Collect. Math. **47**, 3 (1997), 231–268

Continuous wavelet transform on semisimple Lie groups and inversion of the Abel transform and its dual

K. Trimèche

Department of Mathematics, Faculty of Sciences of Tunis, 1060 Tunis, Tunisia

Received September 1, 1995. Revised March 1, 1996

Abstract

In this work we define and study wavelets and continuous wavelet transform on semisimple Lie groups G of real rank ℓ. We prove for this transform Plancherel and inversion formulas. Next using the Abel transform $\mathcal{A}$ on G and its dual $\mathcal{A}^*$, we give relations between the continuous wavelet transform on G and the classical continuous wavelet transform on $\mathbb{R}^\ell$, and we deduce the formulas which give the inverse operators of the operators $\mathcal{A}$ and $\mathcal{A}^*$.

Introduction

Wavelets were introduced by J. Morlet, a French petroleum engineer at ELF-Aquitaine, in connection with the study, of seismic traces. The Mathematical foundations were given in a paper by A. Grossmann and J. Morlet [7]. The harmonic analyst Y. Meyer and many other mathematician became aware of this theory and they recognized many classical results inside it. (See [10], [16], [18], [21]).

Wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics. (See [6], [17], [21], [22]).

The continuous wavelet transform and its Plancherel and inversion Formulas admit a group theoretic interpretation in connection with the $ax + b$ group (group of affine transformations of the real line), the Heisenberg group and in greater generality with locally compact group admitting square integrable representations (See [8], [9], [16], [24]).

It is a natural question to ask, whether there exist wavelets and continuous wavelet transform on real semisimple Lie groups of real rank ℓ.

In this paper we use Harmonic Analysis on G to define and study wavelets and continuous wavelet transform on G. We have find difficulty with groups G because they do not possess dilations, which are fundamental in the classical theory of wavelets on $\mathbb{R}^\ell$.

We have introduced dilations for G on the level of the spherical Fourier transform on G. We establish a Plancherel and inversion formulas for the continuous wavelet transform on G. Next using the Abel transform $\mathcal{A}$ on G and its dual $\mathcal{A}^*$ we give relations between these transforms and the classical continuous wavelet transform on $\mathbb{R}^\ell$, and we deduce the formulas which give the inverse operators of the operators $\mathcal{A}$ and $\mathcal{A}^*$.

This paper is arranged as follow.

In the first section we recall some basic results on the structure of real semisimple Lie groups of real rank ℓ, on spherical functions and on the spherical Schwartz space $\mathcal{C}(K\backslash G/K)$.

We study in the second and third sections the spherical Fourier transform, the Abel transform and the convolution on G.

In the fourth section we define wavelets on G and we study their properties. Next we prove that the Abel transform on G relates these wavelets and classical wavelets on $\mathbb{R}^\ell$.

We define in the fifth section continuous wavelet transform on G and we establish Plancherel and inversion formulas for this transform.

In the sixth section we give inversion formulas for the Abel transform $\mathcal{A}$ on G and its dual $\mathcal{A}^*$, using wavelets on G.

I. Preliminaries

In this section we recall some basic results on real semisimple Lie groups. (See [12], [13]).

1. Structure of real semisimple Lie groups

Let G be a noncompact connected real semisimple Lie group with finite center, $\mathcal{G}$ the Lie algebra of G. Let θ be a Cartan involution of $\mathcal{G}$, $\mathcal{G} = \mathcal{K}+p$ the corresponding Cartan decomposition and K the analytic subgroup of G with Lie algebra $\mathcal{K}$. Let $\mathfrak{a} \subset p$ be a maximal abelian subspace, $\mathfrak{a}^*$ its (real) dual, $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$.

The Killing form of $\mathcal{G}$, induces a scalar product on $\mathfrak{a}$ and hence on $\mathfrak{a}^*$. We denote by $\langle .,. \rangle$ its $\mathbb{C}$-bilinear extension to $\mathfrak{a}^*_{\mathbb{C}}$.

The $\ell = \dim \mathfrak{a}$ is called the real rank of G. Let $e_1, \ldots, e_\ell$ be an orthonormal basis of $\mathfrak{a}$ and $e_1^*, \ldots, e_\ell^*$ the dual basis of $\mathfrak{a}^*_{\mathbb{C}}$. Then every λ in $\mathfrak{a}^*_{\mathbb{C}}$ is uniquely written in the form

$$\lambda = z_1 e_1^* + \ldots + z_\ell e_\ell^* , \quad z_j \in \mathbb{C}, \quad j = 1, 2, \ldots, \ell .$$

Using the basis $e_1, \ldots, e_\ell$ we can identify $\mathfrak{a}$ with $\mathbb{R}^\ell$.

For λ in $\mathfrak{a}^*$ put $\mathcal{G}_\lambda = \{X \in \mathcal{G}/[H, X] = \lambda(H)X , \text{ for all } H \in \mathfrak{a}\}$. If $\lambda \neq 0$ and $\dim \mathcal{G}_\lambda \neq 0$ then λ is called a (restricted) root and $m_\lambda = \dim \mathcal{G}_\lambda$ is called its multiplicity. The set of restricted roots will be denoted by $\sum$. If λ, μ are in $\mathfrak{a}^*$ let H_λ in $\mathfrak{a}$ be determined by $\lambda(H) = \langle H_\lambda, H \rangle$ for H in $\mathfrak{a}$, and put $\langle \lambda, \mu \rangle = \langle H_\lambda, H_\mu \rangle$. Let W be the Weyl group associated with $\sum$ and $|W|$ is cardinality.

Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$ and let $\overline{\mathfrak{a}^+}$ be its closure. We call a root positive if it is positive on $\mathfrak{a}^+$. The corresponding Weyl chamber in $\mathfrak{a}^*$ will be denoted by $\mathfrak{a}^*_+$ and let $\overline{\mathfrak{a}^*_+}$ its closure. Let $\sum^+$ be the set of positive roots. Put $\rho = \frac{1}{2} \sum_{\alpha \in \sum^+} m_\alpha \alpha$.

Let $\sum_0 = \{\alpha \in / \frac{1}{2}\alpha \notin \sum\}$ and put $\sum_0^+ = \sum^+ \cap \sum_0$. Let $\mathcal{N} = \bigoplus_{\alpha \in \sum^+} \mathcal{G}_\alpha$, $\overline{\mathcal{N}} = \theta \mathcal{N}$.

Let A be the analytic subgroup of G with Lie algebra $\mathfrak{a}$. The exponential map is an isomorphism from $\mathfrak{a}$ (considered as an abelian Lie group) onto A. We put $A^+ = \exp \mathfrak{a}^+$. Its closure in G is $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$. Let N (resp $\overline{\mathbf{N}}$) be the analytic subgroup of G with Lie algebra $\mathcal{N}$ (resp $\overline{\mathcal{N}}$).

Let x^+ be the $\overline{\mathfrak{a}^+}$-component of $x \in G$ in the Cartan decomposition $G = K \exp \overline{\mathfrak{a}^+} K$ and let $|x| = \|x^+\|$. Viewed on G/K (or $K \backslash G$), $| \cdot |$ is the distance to the origin $0 = \{K\}$.

Let $H : G \to \mathfrak{a}$ be the Iwasawa projection according to the Iwasawa decomposition $G = \mathrm{KAN}$, i.e. if $g \in G$ then $H(g)$ is the unique element in $\mathfrak{a}$ such that $g \in K \exp H(g) N$.

We normalize the Lebesgue measures dH and $d\lambda$ on $\mathfrak{a}$ and $\mathfrak{a}^*$ such that for the Fourier transform

$$\text{(I.1)} \qquad \mathcal{F}_0(f)(\lambda) = \int_{\mathfrak{a}} f(H) e^{-i\lambda(H)} dH , \quad \lambda \in \mathfrak{a}^*$$

we have the inversion formula

$$\text{(I.2)} \qquad \mathcal{F}_0^{-1}(g)(H) = \int_{\mathfrak{a}^*} g(\lambda) e^{i\lambda(H)} d\lambda , \, H \in \mathfrak{a} , \quad g \in \mathcal{S}(\mathfrak{a}^*)$$

Here $\mathcal{S}(\mathfrak{a}^*)$ denotes the space of C^∞-functions on $\mathfrak{a}^*$ which are rapidly decreasing as their derivatives. On the compact group K the Haar measure dk is normalized such that the total measure is 1. The Haar measure of nilpotent groups $N, \overline{\mathbf{N}}$ are normalized such that $\theta(dn) = d\overline{n}$ and

$$\int_N e^{-2\rho(H(\overline{n}))}\overline{dn} = 1 \,.$$

In the Iwasawa decomposition, the Haar measure dx on G is given by

$$\int_G f(x)dx = \int_K \int_{\mathfrak{a}} \int_N f\big(k(\exp H)n\big)e^{2\rho(H)}dkdHdn\,, \quad f \in \mathcal{D}(G)\,.$$

Here $\mathcal{D}(G)$ denotes the space of C^∞-functions on G with compact support (See [12] p. 273). In the Cartan decomposition, the Haar measure dx of G is given by

$$\int_G f(x)dx = \int_K \int_{\mathfrak{a}^+} \int_K f\big(k_1(\exp H)k_2\big)\delta(H)dk_1dHdk_2\,, \quad f \in \mathcal{D}(G)$$

where

$$\text{(I.3)} \qquad \delta(H) = \prod_{\alpha\in\sum^+} [2\mathrm{sh}\,\alpha(H)]^{m_\alpha}\,.$$

(See [20] p. 268).

We have the following estimate for the density $\delta(H)$:

$$\text{(I.4)} \qquad 0 \le \delta(H) \le e^{2\rho(H)}\,, \quad (H \in \overline{\mathfrak{a}^+})$$

Remark. If G has rank one then, for some α in $\mathfrak{a}^*, \sum$ is equal to $\{\alpha, -\alpha\}$ or $\{\alpha, -\alpha, 2\alpha, -2\alpha\}$. Let H_1 in $\mathfrak{a}$ be such that $\alpha(H_1) = 1$ and write $\mathcal{G}_{\pm1}, \mathcal{G}_{\pm2}$ instead of $\mathcal{G}_{\pm\alpha}\,\mathcal{G}_{\pm2\alpha}$ with dimension $m_{\pm1}, m_{\pm2}$ respectively. Choose the ordering on $\mathfrak{a}^*$ such that α is positive, then $\rho = \frac{1}{2}(m_1 + 2m_2)$.

The Haar measure on G satisfies

$$\int_G f(x)dx = \int_0^\infty f\big(\exp tH_1\big)A(t)dt, \quad f \in \mathcal{D}\big(K\backslash G/K\big)\,.$$

Here $\mathcal{D}(K\backslash G/K)$ denotes the space of C^∞-functions on G which are bi-invariant under K and with compact support, and

$$A(t) = 2^{2\rho}(\mathrm{sh} t)^{2\alpha+1}(\mathrm{ch} t)^{2\beta+1}$$

with

$$\alpha = \frac{1}{2}\big(m_1 + m_2 - 1\big)\,, \quad \beta = \frac{1}{2}\big(m_2 - 1\big)\,, \quad \rho = \alpha + \beta + 1\,.$$

(See [15] p. 14–16 and p. 27).

