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Representation of functions by logarithmic potential and reducibility of analytic functions of several variables *

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Abstract

The necessary and sufficient condition that a given plurisubharmonic or a subharmonic function admits the representation by the logarithmic potential (up to pluriharmonic or a harmonic term) is obtained in terms of the Radon transform. This representation is applied to the problem of representation of analytic functions by products of primary factors.

It is well known that an interest in subharmonic and plurisubharmonic functions is mostly due to the relation of these classes to analytic functions. In the case of one complex variable the Riesz integral representation of subharmonic functions [7, Chapter III] is of particular importance because of its kernel $\ln|z-w|$. This representation itself testifies to a certain relation between subharmonic and analytic functions since it means that any subharmonic function is an integral with respect to the parameter α defining the family of the form $\ln|f_\alpha|$, where f_α is analytic. In several variables that relation disappears and therefore the search for an analogue of such a representation is of great importance. In this connection the author [16–19] considered the problem of representation of subharmonic and plurisubharmonic functions in domains of the space $\mathbb{C}^n$, $n \geq 2$, by the logarithmic potential

$$\int \ln|t - \langle z, w\rangle|\, d\mu(t,w), \tag{1}$$

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where $\mu(t, w)$ is a positive measure defined on the set of hyperplanes.

It should be noted that some properties of the potential (1) were investigated earlier in [3, 4, 13]. In particular, the properties of the potential (1) related to the notion of capacity were studied. Also the properties of this potential were used to study defects of meromorphic mappings and to obtain the average growth estimates for hyperplane sections of analytic sets. The author has discovered that the problem of representation of plurisubharmonic functions by the potential (1) is closely related to the properties of the classic complex Radon transform (concerning the complex Radon transform we refer to [5]). The necessary and sufficient condition that a given plurisubharmonic or a subharmonic function admits the representation by the potential (1) (up to pluriharmonic or a harmonic term) was obtained in terms of the Radon transform. It was shown that a sufficiently smooth and strictly plurisubharmonic function is representable as the potential (1) in some neighborhood of each point. Also it was ascertained that any real-valued function satisfying certain conditions of smoothness may be represented as a difference of potentials (1). However it is known that an arbitrary plurisubharmonic function is not representable as the potential (1) in the whole domain of its definition. In this paper we investigate the problem of representation of functions by the potential

$$\int \ln |P_\alpha| d\mu(\alpha), \tag{2}$$

where, for every α, P_α is a holomorphic polynomial of degree $\leq m$, where m is a fixed integer and μ is a positive measure. In terms of the generalized Radon transform we give the necessary and sufficient condition that a given plurisubharmonic or a subharmonic function admits the representation by the potential (2) (Theorems 1 and 2). Also we apply this representation to the reducibility problem for analytic functions (Theorem 3 and its corollaries). We show that a holomorphic function $L(z)$ is a product (up to factors without zeros) of holomorphic polynomials of degrees $\leq m$ if and only if the function $\ln |L(z)|$ is representable as the potential (2). In contrast with the case of an arbitrary plurisubharmonic function, Corollary 2 of Theorem 3 shows that the local representation of the logarithm of the modulus of an entire function by the potential (2) is equivalent to the global representation. These results justify the Radon transform as a proper tool to investigate the reducibility problem for holomorphic functions.

It is worth mentioning that under certain conditions imposed on an entire function [2, 14] its zero-set is the union of the complex hyperplanes $\{z : \langle a_k, z\rangle = c_k\}_{k=1}^{\infty}$, where $a_k \in \mathbb{R}^n$, $c_k \in \mathbb{C}$.

Now we introduce most of the notation which will be used. We also give some definitions.

For $z, w \in \mathbb{C}^p$, we write $\langle z, w\rangle = \sum z_j w_j$. We put $B(z,R) = \{w \in \mathbb{C}^p \mid |z-w| < R\}$ for $z \in \mathbb{C}^p$ and $R > 0$, $S^{2p-1} = \{z \in \mathbb{C}^p \mid |z| = 1\}$. Throughout this paper we assume that Ω is a domain in $\mathbb{C}^n$. If Ω_1 is a bounded domain whose closure is a compact subset of Ω, we write $\Omega_1 \subset\subset \Omega$. We denote by $\mathcal{D}(\Omega)$ the space of smooth $\mathbb{C}$-valued functions with compact support in Ω. For $\varphi \in \mathcal{D}(\mathbb{C}^n)$, we write $\Delta\varphi$ for the Laplacian. The symbol $d\omega_{2n}(z)$ is used to denote the volume form on $\mathbb{C}^n$:

$$d\omega_{2n}(z) = \left(\frac{i}{2}\right)^n (dz_1 \wedge d\bar{z}_1) \wedge \ldots \wedge (dz_n \wedge d\bar{z}_n).$$

For a holomorphic function $F(z)$, $Z(F)$ denotes the zero-set of F. The term "measure" will refer to positive Radon measures.

If $\varphi \in \mathcal{D}(\mathbb{C}^n)$, the standard complex Radon transform of φ (denoted by $\hat{\varphi}$) is defined by

$$\hat{\varphi}(s,\xi) = \frac{1}{|\xi|^2} \int\limits_{\langle z,\xi\rangle = s} \varphi(z)\, d\lambda(z),$$

where $(s,\xi) \in \mathbb{C} \times (\mathbb{C}^n \setminus 0)$, and $d\lambda(z)$ is the area element on the hyperplane $\{z : \langle z,\xi\rangle = s\}$.

Throughout this paper we fix positive integers m and $n \geq 2$. Let $\mathcal{P}_m$ denote the vector space of all polynomials in the complex variables $z_1, \ldots, z_n$ of degrees $\leq m$. The dimension of $\mathcal{P}_m$ will be denoted by $N+1$. We fix a basis $\{1, P_1(z), \ldots, P_N(z)\}$ of $\mathcal{P}_m$, where the polynomials $P_j(z)$ are homogeneous and $P_j(z) = z_j$ for $1 \leq j \leq n$. $P(z)$ denotes the vector-valued function $(P_1(z), ..., P_N(z))$. Let X denote the topological product $[0,\infty) \times S^{2N-1}$. For an open set $Y \subset X$, we denote by $C_c(Y)$ the space of continuous $\mathbb{C}$-valued functions with compact support in Y. For a set $A \subset \mathbb{C}^n$, we denote by $\hat{A}$ the set of all $(t,w) \in X$ such that the polynomial $t - \langle P(z), w\rangle$ has zeros in A. If $A \subset B$, then obviously $\hat{A} \subset \hat{B}$.

Lemma 1

Let $G \subset \mathbb{C}^n$ be an open set and let $K \subset \mathbb{C}^n$ be compact. Then the set $\hat{G}$ is open and $\hat{K}$ is a compact subset of X.

Proof. Fix $(t_0, w_0) \in \hat{G}$. By definition $t_0 = \langle P(z_0), w_0\rangle$ for some $z_0 \in G$. Suppose, seeking a contradiction, that (t_0, w_0) is not contained in the interior of $\hat{G}$. Then there exists a sequence $\{t_k, w_k\}_{k=1}^{\infty} \subset X$ such that $(t_k, w_k) \to (t_0, w_0)$ and $(t_k, w_k) \notin \hat{G}$. For some $\xi_0 \in S^{2n-1}$ the function $q(\lambda) = t_0 - \langle P(z_0 + \lambda\xi_0), w_0\rangle$ of the variable

$\lambda \in \mathbb{C}$ is not identically zero. Since $q(0) = 0$, it follows from the Hurwitz theorem that, for $k \geq k(\varepsilon)$, the functions $q_k(\lambda) = t_k - \langle P(z_0 + \lambda\xi_0), w_k \rangle$ have zeros in $\{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\}$. Consequently, the functions $t_k - \langle P(z), w_k \rangle$ have zeros in G for $k \geq k_0$, which contradicts the conditions $(t_k, w_k) \notin \hat{G}$.

