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Differential equations on the plane with given solutions

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Abstract

The aim of this paper is to construct the analytic vector fields on the plane with given as trajectories or solutions. In particular we construct the polynomial vector field from given conics (ellipses, hyperbola, parabola, straight lines) and determine the differential equations from a finite number of solutions.

1. Introduction

The relation between coefficients and roots of a polynomial

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 \quad , \quad a_n \neq 0\,,$$

is well known. In particular if $\lambda_1, \ldots, \lambda_n$ are zeros of $P(\lambda)$ such that

$$\Delta = \begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \neq 0\ , \tag{1.1}$$

then P admits the representation

$$P(\lambda) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda & \lambda_1 & \dots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \lambda^n & \lambda_1^n & \dots & \lambda_n^n \end{vmatrix} = \det(B).$$

From here we can easily deduce that $a_n, \dots, a_0$ coincide with the adjoints of $\lambda^n, \dots, 1$ in the matrix B, respectively. These relations between coefficients and solutions can be developed by analogy for a wide variety of differential equations. In this section we will analyze some particular cases.

Functions will be assumed smooth in their domain.

I. Let us give the differential equation

$$Z^{(n)} + \alpha_n(t) Z^{(n-1)} + \dots + \alpha_0(t) = 0, \tag{1.2}$$

where

$$\begin{cases} Z^{(j)} = \dfrac{d}{dt}(Z^{(j-1)}) \quad , \quad Z = x + iy \; , \; i = \sqrt{-1} \; , \\ Z^{(1)} \equiv \dot{Z} \, . \end{cases}$$

It can be shown that if $Z_j : I \subset \mathbb{R} \longrightarrow \mathbb{C}$ then

$$\omega = \big(Z_1(t), \dots\dots\dots, Z_{n+1}(t)\big),$$

are solutions of (1.2) such that

$$\Delta(\omega) \equiv \begin{vmatrix} 1 & \dots & 1 \\ Z_1(t) & \dots & Z_{n+1}(t) \\ \dot{Z}_1(t) & \dots & \dot{Z}_{n+1}(t) \\ \vdots & & \vdots \\ Z_1^{(n-1)}(t) & \dots & Z_{n+1}^{(n-1)}(t) \end{vmatrix} \neq 0 \tag{1.3}$$

for all $t \in I^* \subset I$, in which case we have the following representation

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ Z & Z_1(t) & \dots & Z_{n+1}(t) \\ \dot{Z} & \dot{Z}_1(t) & \dots & \dot{Z}_{n+1}(t) \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ Z^{(n)} & Z_1^{(n)}(t) & \dots & Z_{n+1}^{(n)}(t) \end{vmatrix} \equiv \det(A) = 0 \, . \tag{1.4}$$

From here we can easily deduce the coefficient-solutions relation for the given equation

$$\alpha_k(t) = (-1)^{k+1}\frac{\Delta_k(\omega)}{\Delta(\omega)}, \quad k = 0, \ldots, n, \quad \alpha_{n-1}(t) = -\frac{d}{dt}(ln\Delta(\omega)),$$

where $\Delta_k, \quad k = 0, \ldots, n$ are the adjoints of the (k+1)-element of the first column in matrix A.

We observe that $\Delta_{n+1}(\omega) \equiv (-1)^{n+1}\Delta(\omega)$.

II. Likewise we can prove that if the functions

$$Z_j : I \subset \mathbb{R} \longrightarrow \mathbb{C} \quad , \quad \omega = \big(Z_1(t), \ldots, \ Z_{n+1}(t)\big),$$

such that

$$\Delta(\omega) = \begin{vmatrix} 1 & \ldots & 1 \\ Z_1(t) & \ldots & Z_{n+1}(t) \\ Z_1^2(t) & \ldots & Z_{n+1}^2(t) \\ \vdots & & \vdots \\ Z_1^n(t) & \ldots & Z_{n+1}^n(t) \end{vmatrix} \neq 0 \tag{1.5}$$

for all $t \in I^* \subset I$, satisfy the equation

$$\dot{Z} + a_n(t)Z^n + \ldots + a_0(t) = 0, \tag{1.6}$$

their we can give the representation:

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ Z & Z_1(t) & \ldots & Z_{n+1}(t) \\ Z^2 & Z_1^2(t) & & Z_{n+1}^2(t) \\ \vdots & \vdots & & \vdots \\ Z^n & Z_1^n(t) & \ldots & Z_{n+1}^n(t) \\ \dot{Z} & \dot{Z}_1(t) & \ldots & \dot{Z}_{n+1}(t) \end{vmatrix} \equiv \det(A) = 0. \tag{1.7}$$

From here we can easily deduce that the coefficient a_n can be represented as follows

$$a_n = (-1)^{k+1}\frac{\Delta_k(\omega)}{\Delta(\omega)}, \quad k = 0, .., n. \tag{1.8}$$

III. Likewise we can look for the coefficient-solution relations for the equation of the type

$$\dot{Z} + \sum_{k+j=0}^{n} a_{kj}(t) Z^k \overline{Z}^j = 0, \tag{1.9}$$

where $\overline{Z} = x - iy$.

It is easy to prove that if we have $\frac{(n+1)(n+2)}{2} \equiv m$ solutions of (1.9), $Z_j : I \subset \mathbb{R} \longrightarrow \mathbb{C}$, $\omega = \big(Z_1(t), \ldots, Z_m(t)\big)$, which satisfy

$$\begin{vmatrix} 1 & \ldots & 1 \\ Z_1(t) & \ldots & Z_m(t) \\ \overline{Z}_1(t) & \ldots & \overline{Z}_m(t) \\ Z_1^2(t) & \ldots & Z_m^2(t) \\ |Z_1(t)|^2 & \ldots & |Z_m(t)|^2 \\ \overline{Z}_1^2(t) & \ldots & \overline{Z}_m^2(t) \\ \vdots & & \vdots \\ \overline{Z}_1^n(t) & \ldots & \overline{Z}_m^n(t) \end{vmatrix} \equiv \Delta(\omega) \neq 0 \tag{1.10}$$

for all $t \in I^* \subset I$, then (1.9) can be rewritten as follows

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ Z & Z_1(t) & \ldots & Z_m(t) \\ \overline{Z} & \overline{Z}_1(t) & \ldots & \overline{Z}_m(t) \\ Z^2 & Z_1^2(t) & \ldots & Z_m^2(t) \\ |Z|^2 & |Z_1(t)|^2 & \ldots & |Z_m(t)|^2 \\ \overline{Z}^2 & \overline{Z}_1^2(t) & \ldots & \overline{Z}_m^2(t) \\ \vdots & \vdots & & \vdots \\ \overline{Z}^n & \overline{Z}_1^n(t) & \ldots & \overline{Z}_m^n(t) \\ \dot{Z} & \dot{Z}_1(t) & \ldots & \dot{Z}_m(t) \end{vmatrix} \equiv \det(A) = 0. \tag{1.11}$$

DEFINITION 1. The system of functions

$$\omega = \big(Z_1(t), \ldots, Z_m(t)\big)$$

will be called the fundamental solution of (1.4) or (1.7) or (1.11) if $\Delta(\omega) \neq 0$ for all $t \in I^* \subset I$, and $\Delta(\omega)$ is determined by the formulas (1.3), (1.5), (1.10) respectively.