2. Spherical functions

The spherical functions on G are the functions

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}dk,\ x \in G\,,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}}\,.$$

We collect now some properties of the spherical functions.

i) The function $\varphi_\lambda(x)$ is bi-invariant under K in $x \in G$ and W-invariant in $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

ii) The function $\varphi_\lambda(x)$ is a C^∞-function in x and a holomorphic function in λ.

iii) We have

$$- \varphi_\lambda(e) = 1\,,\ \varphi_\lambda(x) = \varphi_{-\lambda}\big(x^{-1}\big)\,; \quad \varphi_{-\overline{\lambda}}(x) = \overline{\varphi_\lambda(x)}$$
$$- \varphi_\lambda \equiv \varphi_{\lambda'}, \quad \text{if and only if} \quad \lambda' = \mathcal{S}\lambda \quad \text{for some} \quad \mathcal{S} \in W\,.$$

iv) We have the product formula

$$\forall\, x, y \in G,\ \varphi_\lambda(x)\varphi_\lambda(y) = \int_K \varphi_\lambda(xky)dk \tag{I.5}$$

v) We have

$$\Delta\varphi_\lambda = -\big(\|\lambda\|^2 + \|\rho\|^2\big)\varphi_\lambda$$

where Δ is the Laplacian on G/K.

vi) We have

$$e^{-\rho(H)} \le \varphi_0\big(\exp H\big) \le \mathrm{Const.}\big(1 + \|H\|\big)^a e^{-\rho(H)}\,,\ \big(\mathbf{H} \in \overline{\mathfrak{a}^+}\big) \tag{I.6}$$

For some constant $a > 0$.

vii) We have

$$- O < \varphi_{-i\lambda}(\exp H) \le e^{\lambda(H)}\varphi_0(\exp H)\,, \quad \big(H \in \overline{\mathfrak{a}^+}\,, \quad \lambda \in \overline{\mathfrak{a}^*_+}\big)$$
$$- |\varphi_\lambda(x)| \le \varphi_{iIm\lambda}(x)\,, \quad \big(x \in G\,,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}}\big)\,. \tag{I.7}$$

Remark. If G has rank one, it follows from [15] p. 27, that the set of all spherical functions for (G, K) are

$$\varphi_\lambda(a_t) = \varphi_\lambda^{(\alpha,\beta)}(t)\,,\ a_t \in \mathbf{A}, \quad t \in \mathbb{R}$$

where

$$\alpha = \frac{1}{2}\big(m_1 + m_2 - 1\big)\,, \quad \beta = \frac{1}{2}\big(m_2 - 1\big)$$

and $\varphi_\lambda^{(\alpha,\beta)}$ the Jacobi function

$$\varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1\Big(\frac{1}{2}\big(\alpha+\beta+1-i\lambda\big)\,,\ \frac{1}{2}\big(\alpha+\beta+1+i\lambda\big)\,; \quad \alpha+1\,; -\mathrm{sh}^2 t\Big)$$

${}_2F_1$ being the Gaussian hypergeometric function.

3. The spherical Schwartz space $\mathcal{C}(K\backslash G/K)$

Let $U(\mathcal{G})$ be the universal enveloping algebra of $\mathcal{G}$. The elements of $U(\mathcal{G})$ act on $C^\infty(G)$ (the space of C^∞-functions on G), as differential operators, on both sides. Following Harish-Chandra we shall write $f(a;x;b)$ for the action of $(a,b) \in U(\mathcal{G}) \times U(\mathcal{G})$ on f in $C^\infty(G)$ at $x \in G$. Explicitly,

$$f(a\,;\ x\,;\ b) = \Big(\frac{\partial}{\partial s_1}\cdots\frac{\partial}{\partial s_d}\,\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_e}\Big)_{/s_1=\ldots=s_d=t_1=\ldots=t_e=0} \times f\big((\exp s_1X_1)\ldots(\exp s_dX_d)x(\exp t_1Y_1)\ldots(\exp t_eY_e)\big)$$

if $a = X_1\ldots X_d\,,\ b = Y_1\ldots Y_e\,,\ \big(X_1,\ldots,X_d,\,Y_1,\ldots,Y_e \in \mathcal{G}\big)\,.$

The spherical Schwartz space $\mathcal{C}(K\backslash G/K)$ is the space of all functions f in $C^\infty(K\backslash G/K)$ (the subspace of $C^\infty(G)$ consisting of functions bi-invariant under K) which satisfy

$$\mu_{a,b,r}(f) = \sup_{x\in G}\ \big(1+|x|\big)^r\varphi_0^{-1}(x)\,\big|f(a\,;x\,;b)\big| < +\infty$$

for any $a, b \in U(\mathcal{G})$ and any integer $r \geq 0$.

We topologize $\mathcal{C}(K\backslash G/K)$ by means of the semi-norms $\mu_{a,b,r}$.

The space $\mathcal{C}(K\backslash G/K)$ has the following properties

i) $\mathcal{C}(K\backslash G/K)$ is a Fréchet space.

ii) $\mathcal{D}(K\backslash G/K)$ is a dense subspace of $\mathcal{C}(K\backslash G/K)$.

(See [12] p. 252–257 and [1] p. 338).

iii) We denote by $L^p(K\backslash G/K)\,,\ p \in [1,+\infty]\,$, the space of functions f on G, bi-invariant under K, measurable and such that.

$$\|f\|_p = \Big(\int_G |f(x)|^p dx\Big)^{1/p} < +\infty\,, \quad \text{if} \quad p \in [1,+\infty[$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in G}|f(x)| < +\infty\,.$$

From [1] p. 338, the space $\mathcal{C}(K\backslash G/K)$ is a dense subspace of $L^q(K\backslash G/K)$ for $2 \leq q < +\infty$, while it is not contained in $L^q(K\backslash G/K)$ for $1 \leq q < 2$.

II. The spherical Fourier transform and the Abel transform

1. The Harish-Chandra's c-function

The Harish-Chandra's c-function is defined by

$$c(\lambda) = \gamma \prod_{\alpha \in \sum_0^+} c^{(\alpha)}(\lambda) \tag{II.1}$$

where $c^{(\alpha)}(\lambda)$ is the c-function for the rank one space associated with α and

$$\gamma = \prod_{\alpha \in \sum_0^+} \left[c^{(\alpha)}(i\rho)\right]^{-1} .$$

Moreover

$$c^{(\alpha)}(\lambda) = \left[I^{(\alpha)}(i\rho^{(\alpha)})\right]^{-1} I^{(\alpha)}(\lambda) \tag{II.2}$$

where

$$\rho^{(\alpha)} = \frac{1}{2}\big(m_\alpha + 2m_{2\alpha}\big)\alpha$$

and

$$I^{(\alpha)}(\lambda) = \frac{\Gamma\big(\langle i\lambda, \alpha_0\rangle\big)\Gamma\Big(\frac{1}{4}m_\alpha + \frac{1}{2}\langle i\lambda, \alpha_0\rangle\Big)}{\Gamma\Big(\frac{1}{2}m_\alpha + \langle i\lambda, \alpha_0\rangle\Big)\Gamma\Big(\frac{1}{4}m_\alpha + \frac{1}{2}m_{2\alpha} + \frac{1}{2}\langle i\lambda, \alpha_0\rangle\Big)} \tag{II.3}$$

with $\alpha_0 = \frac{\alpha}{\langle \alpha\,,\,\alpha\rangle}$ and Γ the classical gamma function. (See [12] Theorem 4.7.5, p. 175).

The function $c(\lambda)$ has the following properties

i) $c(-\lambda) = \overline{c(\lambda)}$, $(\lambda \in \mathfrak{a}^*)$

ii) $|c(\lambda)|$ is W-invariant on $\mathfrak{a}^*$

iii) The function $\lambda \to c(\lambda)^{-1}c(-\lambda)^{-1}$ is analytic and nonnegative on $\mathfrak{a}^*$.

(See [12] Proposition 4.7.14, p. 182, and Theorem 6.3.4, p. 272).

To rewrite $|c^{(\alpha)}(\lambda)|^{-2}$ we use the well known formulas for the gamma function

$$\Gamma(z)\Gamma(-z) = -\pi(z\sin\pi z)^{-1};\ \Gamma\Big(\frac{1}{2}+z\Big)\Gamma\Big(\frac{1}{2}-z\Big) = \pi(\cos\pi z)^{-1};$$
$$\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma\Big(\frac{1}{2}+z\Big).$$

According to the possibilities for the multiplicities m_α and $m_{2\alpha}$ of a root $\alpha \in \sum_0^+$ we distinguish four cases (See [4] p. 11–12). We give in each case the explicit form of $|c^{(\alpha)}(\lambda)|^{-2}$. We put $\Lambda = \langle \lambda, \alpha_0\rangle$.

i) If $m_{2\alpha} = 0$, m_α even

(II.4)
$$\left|c^{(\alpha)}(\lambda)\right|^{-2} = \left[I^{(\alpha)}(i\rho^{(\alpha)})\right]^2\left(\Big(\frac{m_\alpha}{2}-1\Big)^2+\Lambda^2\right)\left(\Big(\frac{m_\alpha}{2}-2\Big)^2+\Lambda^2\right)\dots(1+\Lambda^2)\Lambda^2$$

ii) If $m_{2\alpha} = 0$, m_α odd

(II.5)
$$\left|c^{(\alpha)}(\lambda)\right|^{-2} = \left[I^{(\alpha)}(i\rho^{(\alpha)})\right]^2\left(\Big(\frac{m_\alpha}{2}-1\Big)^2+\Lambda^2\right) \times\left(\Big(\frac{m_\alpha}{2}-2\Big)^2+\Lambda^2\right)\dots\Big(\frac{1}{4}+\Lambda^2\Big)\Lambda th\pi\Lambda$$

iii) If $m_{2\alpha}$ odd, $\frac{1}{4}m_\alpha \in \mathbb{Z}$

(II.6)
$$\begin{aligned}\left|c^{(\alpha)}(\lambda)\right|^{-2} = 2^{m_\alpha+1}\left[I^{(\alpha)}(i\rho^{(\alpha)})\right]^2&\left(\Big(\frac{m_\alpha}{4}-\frac{1}{2}\Big)^2+\frac{\Lambda^2}{4}\right)\\ &\times\left(\Big(\frac{m_\alpha}{4}-\frac{3}{2}\Big)^2+\frac{\Lambda^2}{4}\right)\dots\Big(\frac{1}{4}+\frac{\Lambda^2}{4}\Big)\\ &\times\left(\Big(\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}-1\Big)^2+\frac{\Lambda^2}{4}\right)\\ &\times\left(\Big(\frac{m_\alpha}{4}+\frac{m_{2\alpha}}{2}-2\Big)^2+\frac{\Lambda^2}{4}\right)\dots\Big(\frac{1}{4}+\frac{\Lambda^2}{4}\Big)\Lambda th\pi\frac{\Lambda}{2}\end{aligned}$$

iv) If $m_{2\alpha}$ odd, $\frac{1}{4}m_\alpha \in \mathbb{Z}+\frac{1}{2}$

(II.7)
$$\left|c^{(\alpha)}(\lambda)\right|^{-2} = 2^{m_\alpha-1}\left[I^{(\alpha)}(i\rho^{(\alpha)})\right]^2\left(\Big(\frac{m_\alpha}{4}-\frac{1}{2}\Big)^2+\frac{\Lambda^2}{4}\right)$$

$$\times\left(\left(\frac{m_\alpha}{4}-\frac{3}{2}\right)^2+\frac{\Lambda^2}{4}\right)\dots\left(1+\frac{\Lambda^2}{4}\right)$$
$$\times\left(\left(\frac{m_\alpha}{4}+\frac{m_{2\alpha-1}}{2}-\frac{1}{2}\right)^2+\frac{\Lambda^2}{4}\right)$$
$$\times\left(\left(\frac{m_\alpha}{4}+\frac{m_{2\alpha-1}}{2}-\frac{3}{2}\right)^2+\frac{\Lambda^2}{4}\right)\dots\left(1+\frac{\Lambda^2}{4}\right)\Lambda^3\coth\pi\frac{\Lambda}{2}$$

Remark. We deduce from relations (II.4),...,(II.7) the following estimate

$$\left|c(\lambda)\right|^{-2}\le \text{const.}(1+\|\lambda\|)^b\,,\quad (\lambda\in\mathfrak{a}^*) \tag{II.8}$$

for some constant $b>0$. (See also [12] Proposition 4.7.15, p. 183).

2. The spherical Fourier transform

Notations. We denote by

- $\mathcal{P}(\mathfrak{a}^*_{\mathbb{C}})^W$ the space of entire functions on $\mathfrak{a}^*_{\mathbb{C}}$, which are W-invariant of exponential type and rapidly decreasing.
- $\mathcal{S}(\mathfrak{a}^*)^W$ the space of C^∞-functions on $\mathfrak{a}^*$, which are W-invariant and rapidly decreasing as their derivatives.
- $L^p(\mathfrak{a}^*,\frac{|c(\lambda)|^{-2}}{|W|}d\lambda)^W, p\in[1,+\infty]$, the space of functions f on $\mathfrak{a}^*$, W-invariant, measurable and such that

$$\|f\|_{L^p}=\left(\frac{1}{|W|}\int_{\mathfrak{a}^*}\left|f(\lambda)\right|^p\left|c(\lambda)\right|^{-2}d\lambda\right)^{1/p}<+\infty,\quad p\in[1,+\infty[$$
$$\|f\|_{L^\infty}=\operatorname*{ess\,sup}_{\lambda\in\mathfrak{a}^*}|f(\lambda)|<+\infty\,.$$

We topologize these spaces with the classical topology.