Let $K \subset \mathbb{C}^n$ be compact. It is easy to see that $\hat{K} \subset [0, R] \times S^{2N-1}$ for some $R > 0$. Thus, to complete the proof of Lemma 1, it remains to show that $\hat{K}$ is closed. Suppose $\{t_k, w_k\}_{k=1}^{\infty} \subset K$ and $(t_k, w_k) \to (t_0, w_0)$. By definition of $\hat{K}$, we have $t_k = \langle P(z_k), w_k \rangle$ for some $z_k \in K$. We may suppose that $z_k \to z_0 \in K$. Then $t_0 = \langle P(z_0), w_0 \rangle$ and $(t_0, w_0) \in \hat{K}$ by definition of $\hat{K}$. Lemma 1 is proved. $\square$

DEFINITION 1. A subharmonic function $u(z)$ on $\Omega \subset \mathbb{C}^n$ will be called an m-logarithmic potential with a harmonic addition ($m - log + h$-potential) if there exists a measure $\mu \geq 0$ on $\hat{\Omega}$ such that for every $\Omega_1 \subset\subset \Omega$ we have

$$u(z) = \int\limits_{\hat{\Omega}_1} \ln |t - \langle P(z), w \rangle| \, d\mu(t, w) + H(\Omega_1, z), \tag{3}$$

where $H(\Omega_1, z)$ is harmonic on Ω_1. The measure μ will be called the $m - log + h$-measure of u. Let $u(z)$ be a plurisubharmonic function on Ω. We say that $u(z)$ is an m-logarithmic potential ($m - log$-potential) and that measure μ is the $m - log$-measure of u if for every $\Omega_1 \subset\subset \Omega$ the representation (3) holds, where the function $H(\Omega_1, z)$ is pluriharmonic on Ω_1.

Lemma 2

For all $(s, w) \in \mathbb{C} \times S^{2n-1}$, $\varphi \in \mathcal{D}(\mathbb{C}^n)$ the following equalities hold:

$$\int\limits_{\mathbb{C}^n} \ln |s - \langle z, w \rangle| \frac{\partial^2 \varphi(z)}{\partial z_i \partial \overline{z}_j} \, d\omega_{2n}(z) = (\pi/2)\hat{\varphi}(s, w) w_i \overline{w}_j, \tag{4}$$

$$\int\limits_{\mathbb{C}^n} \ln |s - \langle z, w \rangle| \Delta\varphi(z) \, d\omega_{2n}(z) = (2\pi)\hat{\varphi}(s, w), \tag{5}$$

where $\hat{\varphi}(s, w)$ is the (complex) Radon transform of φ.

Proof. It is evident that (5) follows from (4). To prove (4), we denote by I the integral on the left-hand side of (4). The change of variables gives

$$I = \int_{\mathbb{C}} \ln|s-\lambda| \hat{\psi}_{ij}(\lambda, w) d\omega_2(\lambda),$$

where $\hat{\psi}_{ij}(\lambda, w)$ is the Radon transform of $\psi_{ij}(z) = \partial^2\varphi(z)/\partial z_i \partial \overline{z}_j$. We have [5, Chapter II]

$$\hat{\psi}_{ij}(\lambda, w) = w_i \overline{w}_j \frac{\partial^2 \hat{\varphi}(\lambda, w)}{\partial \lambda \partial \overline{\lambda}}. \tag{6}$$

For every $\psi(\lambda) \in \mathcal{D}(\mathbb{C})$ we have [7, Chapter III]

$$\int_{\mathbb{C}} \ln|s-\lambda| \frac{\partial^2 \psi(\lambda)}{\partial \lambda \partial \overline{\lambda}} \, d\omega_2(\lambda) = (\pi/2)\psi(s). \tag{7}$$

Since, for every $w \in S^{2n-1}$, $\hat{\varphi}(s, w)$ belongs to $\mathcal{D}(\mathbb{C})$, the assertion being proved follows from (6) and (7). The lemma is proved. □

It should be noted that the Poincare-Lelong formula [6] contains (5) as a special case.

Equality (5) can be used as a definition of the Radon transform. In a natural way this leads to the following.

Definition 2. Let $\varphi(z) \in \mathcal{D}(\mathbb{C}^n)$. In the notation introduced above, the m-Radon transform of φ (denoted by $\tilde{\varphi}$) is defined by

$$\tilde{\varphi}(t, w) = \frac{1}{2\pi} \int_{\mathbb{C}^n} \ln|t - \langle P(z), w\rangle| \Delta\varphi(z) \, d\omega_{2n}(z), \quad (t, w) \in [0, \infty) \times S^{2N-1}.$$

Let $\tilde{\varphi}(t, w)$ be the m-Radon transform of $\varphi \in \mathcal{D}(\mathbb{C}^n)$. Suppose $(w_1, ..., w_n) \in S^{2n-1}$. For $w = (w_1, ..., w_n, 0, ..., 0) \in S^{2N-1}$, we have by Lemma 2 that $\tilde{\varphi}(t, w) = \hat{\varphi}(t, w_1, ..., w_n)$, where $\hat{\varphi}$ is the standard Radon transform of φ. In particular, the 1-Radon transform coincides with the restriction of the standard Radon transform of φ to $\mathbb{R} \times S^{2n-1}$. Therefore, if $\tilde{\varphi}(t, w) \equiv 0$, then $\hat{\varphi}(t, w) = 0$ on $\mathbb{R} \times S^{2n-1}$, so $\varphi(z) \equiv 0$ in view of well-known properties of the Radon transform.

Lemma 3

For all $i, j \in \{1, ..., n\}$ and $\varphi \in \mathcal{D}(\Omega)$ the function

$$\psi_{ij}(t, w) = \int_{\mathbb{C}^n} \ln|t - \langle P(z), w\rangle| \frac{\partial^2 \varphi(z)}{\partial z_i \partial \overline{z}_j} \, d\omega_{2n}(z) \tag{8}$$

belongs to $C_c(\hat{\Omega})$. For every function $\psi \in C_c(\hat{\Omega})$ there exists a nonnegative function $\varphi \in \mathcal{D}(\Omega)$ such that $|\psi(t, w)| \leq \tilde{\varphi}(t, w)$, where $\tilde{\varphi}(t, w)$ is the m-Radon transform of φ.

Proof. Let $\varphi \in \mathcal{D}(\Omega)$. Let $\Omega_1 \subset\subset \Omega$ be a bounded domain that contains the support of φ. For every $(t, w) \notin \hat{\Omega}_1$ the function $\ln |t - \langle P(z), w\rangle|$ is pluriharmonic on Ω_1 by definition of $\hat{\Omega}_1$. Then for $(t, w) \notin \hat{\Omega}_1$ the integral on the right-hand side of (8) equals zero. Therefore the support of ψ_{ij} is contained in $\bar{\hat{\Omega}}_1$. Suppose $(t_k, w_k) \in X$ and $(t_k, w_k) \to (t_0, w_0) \in X$ as $k \to \infty$. Then it is easily seen that

$$\lim_{k\to\infty} \int\limits_K |\ln |t_k - \langle P(z), w_k\rangle| - \ln |t_0 - \langle P(z), w_0\rangle|| \, d\omega_{2n}(z) = 0$$

for every $K \subset\subset \mathbb{C}^n$. From this it follows that ψ_{ij} is continuous on $X = [0, \infty) \times S^{2N-1}$.