Definition 2. Let $\Delta_k, \quad k = 1, \ldots, m+1$ be the adjoints of the elements of the first column in matrix A defined by the formulas (1.4), (1.7) or (1.11). Then the system of functions $\Omega = \big(w_1(t), \ldots, w_n(t)\big)$ such that

$$\begin{cases} \Delta(\Omega) \neq 0 \\ \dfrac{\Delta_j(\omega)}{\Delta(\omega)} = \dfrac{\Delta_j(\Omega)}{\Delta(\Omega)}, \quad j = 1, .., m+1 \end{cases} \tag{1.12}$$

will be called fundamental solution equivalent to ω and we shall write

$$\omega \approx \Omega,$$

where $\Delta_{m+1} \equiv (-1)^m \Delta_m(\omega)$. Of course if

$$\omega \approx \Omega \quad \text{and} \quad \Omega \approx \tilde{\Omega} \Longrightarrow \omega \approx \tilde{\Omega}.$$

As we can observe from definitions 1 and 2 we suppose that for all fundamental solutions there is at least one solution such that

$$\dot{Z}_j(t) \neq 0, \qquad \forall t \in I^{**} \subset I. \tag{1.13}$$

The aim of this article is to study the inverse problem of differential equations: the construction of vector fields on the plane from given properties, such as a finite number of solutions or trajectories [4].

2. Construction of an analytical vector field on the plane from given trajectories

It is well-known that for the analytic vector field on the plane:

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y), \end{cases} \tag{2.1}$$

the G domain of $P(x, y)$ and $Q(x, y)$ is divided into elementary regions [2], such that their boundary is determined from a finite number of singular trajectories. In these regions the non-singular trajectories which are topological equivalent are located.

For the structurally stable dynamic system the singular trajectories can be

i) Stable simple critical points (saddles, nodes, foci).
ii) Stable limit cycles.
iii) α-ω-separatries which may only spread towards a node, a focus, a limit cycle or leave the G domain.

From these results it seems interesting to state the inverse problem: the construction of (2.1) from a finite number of singular trajectories or solutions.

Problem 1 (construction of the analytic vector field from given trajectories).

Let

$$y_j : I \subset \mathbb{R} \longrightarrow \mathbb{R}, \quad j = 1, \ldots, n+1$$
$$x \longrightarrow y_j(x)$$

be the analytic functions on (I) such that

$$\omega = \big(y_1(x), \ldots, y_{n+1}(x)\big)$$

is a system of functions for which the next relation holds

$$\Delta(\omega) = \begin{vmatrix} 1 & \ldots & 1 \\ y_1(x) & \ldots & y_{n+1}(x) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ y_1^n(x) & \ldots & y_{n+1}^n(x) \end{vmatrix} \neq 0, \quad \forall x \in I^* \subset I. \tag{2.2}$$

We require the dynamic system to be constructed from given trajectories.

Firstly we introduce the following.

Definition 3. The analytic function Φ defined on G which vanishes on the curves $y_j(x)$, $j = 1, 2, ..., n+1$, will be called Eruguin's function.

Proposition 2.1

The most general first order differential equation which has the curves $y = y_j$, $j = 1, \ldots, n+1$ as solutions can be represented as follows:

$$y' = -\sum_{k=0}^{n} \frac{\Delta_k(\omega) y^k(x)}{\Delta_{n+1}(\omega)} + \Phi(x, y), \tag{2.3}$$

where Φ is the Eruguin function.

In fact, from (1.6)-(1.8) and considering the properties of Eruguin function we deduced that (2.3) has the given curves as a solution.

Now we will prove that the equation constructed is the most general.

In fact, let us suppose that

$$y' = F(x, y) \tag{2.4}$$

is the other equation for which the given curves are solutions, i.e,

$$F(x, y)|_{y=y_j} = y_j'.$$

By choosing Φ as follows

$$\Phi(x, y) = \sum_{k=0}^{n} \frac{\Delta_k(\omega) y^k(x)}{\Delta_{n+1}(\omega)} + F(x, y)$$

we deduce that this Φ is the Eruguin function.

Equations (2.3) and (2.4) will be called Φ equivalents.