Definition II.1. The spherical Fourier transform $\mathcal{F}$ (sometimes called the Harish-Chandra transform) is defined by

$$\mathcal{F}(f)(\lambda)=\int_G f(x)\varphi_{-\lambda}(x)dx\,,\quad f\in\mathcal{D}(K\backslash G/K)\,.$$

Remark. If G has rank one the spherical Fourier transform can be written in the from

$$\mathcal{F}(f)(\lambda)=\int_0^\infty f(a_t)\varphi_\lambda^{(\alpha,\beta)}(t)A(t)dt\,,\quad f\in\mathcal{D}(K\backslash G/K)$$

we put

$$f(a_t) = f[t]\,,\ a_t \in A\,,\quad t \in \mathbb{R}$$

the function $f[t]$ belongs to the space $\mathcal{D}_*(\mathbb{R})$ of C^∞-functions on $\mathbb{R}$, even and with compact support.

Using this notation we have

$$\mathcal{F}(f)(\lambda) = \int_0^\infty f[t]\varphi_\lambda^{(\alpha,\beta)}(t)A(t)dt\,,\quad f \in \mathcal{D}_*(\mathbb{R}) \tag{II.9}$$

then the spherical Fourier transform of $f[t]$ is the Jacobi transform (See [15] p. 27).

Theorem II.1

i) The transform $\mathcal{F}$ is an isomorphism between $\mathcal{D}(K\backslash G/K)$ and $\mathcal{P}(\mathfrak{a}^*_{\mathbb{C}})^W$.
ii) More precisely, f has support in the ball $\{x \in G/|x| \leq R\}$ if and only if $\mathcal{F}(f)$ is of exponential type R.
iii) The inverse transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(h)(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} h(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda\,,\quad (x \in G)\,. \tag{II.10}$$

(See [13] p. 450 and 454; [1] p. 332).

From [5], 22.9.4 iii), p. 78, we deduce the following result.

Corollary II.1

Let f be a function on G satisfying
– f is continuous and bounded
– f belongs to $L^1(K\backslash G/K)$
– $\mathcal{F}(f)$ belongs to $L^1\big(\mathfrak{a}^*,\ \frac{|c(\lambda)|^{-2}}{|W|}d\lambda\big)^W$
then we have the inversion formula for the transform $\mathcal{F}$:

$$f(x) = \frac{1}{|W|}\int_{\mathfrak{a}^*} \mathcal{F}(f)(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda\,,\quad (x \in G)\,. \tag{II.11}$$

Theorem II.2

The spherical Fourier transform $\mathcal{F}$ is a topological isomorphism between $\mathcal{C}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a}^*)^W$. The inverse transform $\mathcal{F}^{-1}$ is given by the relation (II.10). (See [13] Theorem 6.4.1, p. 273; [1] Theorem 1, p. 331–332).

Theorem II.3

i) Plancherel formula for $\mathcal{F}$

For all f in $\mathcal{C}(K\backslash G/K)$ we have

$$\int_G |f(x)|^2 dx = \frac{1}{|W|} \int_{\mathfrak{a}^*} \big|\mathcal{F}(f)(\lambda)\big|^2 \big|c(\lambda)\big|^{-2} d\lambda\,. \tag{II.12}$$

ii) Plancherel theorem for $\mathcal{F}$

The transform $\mathcal{F}$ extends uniquely to a unitary isomorphism of $L^2(K\backslash G/K)$ onto $L^2\big(\mathfrak{a}^*, \frac{|c(\lambda)|^{-2}}{|W|}\, d\lambda\big)^W$.

(See [13] p. 454; [12] Theorem 6.4.2, p. 274; [1] p. 337).

3. The Abel transform

Notations. *We denote by*

- *$\mathcal{D}(\mathfrak{a})^W$ the space of C^∞-functions on $\mathfrak{a}$, which are W-invariant and with compact support.*
- *$\mathcal{S}(\mathfrak{a})^W$ the space of C^∞-functions on $\mathfrak{a}$, which are W-invariant and rapidly decreasing as their derivatives.*

We topologize these spaces with the classical topology.

Definition II.2. *The Abel transform $\mathcal{A}$ is defined on $\mathcal{D}(K\backslash G/K)$ by*

$$\forall\, H \in \mathfrak{a},\; \mathcal{A}(f)(H) = e^{\rho(H)} \int_N f\big((\exp H)n\big) dn \tag{II.13}$$

(See [13] p. 450; [12] Proposition 3.3.1, p. 107; [1] p. 337).

Proposition II.1

For f in $\mathcal{D}(K\backslash G/K)$ we have

$$\mathcal{F}(f) = \mathcal{F}_0 \circ \mathcal{A}(f) \tag{II.14}$$

where $\mathcal{F}_0$ is the classical Fourier transform given by the relation (I.1).

(See [12] Proposition 3.3.1, 3.3.2, p. 107–108; [1] p. 337–338).

Theorem II.4

i) The transform $\mathcal{A}$ is a topological isomorphism between $\mathcal{D}(K\backslash G/K)$ and $\mathcal{D}(\mathfrak{a})^W$.
ii) More precisely, f has support in the ball $\{x \in G/|x| \le R\}$ if and only if $\mathcal{A}(f)$ has support in the ball $\{H \in \mathfrak{a}/\|H\| \le R\}$.

(See [13] Corollary 7.4, p. 454; [12] Proposition 3.3.2, p. 107; [1] Proposition 5, p. 338).

Theorem II.5

For any f in $\mathcal{C}(K\backslash G/K)$ the function $\mathcal{A}(f)$ defined by the relation (II.13) lies in $\mathcal{S}(\mathfrak{a})^W$ and the map $\mathcal{A}$ is a topological isomorphism between $\mathcal{C}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a})^W$ and we have the relation (II.14).

(See [12] Theorem 6.2.4, the relation (6.2.16) p. 264–265; [1] p. 348).

Remark. In the rank one case an explicit inversion formula for the Abel transform as a Weyl fractional integral has been obtained by T.H. Koornwinder [15]. For other groups see [23], [2], [3], [19].

III. Convolution

1. Generalized translation operators on G

DEFINITION III.1. Let f be a function in $\mathcal{D}(K\backslash G/K)$. For $x, y \in G$. We put

$$T_x(f)(y) = \int_K f(xky)dk\,. \tag{III.1}$$

The operators $T_x, x \in G$, are called generalized translation operators on G.

Properties

i) The function $T_x(f)(y)$ is bi-invariant under K with respect to x and y.
ii) For f in $\mathcal{D}(K\backslash G/K)$ and $x, y \in G$

$$T_e(f)(x) = f(x);\ T_x(f)(y) = T_y(f)(x)$$

iii) For $x, y \in G$.

$$T_x(\varphi_\lambda)(y) = \varphi_\lambda(x)\varphi_\lambda(y)\,. \tag{III.2}$$

Theorem III.1

i) Let f be in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$). For all $x \in G$, the function $T_x(f)$ belongs to $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$) and we have

$$\forall \lambda \in \mathfrak{a}^*, \quad \mathcal{F}\big(T_x(f)\big)(\lambda) = \varphi_\lambda(x)\mathcal{F}(f)(\lambda)\,. \tag{III.3}$$

(See [12] Theorem 6.2.2, p. 262).

ii) Let f be in $L^p(K\backslash G/K), p \in [1,+\infty]$. For all $x \in G$, the function $T_x(f)$ belongs to $L^p(K\backslash G/K), p \in [1,+\infty]$, and we have

$$\|T_x(f)\|_p \leq \|f\|_p \tag{III.4}$$

Theorem III.2

i) For f in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$), the map $(x,y) \to T_x(f)(y)$ is continuous on $G \times G$.
ii) For f in $L^p(K\backslash G/K)$, $p \in [1,+\infty[$, the map $x \to T_x(f)$ is continuous from G into $L^p(K\backslash G/K)$, $p \in [1,+\infty[$.

Proof. i) We deduce the result from the definition III.1

ii) It is sufficient to consider the case where f is in $\mathcal{D}(K\backslash G/K)$. For all $x_0 \in G$, there exists $r > 0$ such that for all $x \in G$ satisfying $|x\, x_0^{-1}| \leq 1$, we have

$$\|T_x(f) - T_{x_0}(f)\|_p \leq \left(\int_B dy\right)^{1/p} \sup_{y\in B} \left|T_x(f)(y) - T_{x_0}(f)(y)\right|$$

where $B = \{y \in G/|y| < r\}$.

From this inequality and the i) we deduce

$$\lim_{x\to x_0} \|T_x(f) - T_{x_0}(f)\|_p = 0\,. \qquad \square$$

Remark. In the rank one case the expression and the properties of the generalized translation operators are given by T.H. Koornwinder [15].

2. Convolution

Definition III.2. The convolution of f and g in $\mathcal{D}(K\backslash G/K)$ is the function $f * g$ defined by

$$f * g(x) = \int_G f(y) g\big(y^{-1}x\big)dy$$

This relation can also be written in the form

$$f * g(x) = \int_G f(y) T_{y^{-1}}(g)(x)dy$$

Theorem III.3

i) The spaces $\mathcal{D}(K\backslash G/K)$, $\mathcal{C}(K\backslash G/K)$, $L^1(K\backslash G/K)$ are commutative convolution algebras and we have

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$$

(See [12] Theorem 6.1.10, p. 255, Theorem 6.2.2, p. 262).
ii) For f in $L^p(K\backslash G/K)$ and g in $L^q(K\backslash G/K)$ with $p, q \in [1, +\infty]$, the function $f * g$ belongs to $L^r(K\backslash G/K)$ with $r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ and we have

$$\|f * g\|_r \leq \|f\|_p \|g\|_q \,.$$

Theorem III.4

Let f be in $L^2(K\backslash G/K)$ and g in $L^p(K\backslash G/K)$, $p \in [1, 2[$, then
i) The function $f * g$ belongs to $L^2(K\backslash G/K)$ and we have

$$\|f * g\|_2 \leq \|\varphi_0\|_q |\, |f\|_2 |\, |g\|_p$$

with $q \in]2, +\infty[$, such that $\frac{1}{p} + \frac{1}{q} = 1$.
ii) We have

$$\mathcal{F}(f * g) = \mathcal{F}(f).\mathcal{F}(g)\,.$$

Proof. i) Let f, g in $\mathcal{D}(K\backslash G/K)$, then by Theorem II.3

$$\|f * g\|_2^2 = \|\mathcal{F}(f).\,\mathcal{F}(g)\|_{L^2}^2 \leq \|\mathcal{F}(g)\|_{L^\infty}^2 \|\mathcal{F}(f)\|_{L^2}^2$$

but from Hölder's inequality we have

$$\forall\, \lambda \in \mathfrak{a}^*, \ \big|\mathcal{F}(g)(\lambda)\big| \leq \|\varphi_0\|_q \|g\|_p$$

then

$$\|f * g\|_2^2 \leq \|\varphi_0\|_q^2 \|g\|_p^2 \|f\|_2^2$$

since $\mathcal{D}(K\backslash G/K)$ is dense in $L^2(K\backslash G/K)$ and $L^p(K\backslash G/K)$ the result follows.

ii) The result is clear. □

Theorem III.5

Let f and g be in $L^2(K\backslash G/K)$. Then the function $f * g$ belongs to $L^q(K\backslash G/K)$, $q \in]2, +\infty[$, and we have

$$\|f * g\|_q \leq \|\varphi_0\|_q \|f\|_2 \|g\|_2$$

Proof. Let h be in $L^p(K\backslash G/K)$ with $p \in [1,2[$, such that $\frac{1}{p}+\frac{1}{q}=1$ and f, g in $\mathcal{D}(K\backslash G/K)$, then from the theorem III.4

$$\left|\int_G f*g(x)h(x)dx\right| \leq \int_G |g(x)|\big(|h|*|f|\big)(x)dx$$
$$\leq \|g\|_2\|(|h|*|f|)\|_2 \leq \|\varphi_0\|_q\|f\|_2\|g\|_2\|h\|_p$$

taking supremum over $\{h \in L^p(K\backslash G/K)/\|h\|_p \leq 1\}$ we get

$$\|f*g\|_q \leq \|\varphi_0\|_q\|f\|_2\|g\|_2. \qquad \square$$

Theorem III.6

For f and g in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$ we have

$$\mathcal{A}(f*g)=\mathcal{A}(f)*_0\mathcal{A}(g)$$

where $*_0$ is the convolution on $\mathfrak{a}$.