Suppose now that $\psi \in C_c(\hat{\Omega})$. Let $\{\Omega_p\}_{p=1}^{\infty}$ be a sequence of bounded domains such that $\bar{\Omega}_p \subset \Omega_{p+1}$ and $\cup\Omega_p = \Omega$. Then by Lemma 1, the sets $\hat{\Omega}_p$ are open and $\{\hat{\Omega}_p\}_{p=1}^{\infty}$ is a covering of $\hat{\Omega}$. The support of ψ is a compact subset of $\hat{\Omega}$, so there exists a number q such that supp$\psi \subset \overline{\hat{\bar{\Omega}}}_q$. Let $\varphi \in \mathcal{D}(\Omega)$ be a nonnegative function such that $\varphi(z) = 1$ on $\overline{\Omega}_{q+1}$. Let $\tilde{\varphi}(t, w)$ be the m-Radon transform of φ. Since the function $\ln |t - \langle P(z), w\rangle|$ is plurisubharmonic, we have $\tilde{\varphi}(t, w) \geq 0$. We will show, by the method of contradiction, that $\tilde{\varphi}(t, w) > 0$ on $\hat{\bar{\Omega}}_q$. Suppose $\tilde{\varphi}(t_0, w_0) = 0$ for some $(t_0, w_0) \in \hat{\bar{\Omega}}_q$. By the choice of the function φ we have

$$\tilde{\varphi}(t_0, w_0) \geq c_n \mu_{(t_0,w_0)}(\bar{\Omega}_{q+1}),$$

where $c_n > 0$ and $\mu_{(t_0,w_0)}$ is the Riesz measure of the function $\ln |t_0 - \langle P(z), w_0\rangle|$. Since $\tilde{\varphi}(t_0, w_0) = 0$, we have $\mu_{(t_0,w_0)}(\bar{\Omega}_{q+1}) = 0$. This means that the function $\ln |t_0 - \langle P(z), w_0\rangle|$ is harmonic on Ω_{q+1}. On the other hand, since $(t_0, w_0) \in \hat{\bar{\Omega}}_q$, we have $\ln |t_0 - \langle P(z_0), w_0\rangle| = -\infty$ for some $z_0 \in \bar{\Omega}_q$.This contradiction shows that $\tilde{\varphi}(t, w) > 0$ on $\hat{\bar{\Omega}}_q$. Since $\tilde{\varphi}(t, w)$ is continuous and since $\overline{\hat{\bar{\Omega}}}_q$ is a compact set, there exists $c_0 > 0$ such that $\tilde{\varphi}(t, w) > c_0$ on $\hat{\bar{\Omega}}_q$. Then, since supp$\psi \subset \hat{\Omega}_q$, we have $|\psi(t, w)| \leq A\tilde{\varphi}(t, w)$ for some $A > 0$. Lemma 3 is proved. □

Theorem 1

Let $u(z) \not\equiv -\infty$ be a subharmonic function on $\Omega \subset \mathbb{C}^n$. In order that $u(z)$ be an $m - log + h$-potential on Ω, it is necessary and sufficient that

$$\int u(z)\Delta\varphi(z) \, d\omega_{2n}(z) \geq 0, \qquad \text{for each } \varphi \in \mathcal{D}(\Omega) \text{ such that } \tilde{\varphi} \geq 0, \tag{9}$$

where $\tilde{\varphi}$ denotes the m-Radon transform of φ.

Proof. **Necessity.** Let $u(z)$ be an $m - log + h$-potential on Ω. Denote by μ the $m - log + h$-measure of u. Suppose that the m-Radon transform of $\varphi \in \mathcal{D}(\Omega)$ is nonnegative. Since $\varphi \in \mathcal{D}(\Omega_1)$ for some $\Omega_1 \subset\subset \Omega$, it follows from Definition 1 and Fubini's theorem that

$$\begin{aligned}
&\int\limits_{\Omega_1} u(z)\Delta\varphi(z)\, d\omega_{2n}(z) \\
=&\int\limits_{\hat{\Omega}_1} \left(\int\limits_{\Omega_1} \Delta\varphi(z) \ln|t - \langle P(z), w\rangle|\, d\omega_{2n}(z) \right) d\mu(t,w) \\
=&2\pi \int\limits_{\hat{\Omega}_1} \tilde{\varphi}(t,w)\, d\mu(t,w) \geq 0.
\end{aligned}$$

The last equality follows from the definition of the m-Radon transform.

Sufficiency Let $u(z) \not\equiv -\infty$ be a subharmonic function on Ω such that (9) holds. Denote by $\mathrm{Re}C_c(\hat{\Omega})$ the vector space of all real-valued functions $\psi \in C_c(\hat{\Omega})$. Let M be the subspace of $\mathrm{Re}C_c(\hat{\Omega})$ formed by the m-Radon transforms $\tilde{\varphi}$ of functions $\varphi \in \mathcal{D}(\Omega)$. Since every function $\varphi \in \mathcal{D}(\Omega)$ is uniquely determined by its m-Radon transform, it follows from Definition 2 that $\tilde{\varphi}$ belongs to M if and only if it is the m-Radon transform of a real-valued function $\varphi \in \mathcal{D}(\Omega)$. We define a functional F on M by

$$\langle F, \tilde{\varphi}\rangle = \frac{1}{2\pi} \int u(z)\Delta\varphi(z)\, d\omega_{2n}(z),$$

where the function $\varphi \in \mathcal{D}(\Omega)$ is chosen in such a way that its m-Radon transform equals $\tilde{\varphi}$. Since φ is uniquely determined by $\tilde{\varphi}$, the functional F is well defined. By our assumption the functional F is positive on M, i.e., $\langle F, \tilde{\varphi}\rangle \geq 0$ for $\tilde{\varphi} \geq 0$. By Lemma 3 for every function $\psi(t,w) \in C_c(\hat{\Omega})$ there exists a function $\tilde{\varphi} \in M$ such that $|\psi(t,w)| \leq \tilde{\varphi}(t,w)$. Then [12, Chapter XI] F can be extended to a positive functional F_1 on $\mathrm{Re}C_c(\hat{\Omega})$. By the Riesz theorem on positive functionals [7, Chapter III] there exists a positive Radon measure on $\hat{\Omega}$ such that

$$\langle F_1, \psi\rangle = \int\limits_{\hat{\Omega}} \psi(t,w)\, d\mu(t,w)$$

for every $\psi(t,w) \in \mathrm{Re}C_c(\hat{\Omega})$. Let $\Omega_1 \subset\subset \Omega$. We set

$$H(\Omega_1, z) = u(z) - \int\limits_{\hat{\Omega}_1} \ln|t - \langle P(z), w\rangle|\, d\mu(t,w). \tag{10}$$