Proposition 2.2

Let Φ_1, Φ_2, ν be arbitrary analytic functions on $G \subset \mathbb{R}^2$ such that

$$\begin{cases} \Phi_l(x, y_j(x)) \equiv 0, \quad l = 1, 2; \quad j = 1, \ldots, n+1 \\ \nu(x, y_j(x)) \neq 0. \end{cases} \tag{2.5}$$

The most general analytic dynamic system with $y_1(x), \ldots, y_{n+1}(x)$, trajectories which satisfy (2.5) is the following

$$\begin{cases} \dot{x} = \nu(x, y)\Delta_{n+1}(\omega) + \Phi_1(x, y) \\ \dot{y} = -\nu(x, y) \sum_{k=0}^{n} \Delta_k(\omega) y^k + \Phi_2(x, y), \end{cases} \tag{2.6}$$

where $\Delta_0(\omega), \ldots, \Delta_n(\omega), \Delta_{n+1}(\omega)$ are the adjoints of the elements in the first column of matrix A:

$$A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ y & y_1(x) & \ldots & y_{n+1}(x) \\ y^2 & y_1^2(x) & \ldots & y_{n+1}^2(x) \\ \vdots & \vdots & & \vdots \\ y^n & y_1^n(x) & \ldots & y_{n+1}^n(x) \\ y' & y_1'(x) & \ldots & y_{n+1}'(x) \end{bmatrix}. \tag{2.7}$$

Proof. Proposition 2.2 follows from proposition 2.1 if the Eruguin function $\Phi(x,y)$ is related to the functions Φ_j, $j=1,2$ by the formula

$$\Phi = \frac{\left(\sum_{k=0}^{n} \frac{\Delta_k(\omega) y^k(x)}{\Delta_{n+1}(\omega)}\Phi_1 + \Phi_2\right)}{\nu(x,y)\Delta_{n+1} + \Phi_1}. \quad \square$$

It is clear that functions Φ_1, Φ_2 can be represented as follows

$$\Phi_l(x,y) = g_l(x,y)\ \det S, \quad l = 1,2 \tag{2.8}$$

$$S = \begin{bmatrix} 1 & 1 & \dots & 1 \\ y & y_1(x) & \dots & y_{n+1}(x) \\ y^2 & y_1^2(x) & \dots & y_{n+1}^2(x) \\ \vdots & \vdots & & \vdots \\ y^n & y_1^n(x) & \dots & y_{n+1}^n(x) \\ y^{n+1} & y_1^{n+1}(x) & \dots & y_{n+1}^{n+1}(x) \end{bmatrix}$$

where g_l, $\quad l = 1,2$ are arbitrary analytical functions. By putting (2.8) in (2.6) it is easy to obtain the following expression for system (2.6):

$$\begin{cases} \dot{x} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ y & y_1(x) & \dots & y_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ y^n & y_1^n(x) & \dots & y_{n+1}^n(x) \\ \nu + g_1 y^{n+1} & g_1 y_1^{n+1}(x) & \dots & g_1 y_{n+1}^{n+1}(x) \end{vmatrix} \equiv E_1 \\ \\ \dot{y} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ y & y_1(x) & \dots & y_{n+1}(x) \\ \vdots & \vdots & & \vdots \\ y^n & y_1^n(x) & \dots & y_{n+1}^n(x) \\ g_2 y^{n+1} & h_1 & & h_{n+1} \end{vmatrix} \equiv E_2 \end{cases} \tag{2.9}$$

where $h_j \equiv -\nu y_j'(x) + g_2 y_j^{n+1}$, $j = 1, 2, .., n+1$.

The natural arbitrariness with which we constructed the dynamic system can be restricted if we impose some complementary conditions, that (2.9) is structurally stable, non-structurally stable, Hamiltonian, polynomial of degree n, etc.

We shall study the following particular cases:

I. Construction of the quadratic stationary vector field from given conics.

II. Construction of the polynomial vector field of degree n from given $n+1$ straight lines.

3. Construction of the quadratic vector field from given conics

We shall study (2.9) for the case $n = 1$, $m = 2$ and under the restriction

$$\begin{cases} E_1 = a_{20}x^2 + a_{11}xy + a_{02}y^2 + \alpha_1 x + \alpha_2 y + \alpha_3 \equiv P(x,y) \\ E_2 = b_{20}x^2 + b_{11}xy + b_{02}y^2 + \beta_1 x + \beta_2 y + \beta_3 \equiv Q(x,y), \end{cases} \tag{3.1}$$