(See [13] Corollary 7.4, p. 454; [12] Proposition 3.3.2, p. 107; [1] p. 348).

IV. Wavelets on G

Definition IV.1. We say that a function g on G, measurable, bi-invariant under K, is a wavelet on G if there is a constant C_g with the property that $0 < C_g < +\infty$ and, for λ almost every where on $\mathfrak{a}^*$,

$$C_g=\int_0^\infty \big|\mathcal{F}(g)(a\lambda)\big|^2\frac{da}{a}$$

Proposition IV.1

i) Let $t \in]0,+\infty[$. There exists a function β_t in $\mathcal{C}(K\backslash G/K)$ such that

$$\forall\,\lambda\in\mathfrak{a}^*\,,\ \mathcal{F}(\beta_t)(\lambda)=\exp\big[-t(\|\lambda\|^2+\|\rho\|^2)\big]$$

ii) The function

$$g(x)=-\frac{d}{dt}\beta_t(x)-\|\rho\|^2\beta_t(x)$$

is a wavelet on G which belongs to $\mathcal{C}(K\backslash G/K)$ and we have

$$C_g = \frac{e^{-2t\|\rho\|^2}}{8t^2}$$

Proof. We deduce the results from the theorem II.2 and the definition IV.1. □

Remark. The properties of the function β_t have been studied by R. Gangolli [11].

If G has rank one, the definition IV.1 can be written more simply as follows: A function g on G measurable, bi-invariant under K, is a wavelet on G if the even function $g[t]$ on $\mathbb{R}$ given by

$$g(a_t) = g[t]\,,\ a_t \in A\,,\ t \in \mathbb{R}$$

satisfies the condition

$$0 < C_g = \int_0^\infty \left|\mathcal{F}(g)(a)\right|^2 \frac{da}{a} < +\infty \tag{IV.1}$$

where $\mathcal{F}$ is the Jacobi transform defined by the relation (II.9).

Proposition IV.2

For the rank one case, we consider an even non zero function g, on $\mathbb{R}$ in $L^2\big([0,+\infty[, A(t)dt\big)$ (the space of square integrable functions on $[0,+\infty[$ with respect to the measure $A(t)dt$) satisfying

$$(*)\qquad \exists\,\alpha > 0 \quad \textit{such that} \quad \mathcal{F}(g)(\lambda) - \mathcal{F}(g)(0) = 0\big(|\lambda|^\alpha\big)\,, \quad \textit{as} \quad \lambda \to 0\,.$$

Then the condition (IV.1) is equivalent to

$$\mathcal{F}(g)(0) = 0\,.$$

Proof.

We assume that the condition (IV.1) is satisfied. If $\mathcal{F}(g)(0) \neq 0$, then there are positive η and M such that

$$\big|\mathcal{F}(g)(\lambda)\big| \geq M\,, \quad \text{for} \quad \lambda \in [0,\eta]$$

thus the integral in (IV.1) would be equal to $+\infty$.

We assume that

$$\mathcal{F}(g)(0) = 0$$

since $g \neq 0$, we deduce from the theorem II.3 that the first inequality in (IV.1) will hold.

From the condition (*) there are positive δ and ε such that

$$\left|\mathcal{F}(g)(\lambda)\right| \geq \varepsilon \lambda^{\alpha}, \quad \text{for} \quad \lambda \in [0, \delta]$$

we deduce from this inequality, relations (II.4),...,(II.7), and the fact that the function $|c(\lambda)|^{-2}$ is increasing on $[0, +\infty[$ that

$$\int_0^{\infty} \left|\mathcal{F}(g)(a)\right|^2 \frac{da}{a} \leq \varepsilon \int_0^{\delta} \frac{da}{a^{1-\alpha}} + \frac{|c(\delta)|^2}{\delta} \int_{\delta}^{+\infty} \left|\mathcal{F}(g)(a)\right|^2 \left|c(a)\right|^{-2} da$$

using the theorem II.3 we obtain

$$\int_0^{\infty} \left|\mathcal{F}(g)(a)\right|^2 \frac{da}{a} \leq \frac{\varepsilon \delta^{\alpha}}{\alpha} + \frac{1}{\delta}\left|c(\delta)\right|^2 \|g\|_2^2$$

thus the integral in (IV.1) is finite. □

Remark. The relation

$$\mathcal{F}(g)(0) = 0\,.$$

can be equivalently written as

$$\int_0^{\infty} g[t] \varphi_0^{(\alpha,\beta)}(t) A(t) dt = 0\,.$$

Proposition IV.3

For arbitrary rank, we consider for $\alpha \in \sum_0^+$ the function $k_{\alpha}(a)$ defined on $]0, +\infty[$ by

$$k_{\alpha}(a) = \sup_{\lambda \in \mathfrak{a}^* \setminus \{0\}} \frac{\left|c^{(\alpha)}\left(\frac{\lambda}{a}\right)\right|^{-2}}{\left|c^{(\alpha)}(\lambda)\right|^{-2}}$$

where $c^{(\alpha)}$ is the function given by the relation (II.2).

Then we have

$$k_{\alpha}(a) = \begin{cases} a^{-2} & , \text{ if } \quad a \geq 1 \\ a^{-(m_{\alpha} + m_{2\alpha})} & , \text{ if } \quad 0 < a < 1 \end{cases}$$

Proof. We deduce the result from relations (II.4),...,(II.7). □

Proposition IV.4

i) The function $k(a)$ defined on $]0,+\infty[$ by

$$k(a) = \sup_{\lambda\in\mathfrak{a}^*\setminus\{0\}} \frac{\left|c(\frac{\lambda}{a})\right|^{-2}}{\left|c(\lambda)\right|^{-2}}$$

where c is the function given by the relation (II.1), satisfies the relation

$$k(a) = \prod_{\alpha\in\sum_0^+} k_\alpha(a)$$

and we have

$$k(a) = \begin{cases} a^{-2\mathrm{card}\sum_0^+} & , \text{ if } \quad a \geq 1 \\ a^{-\dim N} & , \text{ if } \quad 0 < a < 1 \end{cases}$$

ii) When G has a complex structure we have

$$k(a) = a^{-2\mathrm{card}\sum_0^+}, \quad \text{for all} \quad a > 0\,.$$

Proof. i) We deduce the results from relations (II.1), (II.4),...,(II.7) and the proposition IV.3.

ii) We obtain the result from the fact that in this case we have

$$\forall\,\lambda\in\mathfrak{a}^*\,,\ \left|c(\lambda)\right|^{-2} = \left|\pi(i\lambda)\right|^2\pi(\rho)^{-2} \tag{IV.2}$$

where

$$\pi(\lambda) = \sum_{\alpha\in\sum_0^+} \langle\lambda,\alpha\rangle \tag{IV.3}$$

(See [13] Theorem 5.7, p. 432). □

Theorem IV.1

Let g be a wavelet on G in $L^2(K\backslash G/K)$ and $a\in]0,+\infty[$. Then
i) The function $\lambda\to\mathcal{F}(g)(a\lambda)$ belongs to $L^2\big(\mathfrak{a}^*, \frac{|c(\lambda)|^{-2}}{|W|}d\lambda\big)^W$ and we have

$$\|\mathcal{F}(g)(a.)\|_{L^2} \leq \left(\frac{k(a)}{a^\ell}\right)^{1/2}\|g\|_2$$

ii) There exists a function g_a in $L^2(K\backslash G/K)$ such that

$$\forall\,\lambda\in\mathfrak{a}^*,\ \mathcal{F}(g_a)(\lambda)=\mathcal{F}(g)(a\lambda)$$

and we have

$$\|g_a\|_2\leq\left(\frac{k(a)}{a^\ell}\right)^{1/2}\|g\|_2 \tag{IV.4}$$

Proof. i) By change of variables we have

$$\frac{1}{|W|}\int_{\mathfrak{a}^*}\left|\mathcal{F}(g)(a\lambda)\right|^2\left|c(\lambda)\right|^{-2}d\lambda=\frac{1}{a^\ell|W|}\int_{\mathfrak{a}^*}\left|\mathcal{F}(g)(\Lambda)\right|^2\left|c\big(\frac{\Lambda}{a}\big)\right|^{-2}d\Lambda$$

but we have

$$\int_{\mathfrak{a}^*}\left|\mathcal{F}(g)(\Lambda)\right|^2\left|c\big(\frac{\Lambda}{a}\big)\right|^{-2}d\Lambda\leq k(a)\int_{\mathfrak{a}^*}\left|\mathcal{F}(g)(\Lambda)\right|^2\left|c(\Lambda)\right|^{-2}d\Lambda$$

we deduce the result from these relations and the theorem II.3.

ii) The theorem II.3 gives the result. □

Theorem IV.2

i) Let $a\in]0,+\infty[$ and g a wavelet on G in $\mathcal{D}(K\backslash G/K)$ with support in the ball $\{x\in G/|x|\leq R\}$. There exists a function g_a in $\mathcal{D}(K\backslash G/K)$ with support in the ball $\{x\in G/|x|\leq aR)$ such that

$$\forall\,\lambda\in\mathfrak{a}^*,\ \mathcal{F}(g_a)(\lambda)=\mathcal{F}(g)(a\lambda)\,.$$

ii) Let $a\in]0,+\infty[$ and g a wavelet on G in $\mathcal{C}(K\backslash G/K)$. Then there exists a function g_a in $\mathcal{C}(K\backslash G/K)$ such that

$$\forall\,\lambda\in\mathfrak{a}^*,\ \mathcal{F}(g_a)(\lambda)=\mathcal{F}(g)(a\lambda)\,.$$

Proof. We deduce these results from theorems II.1, II.2. □

Theorem IV.3

Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$) and g_a, $a\in]0,+\infty[$, the function given by the theorem IV.2. Then

$$\forall\,x\in G\,,\ g_a(x)=\mathcal{A}^{-1}\circ\mathcal{H}_a\circ\mathcal{A}(g)(x) \tag{IV.5}$$

where $\mathcal{A}$ is the Abel transform and $\mathcal{H}_a$ the operator defined by

(IV.6) $$\forall\, H \in \mathfrak{a}, \quad \mathcal{H}_a(f)(H) = \frac{1}{a^\ell} f\Big(\frac{H}{a}\Big)$$

Proof. From the proposition II.1 and the theorem II.5 we have

$$\forall\, \lambda \in \mathfrak{a}^*, \quad \mathcal{F}_0 \circ \mathcal{A}(g_a)(\lambda) = \mathcal{F}_0 \circ \mathcal{A}(g)(a\lambda)\,.$$

By change of variables we obtain

$$\forall\, \lambda \in \mathfrak{a}^*, \quad \mathcal{F}_0 \circ \mathcal{A}(g_a)(\lambda) = \mathcal{F}_0\big[\mathcal{H}_a \circ \mathcal{A}(g)\big](\lambda)$$

we deduce the result from the injectivity of the Fourier transform $\mathcal{F}_0$ and theorems II.4, II.5. □

Remark. Let g^0 be a function defined on $\mathbb{R}^\ell$. We put for $a \in]0, +\infty[$:

(IV.7) $$\forall\, H \in \mathbb{R}^\ell, \quad g^0_a(H) = \frac{1}{a^\ell} g^0\Big(\frac{H}{a}\Big) = \mathcal{H}_a(g^0)(H)\,.$$

Using this notation, the relation (IV.5) can also be written in the form

(IV.8) $$\forall\, H \in \mathfrak{a}, \quad \mathcal{A}(g_a)(H) = \big(\mathcal{A}(g)\big)_a(H)\,.$$

Theorem IV.4

We suppose that G has complex structure. If g is a wavelet on G in $\mathcal{D}(K\backslash G/K)$ we have for all $a > 0$:

$$\forall\, H \in \mathfrak{a},\ g_a(\exp H) = a^{-(\ell + \mathrm{card} \sum_0^+)} \frac{\delta_0\big(\frac{H}{a}\big)}{\delta_0(H)} g\Big(\exp \frac{H}{a}\Big)$$

where

(IV.9) $$\forall\, H \in \mathfrak{a},\ \delta_0(H) = \prod_{\alpha \in \sum_0^+} 2\mathrm{sh}\, \alpha(H)\,.$$

Proof. When G has complex structure and considered as a real Lie group, we have $m_\alpha = 2$ and $m_{2\alpha} = 0$ for $\alpha \in \sum^+$. In this case the spherical function φ_λ has the following form

$$\varphi_\lambda(\exp\, H) = \pi(\rho)\pi(i\lambda)^{-1}\delta_0(H)^{-1} \sum_{s \in W} (\det s) e^{is\lambda(H)}, \quad \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ H \in \mathfrak{a}$$

where $\delta_0(\cdot)$ and $\pi(\cdot)$ the functions given by relations (IV.9), (IV.3).
(See [23] p. 287–289 and [13] Theorem 5.7, p. 432).