We will show that the function $H(\Omega_1, z)$ is harmonic on Ω_1. Fix some real-valued function $\varphi \in \mathcal{D}(\Omega_1)$ and denote by $\tilde{\varphi}(t,w)$ the m-Radon transform of φ. It follows from Fubini's theorem and Definition 2 that

$$\int\limits_{\Omega_1} H(\Omega_1, z)\Delta\varphi(z)\, d\omega_{2n}(z)$$

$$= \int\limits_{\Omega_1} u(z)\Delta\varphi(z)\, d\omega_{2n}(z) - 2\pi \int\limits_{\hat{\hat{\Omega}}_1} \tilde{\varphi}(t,w)\, d\mu(t,w). \tag{11}$$

By Lemma 3 we have supp$\tilde{\varphi} \subset \hat{\Omega}_1$. Then by the construction of the measure μ, the difference on the right-hand side of (11) equals zero. Then [9, Chapter XI] there exists a harmonic function $\tilde{H}(\Omega_1, z)$ on Ω_1 such that $\tilde{H}(\Omega_1, z) = H(\Omega_1, z)$ almost everywhere on Ω_1. Therefore, in view of (10), $\tilde{H}(\Omega_1, z) - H(\Omega_1, z)$ is a difference of two subharmonic functions and equals zero almost everywhere on Ω_1. Then [15, Chapter I] $\tilde{H}(\Omega_1, z) - H(\Omega_1, z) = 0$ everywhere on Ω_1. Theorem 1 is proved. □

Theorem 1 is an analogue of the Riesz theorem on the integral representation of subharmonic functions of one complex variable. We can state the Riesz theorem as follows:

Let u be a distribution on a domain $\Omega \subset \mathbb{C}$. In order that u be a logarithmic potential, it is necessary and sufficient that

$$\langle \Delta u, \varphi \rangle \geq 0 \qquad \text{for each } \varphi \in \mathcal{D}(\Omega) \text{ such that } \varphi \geq 0, \tag{12}$$

i.e., every subharmonic function of one complex variable is a logarithmic potential.

If $u(z) \not\equiv -\infty$ is a subharmonic function on a domain $\Omega \subset \mathbb{C}^n$,$n \geq 2$, then (12) is not equivalent to (9), because there is a class K of functions $\varphi \in \mathcal{D}(\Omega)$ such that the m-Radon transform of every $\varphi \in K$ is nonnegative, but nonetheless the inequality $\varphi \geq 0$ is not true. We shall see that, for every m, there exists a plurisubharmonic function that is not an $m - log + h$-potential. (See Theorem 3.)

Remark 1. Let $u(z)$ be an $m - log + h$-potential on $\Omega \subset \mathbb{C}^n$ and let μ be a positive measure on $\hat{\Omega}$. It is easy to see that μ is the $m - log + h$-measure of u if and only if, for every $\varphi \in \mathcal{D}(\Omega)$, the following equality holds:

$$\int\limits_{\Omega} u(z)\Delta\varphi(z) d\omega_{2n}(z) = 2\pi \int\limits_{\hat{\Omega}} \tilde{\varphi}(t,w) d\mu(t,w),$$

where $\tilde{\varphi}(t,w)$ is the m-Radon transform of φ. This measure is not unique, even in the case $m = 1$.

Lemma 4

Let Ω be a domain in $\mathbb{C}^n$ and let $\mu \geq 0$ be a positive measure on $\hat{\Omega}$. Then there exists an $m-log+h$-potential $u(z)$ on Ω such that μ is one of the $m-log+h$-measures of u.

Proof. We define a functional ν on $\mathcal{D}(\Omega)$ by

$$\langle \nu, \varphi \rangle = 2\pi \int\limits_{\hat{\Omega}} \tilde{\varphi}(t,w) d\mu(t,w),$$

where $\tilde{\varphi}(t,w)$ is the m-Radon transform of φ. The functional ν is well defined because by Lemma 3 $\tilde{\varphi}$ belongs to $C_c(\hat{\Omega})$ for every function $\varphi \in \mathcal{D}(\Omega)$. We have $\langle \nu, \varphi \rangle \geq 0$ for $\varphi \geq 0$. By the Riesz theorem on positive functionals there exists a positive measure ν on Ω such that

$$\langle \nu, \varphi \rangle = \int\limits_{\Omega} \varphi(z) d\nu(z)$$

for every $\varphi \in \mathcal{D}(\Omega)$. Then [8, Chapter IV] there exists a subharmonic function $u(z)$ on Ω such that $\Delta u = \nu$. By Theorem 1 $u(z)$ is an $m - log + h$-potential on Ω and μ is the $m - log + h$-measure of u. The lemma is proved. $\square$

For a domain $\Omega \subset \mathbb{C}^n$ let $\mathcal{D}^{n-1,n-1}(\Omega)$ denote the space of smooth and compactly supported differential forms on $\mathcal{D}(\Omega)$ of bidegree $(n-1, n-1)$. Every form $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ may be written in unique way as

$$\varphi = \sum_{k,m=1}^{n} \varphi_{km}(z) \wedge \omega_{km}, \tag{13}$$

where $\varphi_{km} \in \mathcal{D}(\Omega)$, and the forms ω_{km} are defined by the equalities

$$\frac{i}{2} dz_k \wedge d\bar{z}_m \wedge \omega_{km} = \left(\frac{i}{2}\right)^n (dz_1 \wedge d\bar{z}_1) \wedge \ldots \wedge (dz_n \wedge d\bar{z}_n).$$

Let us agree to call the functions φ_{km} on the right-hand side of (13) the coefficients of φ. As usual we denote by dd^c the operator $2i\partial\bar{\partial}$.

Definition 3. Let $\varphi \in \mathcal{D}^{n-1,n-1}(\mathbb{C}^n)$. The m-Radon transform of φ is defined by

$$\begin{aligned} \tilde{\varphi}(t,w) &= \frac{1}{2\pi} \langle \ln |t - \langle P(z), w\rangle|, dd^c \varphi \rangle \\ &= \frac{1}{2\pi} \int\limits_{\mathbb{C}^n} \ln |t - \langle P(z), w\rangle| \sum_{i,j=1}^{n} 4 \frac{\partial^2 \varphi_{ij}(z)}{\partial z_i \partial \bar{z}_j} d\omega_{2n}(z), \end{aligned} \tag{14}$$

where, for $i, j = 1, \ldots, n$, the functions $\varphi_{ij}(z)$ are the coefficients of φ.

By Lemma 3 the m-Radon transform of every $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ belongs to $C_c(\hat{\Omega})$. It follows from the Poincare-Lelong formula that, for every $(t,w) \in [0,\infty) \times S^{2N-1}$, the m-Radon transform $\tilde{\varphi}(t,w)$ of $\varphi \in \mathcal{D}^{n-1,n-1}(\mathbb{C}^n)$ equals $[D_{(t,w)}](\varphi)$, where $[D_{(t,w)}]$ is the current of integration over the set $D_{(t,w)} = \{z : t = \langle P(z), w\rangle\}$.

