hence, in an obvious sense, we have the equalities

$$\Pi_r(X, \ell_r) = \mathcal{N}_r(X, \ell_r) = \ell_a^r(X^*)\,,$$

whenever X^* is isomorphic to a subspace of an L^r-space.

b) By analyzing the proofs it is easy to obtain a quantitative version of Theorem 5:

Let X be a Banach space and let $1 < r < +\infty$. Let $c > 0$. The following statements are equivalent:

i) There are a measure μ and a subspace H of $L^r(\mu)$ with $d(X^*, H) \leq c$.

ii) For any finite subset $\{x_1, .., x_m\}$ of X, the inequality

$$\nu_s\left(\sum_{n=1}^{m} e_n^* \otimes x_n : \ell_r \to X\right) \leq c \left(\sum_{n=1}^{m} \|x_n\|^s\right)^{1/s}$$

holds.

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