Using these relations and (IV.2) we obtain for all $a>0,\ \lambda\in\mathfrak{a}^*$ and $H\in\mathfrak{a}$:

$$\text{(IV.10)}\qquad \varphi_{\lambda/a}(\exp H)\Big|c\Big(\frac{\lambda}{a}\Big)\Big|^{-2} = a^{-\mathrm{card}\sum_0^+}\frac{\delta_0\big(\frac{H}{a}\big)}{\delta_0(H)}\big|c(\lambda)\big|^{-2}\varphi_\lambda\Big(\exp\frac{H}{a}\Big).$$

Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$. From the Theorem II.1, the function $g_a,\ a\in]0,+\infty[$, is given by

$$\forall\, H\in\mathfrak{a},\ g_a\ (\exp H) = \frac{1}{|W|}\int_{\mathfrak{a}^*}\mathcal{F}(g)(a\lambda)\varphi_\lambda(\exp H)\big|c(\lambda)|^{-2}d\lambda$$

making a change of variables and using the relation (IV.10) we obtain

$$\forall\, H\in\mathfrak{a},\ g_a(\exp H) = a^{-(\ell+\mathrm{card}\sum_0^+)}\frac{\delta_0\big(\frac{H}{a}\big)}{\delta_0(H)}g\Big(\exp\frac{H}{a}\Big).\qquad\square$$

Remark. From the Theorem II.3. i) and the relation (IV.2) we deduce

$$\|g_a\|_2 = a^{-\left(\frac{\ell}{2}+\mathrm{card}\sum_0^+\right)}\|g\|_2$$

then the inequality (IV.4) is an equality in this case.

Example: Let G denote the group $SO_0(1,3)$. For this group we have the following results.

– The spherical function φ_λ and the function $|c(\lambda)|^{-2}$ are

$$\varphi_\lambda(x) = \frac{\sin\lambda x}{\lambda\,\mathrm{sh}\,x},\quad x\in\mathbb{R}\backslash\{0\},\ \lambda\in\mathbb{C}\backslash\{0\}$$
$$|c(\lambda)|^{-2} = \lambda^2,\ \lambda\in\mathbb{R}.$$

(See [15] p. 14–16 and p. 27).

– The spherical Fourier transform and its inverse are

$$\mathcal{F}(f)(\lambda) = 4\int_0^\infty f(x)\varphi_\lambda(x)\mathrm{sh}^2x\ dx,\ f\in\mathcal{D}_*(\mathbb{R})$$
$$\mathcal{F}^{-1}(h)(x) = \frac{1}{2\pi}\int_0^\infty h(\lambda)\varphi_\lambda(x)\,\lambda^2 d\lambda.$$

(See [15] p. 27).

For this group we have

$$k(a)=\frac{1}{a^2}\,,\quad a\in]0,+\infty[\,.$$

Let g be a wavelet on $G=SO_0(1,3)$ in $\mathcal{D}_*(\mathbb{R})$. The function g_a, $a\in]0,+\infty[$, is given for $x\in\mathbb{R}\backslash\{0\}$ by

$$g_a(x)=\frac{1}{a^2}\,\frac{\mathrm{sh}\frac{x}{a}}{\mathrm{sh}x}g\Big(\frac{x}{a}\Big) \tag{IV.11}$$

we see that g_a is in $\mathcal{D}_*(\mathbb{R})$ and in $\mathcal{S}_*(\mathbb{R})$ (the space of C^∞-functions on $\mathbb{R}$, even and such that for all $p,k\in\mathbb{N}$, $\sup_{x\in\mathbb{R}}\,(1+x^2)^p\,\varphi_0^{-1}(x)\,\big|f^{(k)}(x)\big|<+\infty$).

We have the relation

$$\|g_a\|_2=\frac{1}{a^{3/2}}\|g\|_2$$

then the inequality (IV.4) is also an equality in this case.

We remark that when g is in $L^1([0,+\infty[,4\mathrm{sh}^2xdx)$ (the space of integrable functions on $[0,+\infty[$ with respect to the measure $4\mathrm{sh}^2xdx$) the function g_a, $a\in]1,+\infty[$, given by the relation (IV.11) does not belong to $L^1([0,+\infty[,4\mathrm{sh}^2xdx)$.

Proposition IV.5

Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$)(resp. $L^2(K\backslash G/K)$) and g_a, $a\in]0,+\infty[$, the function given by theorems IV.1, IV.2. Then the map $a\to g_a$ is continuous from $]0,+\infty[$ into $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$)(resp. $L^2(K\backslash G/K)$).

Proof. For g in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$) we deduce the result from the theorem IV.3.

Let g be in $L^2(K\backslash G/K)$. For all $\varepsilon>0$, there exists g^0 in $\mathcal{C}(K\backslash G/K)$ such that

$$\|g-g^0\|_2<\varepsilon\,.$$

Let $a,a_0\in]0,+\infty[$. From the theorem II.3 we have

$$\int_G|g_a^0(x)-g_{a_0}^0(x)\big|^2dx=\frac{1}{|W|}\int_{\mathfrak{a}^*}\big|\mathcal{F}(g^0)(a\lambda)-\mathcal{F}(g^0)(a_0\lambda)\big|^2\big|c(\lambda)\big|^{-2}d\lambda$$

we deduce from this relation and the dominated convergence theorem that the map $a\to g_a^0$ is continuous from $]0,+\infty[$ into $L^2(K\backslash G/K)$.

We have

$$\|g_a-g_{a_0}\|_2\le\|g_a-g_a^0\|_2+\|g_a^0-g_{a_0}^0\|_2+\|g_{a_0}^0-g_{a_0}\|_2$$

but from the relation (IV.2) we have

$$\|g_a - g_a^0\|_2 \leq \left(\frac{k(a)}{a^\ell}\right)^{1/2} \|g - g^0\|_2$$

then

$$\|g_a - g_{a_0}\|_2 \leq \left[\left(\frac{k(a)}{a^\ell}\right)^{1/2} + \left(\frac{k(a_0)}{a_0^\ell}\right)^{1/2}\right] \|g - g^0\|_2 + \|g_a^0 - g_{a_0}^0\|_2$$

we deduce the result from this inequality, the first result and the continuity of the function $k(a)$ on $]0, +\infty[$. □

Example: Let g be a wavelet on $G = SO_0(1,3)$ in $\mathcal{D}_*(\mathbb{R})$ (resp. $\mathcal{S}_*^2(\mathbb{R})$) (resp. $L^2([0,+\infty[, 4\mathrm{sh}^2 x dx))$ and $g_a,\ a \in]0,+\infty[$, the function given by the relation (IV.11). Then the map $a \to g_a$ is continuous from $]0,+\infty[$ into $\mathcal{D}_*(\mathbb{R})$ (resp. $\mathcal{S}_*^2(\mathbb{R})$) (resp. $L^2([0,+\infty[, 4\mathrm{sh}^2 x dx))$.

Definition IV.2. Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$ (resp. $\mathcal{C}(K\backslash G/K)$) (resp. $L^2(K\backslash G/K)$) and $g_a, a \in]0,+\infty[$, the function given by theorems IV.1, IV.2. We define the family of wavelets $g_{a,x}, (a,x) \in]0,+\infty[\times G$, on G by

$$\forall\, y \in G,\ g_{a,x}(y) = \left(\frac{k(a)}{a^\ell}\right)^{-1/2} T_x(g_a)(y)$$

where $T_x, x \in G$, are generalized translation operators on G given by the definition III.1.

Proposition IV.6

Let g be a wavelet on G in $L^2(K\backslash G/K)$. Then the map $(a,x) \to g_{a,x}$ is continuous from $]0,+\infty[\times G$ into $L^2(K\backslash G/K)$.

Proof. Let $(a,x), (a_0,x_0)$ be in $]0,+\infty[\times G$. From the theorem III.1. ii) we deduce

$$\|T_x(g_a) - T_{x_0}(g_{a_0})\|_2 \leq \|g_a - g_{a_0}\|_2 + \|T_x(g_{a_0}) - T_{x_0}(g_{a_0})\|_2$$

we obtain the result from the proposition IV.5, the theorem III.2. ii) and the continuity of $k(a)$. □

Example: The generalized translation operators associated with the group $G = SO_0(1,3)$ are given for $x, y \in \mathbb{R}\backslash\{0\}$, by

$$T_x(f)(y) = \frac{1}{\sqrt{\pi}} \int_{|x-y|}^{x+y} f(t) \frac{\mathrm{sh}\, t}{\mathrm{sh}\, x\, \mathrm{sh}\, y} dt\,, \quad f \in \mathcal{D}_*(\mathbb{R})\,.$$

(See [15] p. 56–60).

Let g be a wavelet on $G = SO_0(1,3)$ and $g_a, a \in]0,+\infty[$, the function given by the relation (IV.11). We have

$$\forall\, y \in \mathbb{R},\ g_{a,x}(y) = a^{3/2} T_x(g_a)(y)$$

and the map $(a,x) \to g_{a,x}$ is continuous from $]0,+\infty[\times\mathbb{R}$ into $L^2([0,+\infty[, 4\mathrm{sh}^2 x dx)$.

V. Continuous wavelet transform on G

DEFINITION V.1. Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$. We define the continuous wavelet transform on G for f in $\mathcal{D}(K\backslash G/K)$ by

$$\Phi_g(f)(a,y) = \int_G f(x)\overline{g}_{a,y}(x)\,dx\,, \quad \text{for all} \quad y \in G\,.$$

This relation can also be written in the form

$$\Phi_g(f)(a,y) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} f * \overline{g}_a(y^{-1})$$

where * is the convolution given by the definition III.2.

Proposition V.1

i) Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$). Then for all $a \in]0,+\infty[$ and for all f in $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$), the map $y \to \Phi_g(f)(a,y)$ belongs to $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$).
ii) Let g be a wavelet on G in $(L^p \cap L^2)(K\backslash G/K)$, $p \in [1,2[$, such that for all $a \in]0,+\infty[$, the function g_a belongs to $L^p(K\backslash G/K)$, $p \in [1,2[$. Then for all f in $L^q(K\backslash G/K)$, $q \in [1,+\infty]$, the map $y \to \Phi_g(f)(a,y)$ belongs to $L^r(K\backslash G/K)$ with $r \in [1,+\infty]$ satisfying $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$.

Proof. We deduce the result from theorems IV.2, III.1. i), III.3. □

Proposition V.2

Let g be a wavelet on G in $L^2(K\backslash G/K)$ and let f be a function in $L^2(K\backslash G/K)$. Then
i) We have

$$\forall\,(a,y) \in]0,+\infty[\times G,\ \big|\Phi_g(f)(a,y)\big| \le \|f\|_2\|g\|_2$$

ii) The map $(a,y) \to \Phi_g(f)(a,y)$ is continuous on $]0,+\infty[\times G$.

iii) For all $a \in]0, +\infty[$ *we have*

$$\lim_{|y|\to+\infty} \Phi_g(f)(a, y) = 0\,.$$

Proof. i) From the definition V.1 we have

$$\forall\,(a, y) \in]0, +\infty[\times G,\ \left|\Phi_g(f)(a, y)\right| \leq \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \|f\|_2 \|g_a\|_2$$

but from the relation (IV.3) we have

$$\|g_a\|_2 \leq \Big(\frac{k(a)}{a^\ell}\Big)^{1/2} \|g\|_2$$

thus

$$\forall\,(a, y) \in]0, +\infty[\times G\,,\ \left|\Phi_g(f)(a, y)\right| \leq \|f\|_2 \|g\|_2$$

ii) We deduce the result from the definition V.1 and the proposition IV.6.

iii) α) Let g^0 be in $\mathcal{C}(K\backslash G/K)$ and let f^0 be a function of $\mathcal{D}(K\backslash G/K)$ which has its support in the ball $B = \{x \in G/|x| \leq R\}$.