Lemma 5

Let $\varphi \in \mathcal{D}^{n-1,n-1}(\mathbb{C}^n)$. Then the m-Radon transform $\tilde{\varphi}(t,w)$ of φ is identically zero if and only if $dd^c\varphi = 0$, i.e., the coefficients φ_{ij} of φ satisfy the equality

$$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi_{ij}(z)}{\partial z_i \partial \bar{z}_j} \equiv 0.$$

Moreover, the function $\tilde{\varphi}(t,w)$ is real-valued if and only if

$$Im\left(\sum_{i,j=1}^{n} \frac{\partial^2 \varphi_{ij}(z)}{\partial z_i \partial \bar{z}_j}\right) \equiv 0. \tag{15}$$

Proof. Obviously it is enough to prove the second statement of the lemma. Let $\tilde{\varphi}(t,w)$ be the m-Radon transform of $\varphi \in \mathcal{D}^{n-1,n-1}(\mathbb{C}^n)$. If the coefficients of φ satisfy (15), then, in view of (14), $\mathrm{Im}(\tilde{\varphi}(t,w)) \equiv 0$. Suppose now that $\mathrm{Im}(\tilde{\varphi}(t,w)) \equiv 0$. Denote by $\hat{\varphi}_{ij}$ the standard Radon transform of φ_{ij}. Setting $w = (w_1,\dots,w_n,0,\dots,0)$ for $(w_1,\dots,w_n) \in S^{2n-1}$, we obtain from (4) that

$$\mathrm{Im}\left(\sum_{i,j=1}^{n} w_i \bar{w}_j \hat{\varphi}_{ij}(t,w_1,\dots,w_n)\right) = 0 \tag{16}$$

for every $t \geq 0$. For all $\psi \in \mathcal{D}(\mathbb{C}^n)$ and $\alpha \in \mathbb{C} \setminus 0$ we have

$$\hat{\psi}(\alpha s, \alpha\xi) \equiv |\alpha|^{-2}\hat{\psi}(s,\xi), \qquad (s,\xi) \in \mathbb{C} \times (\mathbb{C}^n \setminus 0).$$

Then we obtain from (16) that

$$\mathrm{Im}\left(\sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \hat{\varphi}_{ij}(s,\xi)\right) \equiv 0, \qquad (s,\xi) \in \mathbb{C} \times (\mathbb{C}^n \setminus 0).$$

From this it follows that

$$\frac{\partial^2}{\partial s \partial \bar{s}}\mathrm{Im}\left(\sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \hat{\varphi}_{ij}(s,\xi)\right) \equiv \mathrm{Im}\left(\sum_{i,j=1}^{n} \xi_i \bar{\xi}_j \frac{\partial^2 \hat{\varphi}_{ij}(s,\xi)}{\partial s \partial \bar{s}}\right) \equiv 0. \tag{17}$$

Since $\xi_i\bar{\xi}_j\frac{\partial^2\hat{\varphi}_{ij}(s,\xi)}{\partial s\partial\bar{s}}$ is the standard Radon transform of $\frac{\partial^2\varphi_{ij}(z)}{\partial z_i\partial\bar{z}_j}$, it follows from (17) that (15) is valid. The lemma is proved. □

Theorem 2

Let $u(z) \not\equiv -\infty$ be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Then $u(z)$ is an $m-$ log-potential on Ω if and only if

$$\langle u, dd^c\varphi\rangle \geq 0 \qquad \text{for each } \varphi \in \mathcal{D}^{n-1,n-1}(\Omega) \text{ such that } \tilde{\varphi} \geq 0, \tag{18}$$

where $\tilde{\varphi}$ denotes the m-Radon transform of φ.

Proof. The proof of Theorem 2 is similar to that of Theorem 1. The "only if" part is an immediate consequence of Fubini's theorem and Definition 3. Let us prove the "if" part. Let $u(z) \not\equiv -\infty$ be a plurisubharmonic function on Ω such that (18) holds. As in the proof of Theorem 1, $\mathrm{Re}C_c(\hat{\Omega})$ denotes the space of real-valued functions $\psi \in C_c(\hat{\Omega})$. Let L be the subspace of $\mathrm{Re}C_c(\hat{\Omega})$ formed by the m-Radon transforms of forms $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$. By Lemma 5 the m-Radon transform of $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ belongs to L if and only if the coefficients of φ satisfy (15). If $\psi \in \mathcal{D}(\Omega)$ is real-valued, then the m-Radon transform of the form

$$\varphi = \sum_{j=1}^{n} \psi(z) \wedge \omega_{jj}$$

equals the m-Radon transform of the function ψ. Therefore L contains the space M formed by the m-Radon transforms of real-valued functions in $\mathcal{D}(\Omega)$. Then by Lemma 3, for every $\psi \in C_c(\hat{\Omega})$, there exists a function $\tilde{\varphi} \in L$ such that $|\psi| \leq \tilde{\varphi}$. We define a functional F on L by

$$\langle F, \tilde{\varphi}\rangle = \frac{1}{2\pi}\langle u, dd^c\varphi\rangle,$$

where the m-Radon transform of $\varphi \in \mathcal{D}^{n-1,n-1}(\Omega)$ equals $\tilde{\varphi}$. If $\tilde{\varphi} = 0$, then by Lemma 5 we have $dd^c\varphi = 0$. Therefore the functional F is well defined. By our assumption F is positive on L. There exists a positive functional F_1 on $\mathrm{Re}C_c(\hat{\Omega})$ that is an extension of F. This functional F_1 is the m-log-measure of u. The proof is complete. □

It is evident that there is a class of m-log+h-potentials which are not plurisubharmonic. However, we have the following fact:

Lemma 6

Let $u(z)$ be an m-log+h-potential on a domain $\Omega \subset \mathbb{C}^n$ and let μ be the m-log+h measure of u. Suppose that $u(z)$ is harmonic on some neighborhood of $z_0 \in \Omega$. Then there exists $r > 0$ such that $\mu(K(z_0, r)) = 0$, where

$$K(z_0, r) = \{(t, w) \in [0, \infty) \times S^{2N-1} || t - \langle P(z_0), w\rangle| \leq r\}.$$

If $u(z)$ is pluriharmonic on some open subset of Ω, then $u(z)$ is an m-log-potential on Ω and μ is the m-log-measure of u.

Proof. Suppose that $u(z)$ is harmonic on $B(z_0, \varepsilon)$, where $\bar{B}(z_0, \varepsilon) \subset \Omega$. We claim that $K(z_0, r) \subset \hat{B}(z_0, \varepsilon)$ for some $r > 0$. Indeed, in the contrary case, there exists a sequence of positive numbers r_p, $r_p \to 0$, such that $(t_p, w_p) \notin \hat{B}(z_0, \varepsilon)$ for some $(t_p, w_p) \in K(z_0, r_p)$. We may assume that $(t_p, w_p) \to (t_0, w_0) \in [0, \infty) \times S^{2N-1}$. Then $t_0 - \langle P(z_0), w_0\rangle = 0$, so we have $(t_0, w_0) \in \hat{B}(z_0, \varepsilon)$. By Lemma 1 $\hat{B}(z_0, \varepsilon)$ is open, which contradicts that $(t_p, w_p) \notin \hat{B}(z_0, \varepsilon)$. Thus we have $K(z_0, r) \subset \hat{B}(z_0, \varepsilon)$ for some $r > 0$. Since $K(z_0, r)$ is compact, there exists a nonnegative function $\psi_0(t, w) \in C_c(\hat{B}(z_0, \varepsilon))$ that equals 1 on $K(z_0, r)$. By Lemma 3 there is a nonnegative function $\varphi_0(z) \in \mathcal{D}(B(z_0, \varepsilon))$ such that $\psi_0(t, w) \leq \tilde{\varphi}_0(t, w)$, where $\tilde{\varphi}_0(t, w)$ is the m-Radon transform of φ_0. Then, in view of Remark 1, we have

$$\begin{aligned} 0 &= \int u(z) \Delta\varphi_0(z) d\omega_{2n}(z) = 2\pi \int \tilde{\varphi}_0(t, w) d\mu(t, w) \\ &\geq 2\pi \int \psi_0(t, w) d\mu(t, w) \geq 2\pi\mu(K(z_0, r)). \end{aligned}$$

Therefore we have $\mu(K(z_0, r)) = 0$.