As a consequence the following identity holds

$$\begin{cases} \dfrac{\partial^3}{\partial y^3}\big(\nu(x,y)\big)(y_2 - y_1) + \dfrac{\partial^3}{\partial y^3}\Phi_1(x,y) = 0 \\ -\dfrac{\partial^3}{\partial y^3}(\nu\lambda) + \dfrac{\partial^3}{\partial y^3}\Phi_2(x,y) = 0, \end{cases} \tag{3.2}$$

where $\lambda = y_1(x)y_2'(x) - y_2(x)y_1'(x) + \big(y_1'(x) - y_2'(x)\big)y$, $\quad y_j'(x) \equiv \dfrac{d}{dx}y_j$.

These relations take place in particular in the next subcase

i)

$$\begin{cases} \nu(x,y) = \nu(x) \\ g_1(x,y)\big(y_2(x) - y_1(x)\big) = \alpha = \text{const}, \\ g_2(x,y)\big(y_2(x) - y_1(x)\big) = \beta = \text{const}, \end{cases} \tag{3.3}$$

ii)

$$\begin{cases} \nu(x,y) = q(x)(Ax + Cy + B), \\ g_1(x,y)\big(y_2(x) - y_1(x)\big) = \alpha = \text{const}, \\ g_2(x,y)\big(y_2(x) - y_1(x)\big) = \beta = \text{const}. \end{cases} \tag{3.4}$$

For i) by f we shall denote the function

$$f(x) = y_1(x) - y_2(x).$$

From (2.9) and (3.1) we deduce the following equalities (we suppose that $\beta \neq 0$)

$$\begin{cases} y_1 = -\dfrac{a_{11}x + \alpha_2}{2\beta} + \dfrac{f}{2} \\ y_2 = -\dfrac{a_{11}x + \alpha_2}{2\beta} - \dfrac{f}{2}, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \nu(x)f(x) + \dfrac{\beta}{4}f^2(x) = -\Big(a_{20} - \dfrac{a_{11}^2}{4\beta}\Big)x^2 - \Big(\alpha_1 - \dfrac{a_{11}\alpha_2}{2\beta}\Big)x - \alpha_3 + \dfrac{\alpha_2^2}{4\beta} \equiv r_2(x) \\ \dfrac{a_{11}\nu(x)}{\beta}f(x) - \dfrac{\alpha f^2(x)}{2} = -\Big(\dfrac{a_{11}}{\beta}b_{11} - \dfrac{\alpha a_{11}^2}{2\beta^2} - 2b_{20}\Big)x^2 - \Big(\dfrac{\alpha_2 b_{11}}{\beta} + \dfrac{\beta_2 a_{11}}{\beta} - \\ -\dfrac{\alpha a_{11}\alpha_2}{\beta^2} + 2\beta_1\Big)x - \dfrac{\alpha_2\beta_2}{\beta} + \dfrac{\alpha\alpha_2^2}{2\beta^2} + 2\beta_3 \equiv s_2(x) \\ \nu(x)f'(x) = \Big(\dfrac{\alpha a_{11}}{\beta} - b_{11}\Big)x + \dfrac{\alpha\alpha_2}{\beta} - \beta_2 \equiv \ell_1(x). \end{cases} \tag{3.6}$$

After some calculations, and under the conditions $a_{11} + 2\alpha \neq 0$, we deduce the following expression for f:

$$f = \pm 2\sqrt{\frac{a_{11}r_2(x) - \beta s_2(x)}{\beta(a_{11} + 2\alpha)}}$$

By introducing the respective notations we obtain

$$\begin{cases} f(x) = \pm 2\sqrt{px^2 + 2qx + r}\,, \quad px^2 + 2qx + r \geq 0 \\ \nu(x) = C\,\sqrt{px^2 + 2qx + r}\,, \quad p, q, r, C \in \mathbb{R}, \quad C \neq 0 \\ p(q^2 - pr) \neq 0 \end{cases} \tag{3.7}$$