We have

$$\Phi_{g^0}(f^0)(a, y) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \int_B f^0(x)\overline{g^0_a}(y^{-1}x)dx$$

then

$$\begin{aligned}\left|\Phi_{g^0}(f^0)(a, y)\right| &\leq \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \|f^0\|_2 \Big(\int_B \left|g^0_a(y^{-1}x)\right|^2 dx\Big)^{1/2} \\ &\leq \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \|f^0\|_2 \Big(\int_{y^{-1}B} \left|g^0_a(t)\right|^2 dt\Big)^{1/2}\end{aligned}$$

but from the theorem IV.2 ii) the function g^0_a, $a \in]0, +\infty[$, belongs to $\mathcal{C}(K\backslash G/K)$, then from the dominated convergence theorem

$$\lim_{|y|\to+\infty} \int_{y^{-1}B} \left|g^0_a(t)\right|^2 dt = 0$$

thus for all $a \in]0, +\infty[$

$$\lim_{|y|\to+\infty} \Phi_{g^0}(f^0)(a, y) = 0\,.$$

β) We suppose that g and f belong to $L^2(K\backslash G/K)$. From the density of the spaces $\mathcal{D}(K\backslash G/K)$ and $\mathcal{C}(K\backslash G/K)$ in $L^2(K\backslash G/K)$, for all $\varepsilon > 0$ there are a function g^0 in $\mathcal{D}(K\backslash G/K)$ and a function f^0 in $\mathcal{C}(K\backslash G/K)$ such that

$$\|g - g^0\|_2 < \varepsilon \quad \text{and} \quad \|f - f^0\|_2 < \varepsilon\,.$$

We have

$$\big|\Phi_g(f)(a,y)\big| \le \big|\Phi_{g^0}(f^0)(a,y)\big| + \big|\Phi_g(f)(a,y) - \Phi_{g^0}(f^0)(a,y)\big|$$

using the i) we obtain

$$\big|\Phi_g(f)(a,y) - \Phi_{g^0}(f^0)(a,y)\big| \le \|g\|_2\|f - f^0\|_2 + \|f\|_2\|g - g^0\|_2 + \|f - f^0\|_2\|g - g^0\|_2$$

hence we deduce the result. □

Proposition V.3

Let g be a wavelet on G in $L^2(K\backslash G/K)$. Then
i) For all $a \in]0,+\infty[$ and for all f in $L^p(K\backslash G/K)$, $p \in [1,2[$, the map $y \to \Phi_g(f)(a,y)$ belongs to $L^2(K\backslash G/K)$.
ii) For all $a \in]0,+\infty[$ and for all f in $L^2(K\backslash G/K)$ the map $y \to \Phi_g(f)(a,y)$ belongs to $L^q(K\backslash G/K)$, $q \in]2,+\infty[$.

Proof. We obtain these results from theorems IV.1, III.4, III.5. □

Proposition V.4

Let $p \in [1,2[$ and let g be a wavelet on G in $(L^p \cap L^2)(K\backslash G/K)$, such that for all $a \in]0,+\infty[$, the function g_a is in $L^p(K\backslash G/K)$, then for all f in $L^2(K\backslash G/K)$ the map $y \to \Phi_g(f)(a,y)$ belongs to $L^2(K\backslash G/K)$.

Proof. The theorems IV.1, III.4 give the result. □

The following theorems are Plancherel and Parseval formulas for the continuous wavelet transform on G.

Theorem V.1

Let $p \in [1,2[$ and let g be a wavelet on G in $(L^p \cap L^2)(K\backslash G/K)$, such that for all $a \in]0,+\infty[$, the function g_a is in $L^p(K\backslash G/K)$.
i) Plancherel formula for Φ_g

For all f in $L^2(K\backslash G/K)$ we have

$$\|f\|_2^2 = \frac{1}{C_g}\int_0^\infty \int_G \left|\Phi_g(f)(a,y)\right|^2 \frac{k(a)}{a^{\ell+1}} dady$$

ii) Parseval formula for Φ_g

For all f_1, f_2 in $L^2(K\backslash G/K)$ we have

$$\int_G f_1(x)\overline{f_2(x)}dx = \frac{1}{C_g}\int_0^\infty \int_G \Phi_g(f_1)(a,y)\overline{\Phi_g(f_2)(a,y)}\frac{k(a)}{a^{\ell+1}} dady$$

Proof. i) From the definition V.1 and Fubini-Tonelli's theorem we have

$$\begin{aligned}\frac{1}{C_g}\int_0^\infty \int_G \left|\Phi_g(f)(a,y)\right|^2 \frac{k(a)}{a^{\ell+1}} dady &= \frac{1}{C_g}\int_0^\infty \Big(\int_G |f*\overline{g}_a(y^{-1})|^2 dy\Big)\frac{da}{a} \\ &= \frac{1}{C_g}\int_0^\infty \Big(\int_G |f*\overline{g}_a(y)|^2 dy\Big)\frac{da}{a}\end{aligned}$$

we deduce from theorems II.3, III.4

$$\begin{aligned}&\frac{1}{C_g}\int_0^\infty \int_G \left|\Phi_g(f)(a,y)\right|^2 \frac{k(a)}{a^{\ell+1}} dady \\ &= \frac{1}{C_g}\int_0^\infty \left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\mathcal{F}(f)(\lambda)|^2 |\mathcal{F}(\overline{g}_a)(\lambda)|^2 |c(\lambda)|^{-2} d\lambda\right)\frac{da}{a}\end{aligned}$$

then from Fubini-Tonelli's theorem we have

$$\begin{aligned}&\frac{1}{C_g}\int_0^\infty \int_G \left|\Phi_g(f)(a,y)\right|^2 \frac{k(a)}{a^{\ell+1}} dady \\ &= \left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\mathcal{F}(f)(\lambda)|^2 |c(\lambda)|^{-2} d\lambda\right)\left(\frac{1}{C_g}\int_0^\infty |\mathcal{F}(g_a)(\lambda)|^2 \frac{da}{a}\right)\end{aligned}$$

the result follows from theorems II.3, IV.1 and definition IV.1.

ii) We deduce the result from the i). □

The same proof as for theorem V.1 gives the following results.

Theorem V.2

Let $p \in [1, 2[$ and let g be a wavelet on G in $L^2(K\backslash G/K)$.

i) Plancherel formula for Φ_g

For all f in $(L^p \cap L^2)(K\backslash G/K)$, we have

$$\|f\|_2^2 = \frac{1}{C_g}\int_0^\infty \int_G |\Phi_g(f)(a,y)|^2 \frac{k(a)}{a^{\ell+1}} dady$$

ii) Parseval formula for Φ_g

For all f_1, f_2 in $(L^p \cap L^2)(K\backslash G/K)$, we have

$$\int_G f_1(x)\overline{f_2(x)}dx = \frac{1}{C_g}\int_0^\infty \int_G \Phi_g(f_1)(a,y)\overline{\Phi_g(f_2)(a,y)}\frac{k(a)}{a^{\ell+1}} dady$$

Corollary V.1

Let $p \in [1,2[$ and let g be a wavelet on G in $(L^p \cap L^2)(K\backslash G/K)$, such that for all $a \in]0,+\infty[$, the function g_a is in $L^p(K\backslash G/K)$ and positive, then for all f in $L^2(K\backslash G/K)$ we have the inversion formula for the transform Φ_g:

$$f(\cdot) = \frac{1}{C_g}\int_0^\infty \int_G \Phi_g(f)(a,y) g_{a,y}(\cdot)\frac{k(a)}{a^{\ell+1}} dady$$

weakly in $L^2(K\backslash G/K)$.

Proof. From the theorem V.1. ii) and the definition V.1 we have for all h in $L^2(K\backslash G/K)$:

$$\int_G f(x)\overline{h}(x)dx = \frac{1}{C_g}\int_0^\infty \int_G \Phi_g(f)(a,y)\left(\int_G \overline{h}(x) g_{a,y}(x)dx\right)\frac{k(a)}{a^{\ell+1}} dady$$

but from Fubini-Tonelli's theorem and the theorem V.2. ii) we have

$$\int_0^\infty \int_G \int_G |\Phi_g(f)(a,y)||\overline{h}(x)| g_{a,y}(x)\frac{k(a)}{a^{\ell+1}} dadydx$$
$$\leq \int_0^\infty \int_G \Phi_g(|f|)(a,y)\Phi_g(|\overline{h}|)(a,y)\frac{k(a)}{a^{\ell+1}} dady < +\infty\,.$$

Then from Fubini's theorem we deduce

$$\int_G f(x)\overline{h}(x)dx = \int_G \left(\frac{1}{C_g}\int_0^\infty \int_G \Phi_g(f)(a,y) g_{a,y}(x)\frac{k(a)}{a^{\ell+1}} dady\right)\overline{h}(x)dx$$

and the result follows. □

By theorem V.1 the continuous wavelet transform Φ_g on G is an isometry of the Hilbert space $L^2(K\backslash G/K)$ into the Hilbert space $L^2(]0,+\infty[\times G, \frac{k(a)}{a^{\ell+1}C_g}dady)$ (the space of functions on $]0,+\infty[\times G$, bi-invariant under K with respect to the second variable, and square integrable on $]0,+\infty[\times G$ with respect to the measure $\frac{k(a)}{a^{\ell+1}C_g}dady$). For the characterization of the image of Φ_g we interpret the vectors $g_{a,x}$, $(a,x)\in]0,+\infty[\times G$, as a set of coherent states in the Hilbert space $L^2(K\backslash G/K)$. (See [14] section I.2; [16] p. 37–38).

Definition V.2. A set of coherent states in a Hilbert space $\mathcal{H}$ is a subset $\{g_\ell\}_{\ell\in\mathcal{L}}$ of $\mathcal{H}$ such that
i) $\mathcal{L}$ is a locally compact topological space and the mapping $\ell \to g_\ell : \mathcal{L} \to \mathcal{H}$ is continuous.
ii) There is a positive Borel measure $d\ell$ on $\mathcal{L}$ such that, for f in $\mathcal{H}$,

$$\|f\|^2 = \int_{\mathcal{L}} |(f,g_\ell)|^2 d\ell\,.$$

Theorem V.3

Let $\{g_\ell\}_{\ell\in\mathcal{L}}$ be a set of coherent states in a Hilbert space $\mathcal{H}$. Define the isometry Φ of $\mathcal{H}$ into $L^2(\mathcal{L},d\ell)$ by

$$\Phi(f)(\ell) = (f,g_\ell)\,,\quad f\in\mathcal{H}\,.$$

Let F be in $L^2(\mathcal{L},d\ell)$. Then F belongs to $\Phi(\mathcal{H})$ if and only if

$$F(\ell) = \int_{\mathcal{L}} F(\ell')\big(g_{\ell'},g_\ell\big)d\ell'\,.$$

Let now $\mathcal{H} = L^2(K\backslash G/K)$, $\mathcal{L} =]0,+\infty[\times G$. Choose a wavelet g on G satisfying the assumptions of the theorem V.1, and let $g_\ell = g_{a,x}$ be given by the definition IV.2 if $\ell = (a,x)\in\mathcal{L}$. Then we have a set of coherent states. Indeed, the i) of the definition V.2 is satisfied because of proposition IV.6 and the ii) of definition V.2 is satisfied for the measure $\frac{k(a)}{a^{\ell+1}C_g}dady$ on $]0,+\infty[\times G$(See theorem V.1. i)).

Theorem V.4

Let Φ_g be the continuous wavelet transform on G, with g a wavelet on G satisfying the assumptions of the theorem V.1. Let F be in $L^2(]0,+\infty[\times G, \frac{k(a)}{a^{\ell+1}C_g}dady)$.

Then there exists a function f in $L^2(K\backslash G/K)$ such that

$$F = \Phi_g(f)$$

if and only if

$$F(a,x)=\frac{1}{C_g}\int_0^\infty\int_G F(a',x')\left(\int_G g_{a',x'}(y)\overline{g}_{a,x}(y)dy\right)\frac{k(a')}{(a')^{\ell+1}}da'dx'$$

Proof. Apply theorem V.3 with $\mathcal{H}=L^2(K\backslash G/K)$, $\mathcal{L}=]0,+\infty[\times G$, coherent states $g_{a,x}$ and measure $d\ell$ given by $\frac{k(a)}{a^{\ell+1}C_g}dadx$. □

I give now an other inversion formula for the continuous wavelet transform Φ_g on G.