Suppose now that $u(z)$ is pluriharmonic on $B(z_0, \varepsilon)$. Let $\Omega_1 \subset\subset \Omega$ be a bounded domain such that $\bar{B}(z_0, \varepsilon) \subset \Omega_1$. Then the function

$$H(\Omega_1, z) = \int\limits_{\hat{\bar{\Omega}}_1} \ln|t - \langle P(z), w\rangle| \, d\mu(t, w) - u(z)$$

is harmonic on Ω_1 and plurisubharmonic on $B(z_0, \varepsilon)$. Then $H(\Omega_1, z)$ is pluriharmonic on $B(z_0, \varepsilon)$. For every $i, j \in \{1, \ldots, n\}$ the function $\frac{\partial^2 H(\Omega_1, z)}{\partial z_i \partial \bar{z}_j}$ is real-analytic on Ω_1 and vanishes on $B(z_0, \varepsilon)$. Therefore $\frac{\partial^2 H(\Omega_1, z)}{\partial z_i \partial \bar{z}_j}$ is identically zero. This means that $H(\Omega_1, z)$ is pluriharmonic on Ω_1. Suppose now that $\Omega_1 \subset\subset \Omega$ is arbitrary. Let $\Omega_2 \subset\subset \Omega$ be a domain that contains $\bar{\Omega}_1 \cup \overline{B(z_0, \varepsilon)}$. We set

$$H(\Omega_j, z) = \int\limits_{\hat{\bar{\Omega}}_j} \ln |t - \langle P(z), w\rangle| \, d\mu(t, w) - u(z), \qquad j = 1, 2.$$

By what has been proved, the function $H(\Omega_2, z)$ is pluriharmonic on Ω_2. We have

$$H(\Omega_1, z) = - \int\limits_{\hat{\bar{\Omega}}_2 \setminus \hat{\bar{\Omega}}_1} \ln |t - \langle P(z), w\rangle| \, d\mu(t, w) + H(\Omega_2, z). \tag{19}$$

Since, for every $(t, w) \notin \hat{\bar{\Omega}}_1$, the function $t - \langle P(z), w\rangle$ has no zeros in $\bar{\Omega}_1$, the integral on the right-hand side of (19) is pluriharmonic on Ω_1. Therefore $H(\Omega_1, z)$ is pluriharmonic on Ω_1. The proof is complete. $\square$

From Lemma 6 and Theorem 1 we obtain

Corollary

Let $L(z) \not\equiv 0$ be a holomorphic function on a domain $\Omega \subset \mathbb{C}^n$. In order that $\ln |L(z)|$ be an m-log-potential on Ω, it is necessary and sufficient that

$$\int \ln |L(z)| \Delta\varphi(z) \, d\omega_{2n}(z) \geq 0$$

for every function $\varphi \in \mathcal{D}(\Omega)$ whose m-Radon transform is nonnegative.

Theorem 3

Let $L(z) \not\equiv 0$ be a holomorphic function on $\Omega \subset \mathbb{C}^n$. Then $\ln |L(z)|$ is an m-log-potential on Ω if and only if there exist sequences of positive integers $\{n_k\}_{k=1}^{\infty}$ and irreducible polynomials $\{P_k(z)\}_{k=1}^{\infty}$ of degrees $\leq m$ such that, for every $\Omega_1 \subset\subset \Omega$, the following formula holds:

$$L(z) = g(\Omega_1, z) \prod_{k \in I(\Omega_1)} (P_k(z))^{n_k} \qquad z \in \Omega_1, \tag{20}$$

where $I(\Omega_1)$ is a finite set and $g(\Omega_1, z)$ is holomorphic and nowhere zero on Ω_1.

Remark 2. For every irreducible polynomial $P(z)$, $Z(P)$ is an irreducible analytic subset of $\mathbb{C}^n$. (For the definition of complex analytic sets and their fundamental properties we refer to E.M. Chirka [1].) Then, by the uniqueness theorem for irreducible analytic subsets, $Z(P)$ is nowhere dense in the zero-set of every polynomial $Q(z)$ which is not divisible by P. In particular, for every j, $Z(P)$ is nowhere dense in $Z(\partial P/\partial z_j)$. This means that every function $L(z) \in H(\Omega)$ with $Z(L)\cap\Omega \supset Z(P)\cap\Omega$ is divisible by P.

Remark 3. In the case of an arbitrary domain Ω, the representation (20) and the equality

$$Z(L)\cap\Omega = \bigcup_{k=1}^{\infty} Z(P_k)\cap\Omega, \tag{21}$$

are not equivalent. There exist a domain $\Omega \subset \mathbb{C}^n$ and a function $L(z) \in H(\Omega)$ satisfying (21) such that, for some $\Omega_1 \subset\subset \Omega$, the representation (20) also holds but the function $g(\Omega_1, z)$ on the right-hand side of (20) has zeros in Ω_1 and is not divisible by any polynomial P. If $L(z)$ is an entire function, then, for every domain $\Omega \subset \mathbb{C}^n$, (20) is equivalent to (21) because, by the uniqueness theorem for irreducible analytic subsets, the relation $Z(P_k)\cap\Omega \subset Z(L)\cap\Omega$ implies $Z(P_k) \subset Z(L)$. This means that, if (21) holds, then, for every $\Omega_1 \subset\subset \Omega$, the function $g(\Omega_1, z)$ is entire. Thus if $g(\Omega_1, z)$ has zeros in Ω_1, then by (21) it is divisible by P_k for some k.

An important tool in the proof of Theorem 3 is the Lelong number [11]. We recall it here:

Let $u(z)$ be a plurisubharmonic function on some neighborhood V_a of $a \in \mathbb{C}^n$. We denote by $\lambda(a, r, u)$ the average of u over the sphere of radius r about a. The Lelong number $\nu_u(a)$ is defined by

$$\nu_u(a) = \lim_{r\to 0} \frac{\lambda(a, r, u)}{\ln r}.$$

If $L(z) \not\equiv 0$ is a holomorphic function on V_a, then the Lelong number $\nu_u(a)$ of the function $u(z) = \ln|L(z)|$ equals the order of the zero of L at the point a (if $L(a) \neq 0$, then $\nu_u(a) = 0$).

Proof of Theorem 3. The "if" part follows immediately from the definition of m-log-potentials. We shall prove the "only if " part. Let $L(z) \not\equiv 0$ be a holomorphic function such that $\ln|L(z)|$ is an m-log-potential on Ω. Assume, without loss of generality, that $0 \in \Omega$ and $L(0) \neq 0$.