Evidently, if

$$a_{11} + 2\alpha_1 = 0, \tag{3.8}$$

then

$$-2\alpha r_2(x) = s_2(x)\,\beta.$$

The functions $f(x)$ and $\nu(x)$ have the same form as in the previous case. From (3.5) and (3.7) we see that $y_1(x), y_2(x), \nu, g_1, g_2$ are such that

$$\begin{cases} y_1(x) = \dfrac{ax + b}{2} + \sqrt{px^2 + 2qx + r} \\ y_2(x) = \dfrac{ax + b}{2} - \sqrt{px^2 + 2qx + r}, \quad p\,(q^2 - pr) \neq 0 \end{cases} \tag{3.9}$$

$$\begin{cases} \nu(x) = C\sqrt{px^2+2qx+r}\ , \quad C \neq 0 \\ -2g_1(x)\sqrt{px^2+2qx+r} = \alpha = \ \text{const}, \\ -2g_2(x)\sqrt{px^2+2qx+r} = \beta = \ \text{const}, \end{cases} \tag{3.10}$$

so the dynamic system which has the conics (ellipses, hyperbola) (3.9) as trajectories is the following

$$\begin{cases} \dot{x} = -2C(px^2+2qx+r) + \beta\Big(\big(y - \dfrac{ax+b}{2}\big)^2 - (px^2+2qx+r)\Big) \equiv P(x,y) \\ \dot{y} = -2C\Big((px^2+2qx+r)\dfrac{a}{2} + (px+q)\big(y - \dfrac{ax+b}{2}\big)\Big) \\ \quad +\alpha\Big(\big(y - \dfrac{ax+b}{2}\big)^2 - (px^2+2qx+r)\Big) \equiv Q(x,y). \end{cases} \tag{3.11}$$

For the proof, we put (3.9) and (3.10) into (2.9) (with $n=1$).

It is interesting to observe that condition (3.8) for system (3.11) takes the form

$$2\alpha = \beta a.$$

Under this restriction on the α, β parameters and introducing the notations

$$\begin{cases} \lambda^2 = 4p\beta C^2\ell_1^2\ , \qquad \ell_1 = y_0 - \dfrac{ax_0+b}{2}, \\ \ell_2 = px_0^2 + 2qx_0 + r, \end{cases}$$

where (x_0, y_0) is such that

$$P(x_0,y_0) = Q(x_0,y_0) = 0,$$

i.e,

$$\begin{cases} x_0 = -\dfrac{q}{p}, \\ y_0 = \dfrac{ax_0+b}{2} \pm \sqrt{\dfrac{(\beta+2C)(pr-q^2)}{p\beta}}, \end{cases}$$

and making the change

$$\begin{cases} y = y_0 + \dfrac{\lambda\eta}{2\ell_1} + \alpha\xi \\ x = x_0 + \beta\xi \\ t^* = -\lambda t, \end{cases}$$

we obtain the following expression for (3.9)

$$\begin{cases} \eta_1 = \dfrac{2\ell_1}{\lambda}(-\ell_1 + \sqrt{p\beta^2\xi^2 + \ell_2}) \\ \eta_2 = \dfrac{2\ell_1}{\lambda}(-\ell_1 - \sqrt{p\beta^2\xi^2 + \ell_2}). \end{cases}$$

Hence we deduce that (3.11) takes the form

$$\begin{cases} \dfrac{d\eta}{dt^*} = \xi + \dfrac{2Cp\beta}{\lambda}\xi\eta \\ \dfrac{d\xi}{dt^*} = -\eta - \dfrac{Cp\beta}{\lambda}\eta^2 + \dfrac{p\beta(2C+\beta)}{\lambda}\xi^2. \end{cases} \tag{3.12}$$

From the Bautin theorem [4] it is easy to prove that (x_0, y_0), under given conditions are center points in (3.12).

We can study case ii) in a similar fashion.