Theorem V.5

Let g be a wavelet on G in $L^2(K\backslash G/K)$. If f is a function defined on G satisfying
– f is continuous and bounded
– f belongs to $L^1(K\backslash G/K)$
– $\mathcal{F}(f)$ belongs to $L^1\big(\mathfrak{a}^*,\frac{|c(\lambda)|^2}{|W|}d\lambda\big)^W$
then we have

$$f(x)=\frac{1}{C_g}\int_0^\infty\left(\int_G\Phi_g(f)(a,y)g_{a,y}(x)dy\right)\frac{k(a)}{a^{\ell+1}}da. \tag{V.1}$$

Where, for each $x\in G$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

Proof. Fix $x\in G$. From the definition IV.1 and the inversion formula for the spherical Fourier transform (Corollary II.1), together with Fubini's theorem we get

$$f(x)=\frac{1}{C_g}\int_0^\infty\left(\frac{1}{|W|}\int_{\mathfrak{a}^*}\big|\mathcal{F}(g_a)(\lambda)\big|^2\mathcal{F}(f)(\lambda)\varphi_\lambda(x)\big|c(\lambda)\big|^{-2}d\lambda\right)\frac{da}{a}$$

where the inner integral on the right hand side is absolutely convergent for all $a\in]0,+\infty[$, since from the theorem IV.1, $\mathcal{F}(g_a)$ is in $L^2\big(\mathfrak{a}^*,\frac{|c(\lambda)|^2}{|W|}d\lambda\big)^W$, $\mathcal{F}(f)$ is bounded, and

$$\forall\, x\in G\,,\quad \forall\,\lambda\in\mathfrak{a}^*,\ |\varphi_\lambda(x)|\le\varphi_0(x)\,.$$

What remains to be proved is that, for all $a\in]0,+\infty[$, and $x\in G$,

$$\frac{1}{|W|}\int_{\mathfrak{a}^*}|\mathcal{F}(g_a)(\lambda)|^2\mathcal{F}(f)(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda=\frac{k(a)}{a^\ell}\int_G\Phi_g(f)(a,y)g_{a,y}(x)dy \tag{V.2}$$

with absolutely convergent integral on the right hand side. This absolute convergence follows because from the proposition V.3. ii) the function $y\to\Phi_g(f)(a,y)$

is in $L^2(K\backslash G/K)$, while from the definition IV.2 the function $y \to g_{a,y}$ is also in $L^2(K\backslash G/K)$.

We can write (V.2) as

$$\text{(V.3)} \qquad \frac{1}{|W|}\int_{\mathfrak{a}^*} \mathcal{F}(g_a)(\lambda)\overline{\mathcal{F}(g_a)}(\lambda)\mathcal{F}(f)(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda = g_a * (f * \overline{g}_a)(x)\,.$$

Now this relation follows because both of its sides are equal to

$$\frac{1}{|W|}\int_{\mathfrak{a}^*} \mathcal{F}(g_a)(\lambda)\mathcal{F}(f * \overline{g}_a)(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda$$

for the left hand side of (V.3) this is clear from the fact that $\overline{g}_a$ belongs to $L^2(K\backslash G/K)$, while this follows for the right hand side of (V.3) by an application of the Parseval formula for the spherical Fourier transform. □

The same proof as for the theorem V.5 gives the following result.

Theorem V.6

Let g be a wavelet on G in $(L^1 \cap L^2)(K\backslash G/K)$ such that for all $a \in]0, +\infty[$, the function g_a belongs to $L^1(K\backslash G/K)$. If f is a function in $\mathcal{C}(K\backslash G/K)$, then we have the inversion formula (V.1).

Remark. i) We can also obtain the inversion formula (V.1) if we take in the theorem V.5 the function f in $\mathcal{D}(K\backslash G/K)$.

ii) The inversion formula (V.1) is also true for g and f in $\mathcal{C}(K\backslash G/K)$.

VI. Inversion of the Abel transform and its dual using wavelets on G

1. Classical wavelets on $\mathbb{R}^\ell$

We define for regular functions on $\mathbb{R}^\ell$, the classical continuous wavelet transform S on $\mathbb{R}^\ell$ by

$$S_{g^0}(f)(a,y) = \int_{\mathbb{R}^\ell} f(x)\overline{g}^0_{a,y}(x)dx\,, \quad \text{for all} \quad y \in \mathbb{R}^\ell$$

where $g^0_{a,y}$, $a \in]0, +\infty[$, $y \in \mathbb{R}^\ell$, is given by

$$g^0_{a,y}(x) = a^{\ell/2}\tau_x(g^0_a)(-y)$$

with τ_x is the translation operator defined by

$$\tau_x(h)(y) = h(x+y)$$

and

$$g_a^0(x) = \frac{1}{a^\ell} g^0\Big(\frac{x}{a}\Big)$$

the function g^0 is a classical wavelet on $\mathbb{R}^\ell$, i.e. a function on $\mathbb{R}^\ell$, satisfying the condition: there exists a constant C_{g^0} such that

– $0 < C_{g^0} < +\infty$.

– For λ almost everywhere on $\mathbb{R}^\ell$ we have

$$C_{g^0} = \int_0^\infty \big|\mathcal{F}_0(g^0)(a\lambda)\big|^2 \frac{da}{a} .$$

The results obtained for the transform S are proved in [16]. In particular we have the following inversion formula:

For g^0 square integrable on $\mathbb{R}^\ell$ with respect to the Lebesgue measure dx we have

$$\text{(VI.1)} \qquad f(x) = \frac{1}{C_{g^0}} \int_0^\infty \left(\int_{\mathbb{R}^\ell} S_{g^0}(f)(a,y) g^0_{a,x}(y)dy \right) \frac{da}{a^{\ell+1}} , \ a.e.$$

when f and $\mathcal{F}_0(f)$ are integrable on $\mathbb{R}^\ell$ with respect to the Lebesgue measure dx.

We remark that in this formula, for each $x \in \mathbb{R}^\ell$, both the inner integral and the outer integral are absolutely convergent, but possible not the double integral.

Proposition VI.1

Let g be a wavelet on G in $\mathcal{D}(K\backslash G/K)$(resp. $\mathcal{C}(K\backslash G/K)$) and g_a, $a \in]0,+\infty[$, the function given by the theorem IV.2. Then

i) We have the relation

$$\forall\, H \in \mathfrak{a},\ \mathcal{A}(g_a)(H) = \frac{1}{a^\ell} \mathcal{A}(g)\Big(\frac{H}{a}\Big) .$$

ii) The function $\mathcal{A}(g)$ is a classical wavelet on $\mathbb{R}^\ell$.

Proof. We deduce these results from the theorem IV.3. □

2. The dual Abel transform

Notations. We denote by

– $\mathcal{E}(\mathfrak{a})$ the space of C^∞-functions on $\mathfrak{a}$. We topologize this space with the classical topology.

– $\mathcal{S}^0(\mathfrak{a}^*)^W$ the subspace of $\mathcal{S}(\mathfrak{a}^*)$ consisting of functions f such that

$$\forall\, \alpha \in \mathbb{N}^\ell\,,\ \frac{\partial^{|\alpha|}}{\partial\lambda_1^{\alpha_1}\ldots\partial\lambda_\ell^{\alpha_\ell}} f(\lambda)/_{\lambda=0} = 0\,.$$

– $\mathcal{S}_0(\mathfrak{a})^W$ (resp. $\mathcal{C}_0(K\backslash G/K)$) the subspace of $\mathcal{S}(\mathfrak{a})^W$ (resp. $\mathcal{C}(K\backslash G/K)$) consisting of functions f such that the function $\mathcal{F}_0(f)$ (resp. $\mathcal{F}(f)$) belongs to $\mathcal{S}^0(\mathfrak{a}^*)^W$.

Definition VI.1. The dual Abel transform $\mathcal{A}^*$ is defined on $\mathcal{E}(\mathfrak{a})$ by

$$\forall\, x \in G,\ \mathcal{A}^*(f)(x) = \int_K f(H(xk))e^{-\rho(H(xk))}dk\,.$$

Proposition VI.2

i) The operator $\mathcal{A}^*$ is linear and continuous from $\mathcal{E}(\mathfrak{a})$ into $\mathcal{C}^\infty(K\backslash G/K)$.
ii) We have

$$\forall\, x \in G,\ \forall\, \lambda \in \mathfrak{a}^*_{\mathbb{C}},\ \varphi_\lambda(x) = \mathcal{A}^*\big(e^{i\lambda(\cdot)}\big)(x)\,.$$

Proposition VI.3

i) The Fourier transform $\mathcal{F}_0$ is a topological isomorphism from $\mathcal{S}_0(\mathfrak{a})^W$ onto $\mathcal{S}^0(\mathfrak{a}^*)^W$.
ii) The spherical Fourier transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{C}_0(K\backslash G/K)$ onto $\mathcal{S}^0(\mathfrak{a}^*)^W$.
iii) The Abel transform $\mathcal{A}$ is a topological isomorphism from $\mathcal{C}_0(K\backslash G/K)$ onto $\mathcal{S}_0(\mathfrak{a}^*)^W$.

Proof. We deduce the results from the properties of the Fourier transform $\mathcal{F}_0$ and the theorems II.2, II.5. □

Proposition VI.4

The operator $\mathcal{K}_0$(resp. $\mathcal{K}_1$) defined on $\mathcal{S}_0(\mathfrak{a})^W$(resp. $\mathcal{C}_0(K\backslash G/K)$) by

$$\forall\, H \in \mathfrak{a},\ \mathcal{K}_0(f)(H) = \mathcal{F}_0^{-1}\left[\frac{|c(\lambda)|^{-2}}{|W|}\mathcal{F}_0(f)\right](H)$$
$$\left(\text{resp.}\forall\, x \in G,\, \mathcal{K}_1(f)(x) = \mathcal{F}^{-1}\left[\frac{|c(\lambda)|^{-2}}{|W|}\mathcal{F}(f)\right](x)\right)$$

is a topological isomorphism from $\mathcal{S}_0(\mathfrak{a})^W$(resp. $\mathcal{C}_0(K\backslash G/K)$) onto itself.

Proof. The relations (II.4),...,(II.7) and the proposition VI.3 give the results. □

Theorem VI.1

The operator $\mathcal{A}^*$ is a topological isomorphism from $\mathcal{S}_0(\mathfrak{a})^W$ onto $\mathcal{C}_0(K\backslash G/K)$ and we have the inversion formulas.

– For f in $\mathcal{C}_0(K\backslash G/K)$, $f = \mathcal{A}^*\mathcal{K}_0\mathcal{A}(f)$, $f = \mathcal{K}_1\mathcal{A}^*\mathcal{A}(f)$.
– For f in $\mathcal{S}_0(\mathfrak{a})^W$, $f = \mathcal{A}\mathcal{K}_1\mathcal{A}^*(f)$, $f = \mathcal{K}_0\mathcal{A}\mathcal{A}^*(f)$.

Proof. We obtain the relation

$$f = \mathcal{A}^*\mathcal{K}_0\mathcal{A}(f), \quad f \in \mathcal{C}_0(K\backslash G/K) \tag{VI.2}$$

from the inversion formula (II.10) of the spherical Fourier transform $\mathcal{F}$ and the propositions VI.2. ii), VI.3 and VI.4.

We deduce the relation

$$f = \mathcal{A}\mathcal{K}_1\mathcal{A}^*(f), \quad f \in \mathcal{S}_0(\mathfrak{a})^W$$

from the relations (VI.3) and (II.14).