Let μ be the m-log-measure of $\ln|L(z)|$. We denote by B the set of all $(t, w) \in \hat{\Omega}$ such that $L(z)$ vanishes on $\{z \in \Omega | t = \langle P(z), w\rangle\}$. It is easy to see that B is (relatively) closed in $\hat{\Omega}$. We claim that $\mu(\hat{\Omega} \setminus B) = 0$. Indeed, for every $(t_0, w_0) \in \hat{\Omega} \setminus B$, there exists a point $z_0 \in \Omega$ such that $t_0 = \langle P(z_0), w_0 \rangle$ and $L(z_0) \neq 0$. Then $\ln |L(z)|$ is pluriharmonic on some neighborhood of z_0. Then by Lemma 6 there exists $r > 0$ such that

$$A = \{(t, w) | \; |t - \langle P(z_0), w \rangle| < r\} \subset \hat{\Omega}$$

and $\mu(A) = 0$. The set A is open and contains the point (t_0, w_0). Thus, for every $(t, w) \in \hat{\Omega} \setminus B$, there exists an open set $A \subset \hat{\Omega}$ such that $(t, w) \in A$ and $\mu(A) = 0$. Therefore $\hat{\Omega} \setminus B$ can be covered by a countable union of sets A_k with $\mu(A_k) = 0$. Then $\mu(\hat{\Omega} \setminus B) = 0$. If $B = \emptyset$, then $\mu(\hat{\Omega}) = 0$ and $\ln |L(z)|$ is pluriharmonic on Ω,i.e., $L(z)$ has no zeros in Ω. Suppose now that $B \neq \emptyset$. By what has been proved, we have for every $\Omega_1 \subset\subset \Omega$ that

$$\ln |L(z)| = \int\limits_{\widehat{\Omega_1} \cap B} \ln |t - \langle P(z), w \rangle| \, d\mu(t, w) + H(\Omega_1, z),$$

where $H(\Omega_1, z)$ is pluriharmonic on Ω_1.

Without loss of generality we can assume that $m > 1$. Denote by $\tilde{B}_{m-1}$ the set of all $(t, w) \in B$ such that $\ln |t - \langle P(z), w \rangle|$ is an $(m-1)$-log-potential on $\mathbb{C}^n$. Since B is closed in $\hat{\Omega}$, it follows from Theorem 2 that $\tilde{B}_{m-1}$ is also closed in $\hat{\Omega}$. Let $B_m = B \setminus \tilde{B}_{m-1}$. For every $(t, w) \in B_m$ the polynomial $t - \langle P(z), w \rangle$ is irreducible, for in the contrary case $\ln |t - \langle P(z), w \rangle|$ is an $(m-1)$-log-potential. Since, for every $(t, w) \in B$, $L(z)$ vanishes on $\{z \in \Omega | t = \langle P(z), w \rangle\}$, it follows from Remark 2 that every finite product

$$\prod (t_j - \langle P(z), w_j \rangle) \;, \; (t_j, w_j) \in B_m,$$

divides $L(z)$. Then, for every $\Omega_1 \subset\subset \Omega$, the intersection $B_m \cap \hat{\hat{\Omega}}_1$ is a finite set of points. Therefore the restriction ν_m of the measure μ to B_m equals

$$\nu_m = \sum_{k=1}^{\infty} \alpha_{k,m} \delta(t_{k,m}, w_{k,m}),$$

where $\alpha_{k,m} > 0$ and $\delta(t_{k,m}, w_{k,m})$ denotes the Dirac measure at the point $(t_{k,m}, w_{k,m})$.

By Lemma 4 there exists an m-log+h-potential $v(z)$ on Ω (an m-log-potential with a harmonic addition) such that ν_m is one of the m-log+h-measures of $v(z)$. Then, for every $\Omega_1 \subset\subset \Omega$, we have

$$\ln|L(z)| - v(z) = \int\limits_{\hat{\tilde{\Omega}}_1 \cap \tilde{B}_{m-1}} \ln|t - \langle P(z), w\rangle| \, d\mu(t,w) + h(\Omega_1, z), \tag{22}$$

where $h(\Omega_1, z)$ is harmonic on Ω_1. Since, for every $(t,w) \in \tilde{B}_{m-1}$, the function $\ln|t - \langle P(z), w\rangle|$ is an $(m-1)$-log-potential on $\mathbb{C}^n$, it follows from Theorem 1 and the representation (22) that $\ln|L(z)| - v(z)$ is an $(m-1)$-log+h-potential on Ω.

In the case of k-log-potentials, $1 \le k < m$, we can use the same kernel $\ln|t - \langle P(z), w\rangle|$ as well as in the case of m-log-potentials. In other words, we can assume that, for $1 \le k < m$, every k-log-potential is an m-log-potential whose m-log-measure is concentrated on S_k, where S_k denotes the set of all $(t,w) \in [0,\infty) \times S^{2N-1}$ such that $\langle P(z), w\rangle$ is a polynomial of degree $\le k$. Thus, there is a measure μ_{m-1} on $\hat{\Omega}$ such that, for every $\Omega_1 \subset\subset \Omega$,

$$\ln|L(z)| - v(z) = \int\limits_{\hat{\hat{\Omega}}_1 \cap S_{m-1}} \ln|t - \langle P(z), w\rangle| \, d\mu_{m-1}(t,w) + h(\Omega_1, z),$$

where the function $h(\Omega_1, z)$ is harmonic on Ω_1. Then by the construction of the potential $v(z)$ we have

$$\begin{aligned} \ln|L(z)| = & \int\limits_{\hat{\hat{\Omega}}_1 \cap S_{m-1}} \ln|t - \langle P(z), w\rangle| \, d\mu_{m-1}(t,w) \\ & + \sum_{(t_{k,m}, w_{k,m}) \in \hat{\hat{\Omega}}_1} \alpha_{k,m} \ln|t_{k,m} - \langle P(z), w_{k,m}\rangle| + h_1(\Omega_1, z), \end{aligned} \tag{23}$$

where $h_1(\Omega_1, z)$ is also harmonic on Ω_1. Then by definition, $\mu_{m-1} + \nu_m$ is one of the m-log+h-measures of $\ln|L(z)|$. Since $\ln|L(z)|$ is pluriharmonic on some neighborhood of the origin, it follows from Lemma 6 that $\mu_{m-1} + \nu_m$ is one of the m-log-measures of $\ln|L(z)|$,i.e., for every $\Omega_1 \subset\subset \Omega$, the function $h_1(\Omega_1, z)$ on the right-hand side of (23) is pluriharmonic on Ω_1. By what has been proved, the measure $\mu_{m-1} + \nu_m$ is concentrated on B. Let B_{m-1} be the set of all $(t,w) \in S_{m-1} \cap B$ such that $\ln|t - \langle P(z), w\rangle|$ is not an $(m-2)$-log-potential on $\mathbb{C}^n$. Since, for every $(t,w) \in B_{m-1}$, the polynomial $t - \langle P(z), w\rangle$ is irreducible, it follows, exactly as above, that B_{m-1} is a discrete set such that, for every $\Omega_1 \subset\subset \Omega$, the intersection

$B_{m-1} \cap \hat{\hat{\Omega}}_1$ is a finite set. Therefore the restriction ν_{m-1} of the measure μ_{m-1} to B_{m-1} equals

$$\nu_{m-1} = \sum_{k=1}^{\infty} \alpha_{k,m-1} \delta(t_{k,m-1}, w_{k,m-1}).$$

Then as above, we see that one of the m-log-measures of $\ln |L(z)|$ is of the form $\nu_m + \nu_{m-1} + \mu_{m-2}$, where the measure μ_{m-2} is concentrated on $B \cap S_{m-2}$. Repeating the above procedure for $m-2, m-3, \ldots, 1$, we see that the function $\ln |L(z)|$ has the m-log-measure

$$\nu = \sum_{p=1}^{\infty} \alpha_p \delta(t_p, w_p),$$

where $\alpha_p > 0$, and, for every $\Omega_1 \subset\subset \Omega$, the intersection $\{t_p, w_p\}_{p=1}^{\infty} \cap \hat{\hat{\Omega}}_1$ is a finite set. Moreover, for every $p \in \mathbb{N}$ the polynomial $Q_p(z) = t_p - \langle P(z), w_p \rangle$ is irreducible, and $(t_p, w_p) \in B$. Since $L(0) \neq 0$, we have $t_p \neq 0$ for every $p \in \mathbb{N}$. This means that for $p_1 \neq p_2$ the function $Q_{p_1}(z)/Q_{p_2}(z)$ is not constant.