The other relations and results are clear. □

Theorem VI.2

Let g be a wavelet on G in $\mathcal{C}_0(K\backslash G/K)$.
i) For all f in $\mathcal{C}_0(K\backslash G/K)$ we have for $y \in G$.

$$\Phi_g(f)(a,y) = \big(k(a)\big)^{-1/2}\mathcal{A}^{-1}\big[S_{\mathcal{A}(g)}\big(\mathcal{A}(f)\big)(a,.)\big](y)\,.$$

ii) For all f in $\mathcal{S}_0(\mathfrak{a})^W$ we have for $H \in \mathfrak{a}$

$$S_{\mathcal{A}(g)}(f)(a,H) = \big(k(a)\big)^{1/2}(\mathcal{A}^*)^{-1}\big[\Phi_g\big(\mathcal{A}^*(f)\big)(a,.)\big](H)\,.$$

Proof. i) We have for all $a \in]0,+\infty[$

$$\forall\, y \in G, \quad \Phi_g(f)(a,y) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} f * \overline{g}_a(y^{-1})$$

then

$$\forall\, \lambda \in \mathfrak{a}^*, \ \mathcal{F}\big[\Phi_g(f)(a,.)\big](\lambda) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2}\mathcal{F}(f)(\lambda)\mathcal{F}(\overline{g}_a)(\lambda)\,.$$

From propositions II.1, V.1. i) and theorems II.4, II.5, III.6 we deduce

$$\forall\, H \in \mathfrak{a},\ \mathcal{A}\big[\Phi_g(f)(a,.)\big](H) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \mathcal{A}(f) *_0 \mathcal{A}(\overline{g}_a)(H)$$

but from the relation (IV.8) we have

$$\forall\, H \in \mathfrak{a},\ \mathcal{A}(\overline{g}_a)(H) = \overline{\big(\mathcal{A}(g)\big)_a}(H)$$

then

$$\begin{aligned}\forall\, H \in \mathfrak{a},\ \mathcal{A}\big[\Phi_g(f)(a,.)\big](H) &= \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} \mathcal{A}(f) *_0 \overline{\big(\mathcal{A}(g)\big)_a}(H) \\ &= \big(k(a)\big)^{-1/2}\big[a^{\ell/2}\mathcal{A}(f) *_0 \overline{\big(\mathcal{A}(g)\big)_a}\big](H) \\ &= \big(k(a)\big)^{-1/2}\big[S_{\mathcal{A}(g)}(\mathcal{A}(f))(a,H)\big]\,.\end{aligned}$$

where S is the classical continuous wavelet transform on $\mathbb{R}^\ell$. Thus

$$\forall\, y \in G\,,\ \Phi_g(f)(a,y) = \big(k(a)\big)^{-1/2}\mathcal{A}^{-1}\big[S_{\mathcal{A}(g)}(\mathcal{A}(f))(a,.)\big](y)\,.$$

ii) We deduce the result from the i) and the theorem VI.1. □

Theorem VI.3

Let g be a wavelet on G in $\mathcal{C}_0(K\backslash G/K)$.
i) For all f in $\mathbb{C}_0(K\backslash G/K)$ we have for $y \in G$

$$\Phi_g(f)(a,y) = \big(k(a)\big)^{-1/2}\mathcal{A}^*\big[\tilde{S}_{\mathcal{K}_0(\mathcal{A}(g)_a)}(\mathcal{A}(f))\big](y)$$

ii) For all f in $\mathcal{S}_0(\mathfrak{a})^W$ we have for $H \in \mathfrak{a}$

$$S_{\mathcal{A}(g)}(f)(a,H) = \big(k(a)\big)^{1/2}\mathcal{A}\big[\tilde{\Phi}_{\mathcal{K}_1(g_a)}(\mathcal{A}^*(f))\big](H)$$

where $\tilde{S}_{\mathcal{K}_0(G_a)}$ and $\tilde{\Phi}_{\mathcal{K}_1(G_a)}$ are the operators defined by

$$\begin{aligned}&\forall\, H \in \mathfrak{a},\ \tilde{S}_{\mathcal{K}_0(G_a)}(F)(H) = a^{\ell/2} F *_0 \overline{\mathcal{K}_0(G_a)}(H) \\ &\forall\, y \in G,\ \tilde{\Phi}_{\mathcal{K}_1(G_a)}(F)(y) = \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2} F * \overline{\mathcal{K}_1(G_a)}(y^{-1})\,.\end{aligned}$$

Proof. i) From theorems VI.1, VI.2, i) we have for $y \in G$:

$$\begin{aligned}\Phi_g(f)(a,y) &= \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2}\mathcal{A}^{-1}\big[\mathcal{A}(f) *_0 \overline{\mathcal{A}(g)_a}\,\big](y) \\ &= \Big(\frac{k(a)}{a^\ell}\Big)^{-1/2}\mathcal{A}^*\mathcal{K}_0\big[\mathcal{A}(f) *_0 \overline{\mathcal{A}(g)_a}\,\big](y)\end{aligned}$$

but the definition of the operator $\mathcal{K}_0$ gives

$$\mathcal{K}_0\big[\mathcal{A}(f) *_0 \overline{\mathcal{A}(g)_a}\,\big] = \mathcal{A}(f) *_0 \overline{\mathcal{K}_0(\mathcal{A}(g)_a)}$$

thus

$$\Phi_g(f)(a,y) = \big(k(a)\big)^{-1/2}\mathcal{A}^*\big[\tilde{S}_{\mathcal{K}_0(\mathcal{A}(g)_a)}(\mathcal{A}(f))\big](y)\,.$$

ii) Using theorems VI.1, VI.2. ii) we obtain for $H \in \mathfrak{a}$:

$$\begin{aligned} S_{\mathcal{A}(g)}(f)(a,H) &= a^{\ell/2}(\mathcal{A}^*)^{-1}\big[\mathcal{A}^*(f) * \overline{g}_a\big](H) \\ &= a^{\ell/2}\mathcal{A}\,\mathcal{K}_1\big[\mathcal{A}^*(f) * \overline{g}_a\big](H) \end{aligned}$$

but from the definition of the operators $\mathcal{K}_1$ we obtain

$$\mathcal{K}_1\big[\mathcal{A}^*(f) * \overline{g}_a\big] = \mathcal{A}^*(f) * \overline{\mathcal{K}_1(g_a)}$$

thus

$$S_{\mathcal{A}(g)}(f)(a,H) = \big(k(a)\big)^{1/2}\mathcal{A}\big[\tilde{\Phi}_{\mathcal{K}_1(g_a)}(\mathcal{A}^*(f))\big](H)\,. \qquad \square$$

Remark. The transform $\tilde{S}_{\mathcal{K}_0(G_a)}$ (resp. $\tilde{\Phi}_{\mathcal{K}_1(G_a)}$) is not necessarily a classical continuous wavelet transform on $\mathbb{R}^\ell$ (resp. a continuous wavelet transform on G), because in general we have not

$$\mathcal{K}_0(G_a) = h_0(a)\mathcal{K}_0(G)_a$$

$$\big(\text{resp. } \mathcal{K}_1(G_a) = h_1(a)\mathcal{K}_1(G)_a\big)$$

where h_0 (resp. h_1) is a function on $]0,+\infty[$.

Theorem VI.4

Let g be a wavelet on G in $\mathcal{C}_0(K\backslash G/K)$. Then for all f in $\mathcal{S}_0(\mathfrak{a})^W$ we have the following relation which gives the inverse operator of the operator $\mathcal{A}$: For all $x \in G$

$$\mathcal{A}^{-1}(f)(x) = \frac{1}{C_g}\int_0^\infty \left(\int_G \mathcal{A}^*\big[\tilde{S}_{\mathcal{K}_0(\mathcal{A}(g)_a)}(f)\big](y)g_{a,x}(y)dy\right)\frac{(k(a))^{1/2}}{a^{\ell+1}}\,da$$

Proof. The theorem VI.3 i) and the last remark of the section V give the result. $\square$

Theorem VI.5

Let g be a wavelet on G in $\mathbb{C}_0(K\backslash G/K)$. Then for all f in $\mathcal{C}_0(K\backslash G/K)$ we have the following relation which gives the inverse operator of the operator $\mathcal{A}^*$: For all $H' \in \mathfrak{a}$

$$(\mathcal{A}^*)^{-1}(f)(H') = \frac{1}{C_{\mathcal{A}(g)}}\int_0^\infty \left(\int_{\mathfrak{a}} \mathcal{A}\big[\tilde{\Phi}_{\mathcal{K}_1(g_a)}(f)\big](H)g_{a,H'}(H)dH\right)\frac{(k(a))^{1/2}}{a^{\ell+1}}\,da\,.$$

Proof. We deduce the result from the theorem VI.3 ii) and the relation (VI.1). $\square$

Acknowledgement. The author is thankful to the referee for some valuable comments.

References

1. J.P. Anker, The spherical Fourier transform of rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi and Varadarajan, *J. Funct. Anal.* **96** (1991), 331–349.
2. K. Aomoto, Sur les transformations d'horisphére et les équations intégrales qui s'y rattachent, *J. Fac. Sci. Univ. Tokyo* **14** (1967), 1–23.
3. R.J. Beerends, The Abel transform and shift operators, *Compositio Math.* **66** (1988), 145–197.
4. R.J. Beerends, The Fourier transform of Harish-Chandra's c-function and inversion of the Abel transform, *Math. Ann.* **277** (1987), 1–23.
5. J. Dieudonné, *Éléments d'Analyse Tome VI, Chapitre XXII, Analyse Harmonique,* Cahiers Scientifiques, Gauthier Villars Éditeur, 1975.
6. P. Goupillaud, A. Grossmann and J. Morlet, Cycle octave and related transforms in seismic signal analysis, *Geoexploration* **23** (1984-1985), 85–102.
7. A. Grossmann and J. Morlet, Decomposition of Hardy functions into square integrable wavelets of constant shape, *SIAM. J. Math. Anal.* **15** (4) (1984), 723–736.
8. A Grossmann, J. Morlet and T. Paul, Transforms associated to square integrable group representations I. General results, *J. Math. Phys.* **26** (1985), 2437–2479.
9. A. Grossmann, J. Morlet and T. Paul, Transforms associated to square integrable group representations II. Examples, *Ann. Inst. H. Poincaré,* **45** (1986), 293–309.
10. A. Grossmann, R. Kronland-Martinet and J. Morlet, *Reading and understanding continuous wavelet transforms,* In wavelets, J.M. Combes, A. Grossmann, Ph. Tchamitchian, eds., Springer, 1989, 2–20.
11. R. Gangolli, Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces, *Acta Math.* **121** (1968), 151–192.
12. R. Gangolli and V.S. Varadarajan, *Harmonic Analysis of spherical functions on real reductive groups,* Ergebnisse der Mathematik under ihrer Grenzgebiete, **101**, Springer-Verlag, Berlin, New-York, 1988.
13. S. Helgason, *Groups and Geometric Analysis, Integral Geometry, Invariant Differential operators, and Spherical Functions,* Academic Press, New-York, 1984.
14. J.A. Klauder and B.S. Skagerstam, A coherent state primer, In Coherent states, J.R. Klauder and B.S. Skagerstam, eds., *World Scientific* (1985), 1–115.
15. T.H. Koornwinder, Jacobi functions and Analysis on noncompact semisimple Lie groups, "Special functions: group theoretical aspects and applications", (R.A. Askey, T.H. Koornwinder and W. Schempp. eds.,) Reidel, Dordrecht, 1984.
16. T.H. Koornwinder, The continuous wavelet transform, Series in Approximations and Decompositions.Vol. 1. Wavelets: An elementary treatment of theory and Aplications. Edited by Tom H. Koornwinder; *World Scientific,* 1993, 27–48.
17. R. Kronland-Martinet, J. Morlet and A. Grossmann, Analysis of sound patterns through wavelet transforms, *Internat., J. Pattern recognition Artificial Intelligencel,* **2** (1987), 273–302.
18. P.G. Lemarié, Introduction à la théorie des ondelettes, In "les ondelettes en 1989", P.G. Lemarié, ed. *Lecture Notes in Math.* **1438**, Springer, 1990.
19. C. Meaney, The inverse Abel transform for $SU(p,q)$, *Ark. Math.* **24** (1968), 131–140.

20. R. Mneimne, Equation de la chaleur sur un espace Riemanien symétrique et formule de Plancherel, *Bull. Sci. Math.* **107** (1983), 261–287.
21. Y. Meyer, *1-Ondelettes et opérateurs I-Ondelettes, 2-Ondelettes et opérateurs II-Opérateurs de Calderon-Zygmund*, Actualités Mathématiques-Hermann-Editeurs des Sciences et des Arts, 1990.
22. T. Paul, Functions analytic on the half plane as quantum mechanical States, *J. Math. Phys.* **25** (1984), 3252–3263.
23. F. Rouvière, Sur la transformation d'Abel des groupes de Lie Semi-simples de rang un, *Ann. Scuola Norm. Sup. Pisa* **10** (1983), 263–290.
24. H.G. Stark, Continuous wavelet transform and continuous Multiscale Analysis, *J. Math. Anal. Appl.* **169** (1992), 179–196.