The next step is to prove that the coefficients α_p are positive integers. For $\Omega_1 \subset\subset \Omega$ we denote by $I(\Omega_1)$ the set of all $p \in \mathbb{N}$ such that $(t_p, w_p) \in \hat{\hat{\Omega}}_1$. As above, $Q_p(z)$ denotes the polynomial $t_p - \langle P(z), w_p \rangle$. Fix $p_0 \in \mathbb{N}$. For some $\Omega_1 \subset\subset \Omega$ we have $(t_{p_0}, w_{p_0}) \in \hat{\Omega}_1$. Since $I(\Omega_1)$ is a finite set, by Remark 2 there is a point $z_0 \in \{z \in \Omega_1 |\ Q_{p_0}(z) = 0\}$ lying outside the union

$$\left(\bigcup_{p \in I(\Omega_1) \setminus \{p_0\}} \{z \in \Omega_1 |\ Q_p(z) = 0\} \right) \bigcup \left\{ z \in \Omega_1 |\ \frac{\partial Q_{p_0}(z)}{\partial z_1} = 0 \right\}.$$

We have

$$\begin{aligned} \ln |L(z)| &= \alpha_{p_0} \ln |t_{p_0} - \langle P(z), w_{p_0} \rangle| \\ &\quad + \sum_{p \in I(\Omega_1) \setminus \{p_0\}} \alpha_p \ln |t_p - \langle P(z), w_p \rangle| + H_1(\Omega_1, z), \end{aligned} \tag{24}$$

where $H_1(\Omega_1, z)$ is pluriharmonic on Ω_1. Let $\nu_p(z)$ denote the Lelong number of $\ln |t_p - \langle P(z), w_p \rangle|$ at the point z. By the choice of the point z_0 we have $\nu_p(z_0) = 0$ for $p \in I(\Omega_1) \setminus \{p_0\}$, and $\nu_{p_0}(z_0) = 1$. Then from (24) we obtain

$$\alpha_{p_0} = \nu_L(z_0),$$

where $\nu_L(z_0)$ is the Lelong number of $\ln|L(z)|$ at the point z_0. Thus, for every $p \in \mathbb{N}$, we have $\alpha_p = n_p$, where $n_p \in \mathbb{N}$. For $\Omega_1 \subset\subset \Omega$ we set

$$g(\Omega_1, z) = \frac{L(z)}{\prod\limits_{(t_p, w_p) \in I(\Omega_1)} (t_p - \langle P(z), w_p \rangle)^{n_p}}.$$

It follows from (24) that $\ln|g(\Omega_1, z)|$ is pluriharmonic on Ω_1. Therefore the function $g(\Omega_1, z)$ is holomorphic and nowhere zero on Ω_1. Theorem 3 is proved. $\square$

Corollary 1

Let $P(z)$ be a holomorphic polynomial whose degree equals *$m, m > 1$. Then $P(z)$ is reducible if and only if $\ln|P(z)|$ is an $(m-1)$-log-potential.*

In [19] it is shown that every smooth strictly plurisubharmonic function is an 1-log-potential on some neighborhood of every point. On the other hand, it is easy to find such a function which is not an 1-log-potential on the whole space $\mathbb{C}^n$. This means that the statement of Theorem 2 is not of local nature. However, if we consider a function of the form $u(z) = \ln|L(z)|$, where $L(z)$ is an entire function, then the situation turns out to be quite different:

Corollary 2

Let Ω be a domain in $\mathbb{C}^n$, and let $L(z) \not\equiv 0$ be an entire function. Then the function $\ln|L(z)|$ is an m-log-potential on Ω if and only if it is an m-log-potential on some neighborhood of every $z \in \Omega$.

It is easy to see that Corollary 2 is a consequence of the following result:

Lemma 7

Let $L(z) \not\equiv 0$ be an entire function. Let D_1, D_2 be domains such that $D_1 \cap D_2 \neq \emptyset$. Suppose that the function $\ln|L(z)|$ is an m-log-potential both on D_1 and on D_2. Then it is an m-log-potential on $D_1 \cup D_2$.

Proof of Lemma 7. By Theorem 3 we have

$$Z(L) \cap D_j = \bigcup_{k \in I_j} \left(Z(P_k) \cap D_j \right), \quad j = 1, 2, \tag{25}$$

where I_1 and I_2 are countable sets and $P_k(z)$ are irreducible polynomials of degrees $\leq m$. As remarked above, the relation $Z(P_k) \cap D_j \subset Z(L) \cap D_j$ implies $Z(P_k) \subset Z(L)$. Therefore we have

$$Z(L) \supset \bigcup_{k \in I_1 \cup I_2} Z(P_k).$$

It then follows from (25) that

$$Z(L) \cap (D_1 \cup D_2) = \bigcup_{k \in I_1 \cup I_2} Z(P_k) \cap (D_1 \cup D_2)\,.$$

Then by Remark 3, $\ln |L(z)|$ is an m-log-potential on $D_1 \cup D_2$. Lemma 7 is proved, and with it also Corollary 2. $\square$

Corollary 2 and its proof remain valid if we assume only that the function $L(z)$ is holomorphic on a domain Ω such that, for every irreducible polynomial P of degree $\leq m$, the intersection $\Omega \cap Z(p)$ is an irreducible analytic subset of Ω (instead of assuming that $L(z)$ is entire). We then have the following result.

Corollary 3

$L(z) \not\equiv 0$ be a holomorphic function on a convex domain Ω in $\mathbb{C}^n$. Then the function $\ln |L(z)|$ is an 1-log-potential on Ω if and only if it is an 1-log-potential on some neighborhood of every $z \in \Omega$.

We now show that Corollary 3 is false if Ω is not convex. Let D_1 and D_2 be simply connected plane domains such that $D_1 \cap D_2 = S_1 \cup S_2$, where S_1 and S_2 are convex open sets with $\bar{S}_1 \cap \bar{S}_2 = \emptyset$. Let $l = \{(z_1, z_2) \in \mathbb{C}^2 |\ z_1 = z_2\}$ and $D = D_1 \times D_2 \subset \mathbb{C}^2$. The intersection $l \cap D$ is an analytic subset of D consisting of two irreducible components $A_j = \{(\lambda, \lambda) \in \mathbb{C}^2 |\ \lambda \in S_j\}$, $j = 1, 2$. Then [10, Theorem 6.1.8] there exists a function $L(z) \in H(D)$ such that $A_1 = Z(L) \cap D$ and L divides every function $g \in H(\Omega)$ which vanishes on A_1. Then $\ln |L(z)|$ is an 1-log-potential on some neighborhood of every $z \in D$ because for every $z \in A_1$ the functions $L(z)/(z_1 - z_2)$ and $(z_1 - z_2)/L(z)$ are holomorphic on some neighborhood of z. Suppose that $\ln |L(z)|$ is an 1-log potential on D. Then from Theorem 3 it follows that $L(z)$ vanishes on A_2 which contradicts the equality $A_1 = Z(L) \cap D$.

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