After some calculations it is easy to prove that for $n = 1$,

$$\begin{cases} y_1 = \dfrac{ax+b}{2} + \sqrt{2qx+r} \\ y_2 = \dfrac{ax+b}{2} - \sqrt{2qx+r} \end{cases} \tag{3.13}$$

and

$$\begin{cases} \nu(x,y) = (Ax + Cy + B)\sqrt{2qx+r} \\ -2g_1(x,y)\sqrt{2qx+r} = \alpha = \text{const}, \\ -2g_2(x,y)\sqrt{2qx+r} = \beta = \text{const}, \end{cases}$$

the equations (2.9) take the form

$$\begin{cases} \dot{x} = -2(Ax + Cy + B)(2qx + r) + \beta\Big(\big(y - \dfrac{ax+b}{2}\big)^2 - (2qx+r)\Big) \\ \dot{y} = -(Ax + Cy + B)\Big((2qx+r)a + 2q\big(y - \dfrac{ax+b}{2}\big)\Big) \\ \quad +\alpha\Big(\big(y - \dfrac{ax+b}{2}\big)^2 - (2qx+r)\Big). \end{cases} \tag{3.14}$$

It is clear that (3.14) is the quadratic dinamical system which has the parabola $(p = 0)$ (3.13) as trajectories. Now we shall study the case when $q^2 - pr = 0$, i.e., when the given trajectories are straight lines:

$$\begin{cases} y = k_1x + b_1 \\ y = k_2x + b_2 \end{cases}, \quad k_1 \neq k_2 \tag{3.15}$$

After some calculations it is easy to deduce that the stationary dynamic system which has the given straight lines (3.15) as trajectories is the following

$$\begin{cases} \dot{\omega}_1 = \omega_1\big(\nu + \big(\dfrac{g_2 - g_1 k_1}{k_2 - k_1}\omega_2\big)\big) \\ \dot{\omega}_2 = \omega_2\big(\nu + \big(\dfrac{g_2 - g_1 k_2}{k_2 - k_1}\big)\omega_1\big), \end{cases}$$

where $\omega_j \equiv y - k_j x - b_j, \quad j = 1, 2$. By choosing ν, g_1, g_2 from the equalities

$$\begin{cases} \nu + \dfrac{g_2 - g_1\, k_1}{k_2 - k_1}\,\omega_2 = a_1\omega_1 + a_2\omega_2 + a_3 \\ \nu + \dfrac{g_2 - g_1\, k_2}{k_2 - k_1}\,\omega_1 = b_1\omega_1 + b_2\omega_2 + b_3 \end{cases}$$

we obtain the quadratic system which has (3.15) as trajectories.

4. Construction of the polynomial vector field of degree n from given $n+1$ straight lines

The aim of this chapter is to state and analyze the problem of constructing the polynomial vector field from given straight lines. We shall analyze the conjecture about the maximum number of invariant straight lines which admit the polynomial vector field: "The stationary vector field of degree n has $2n+2$ invariant straight lines if n is odd and has $2n+1$ if n is even". We give a geometrical formulation of this conjecture (in a particular case) based on proposition 6.1 (which we will prove in the next section) and the above method for the construction of the vector field. We study the conjecture for the case when there are n invariant parallel straight lines. The proposed method is illustrated for the quadratic, cubic, and quartic vector field.

Problem 2 Let

$$\omega = (\alpha_1 x + \beta_1, \dots, \alpha_{n+1} x + \beta_{n+1}),\ \ \alpha_j, \beta_j \in \mathbb{R}, \tag{4.1}$$

be a system of functions which satisfies (2.2). We would like to determine the conditions under which system (2.9) is polynomial of degree n on the variables x and y.

We propose the solution to this problem only for the case in which

$$g_l(x, y) \equiv 0, \quad l = 1, 2\, \dots, n+1\,,$$

which is the case when there are n invariant parallel straight lines. Under these conditions system (2.9) takes the form:

$$\begin{cases} \dot{x} = (-1)^{n+1}\nu(x,y)\prod_{1\le \ell<k\le n+1}\big((\alpha_k-\alpha_\ell)x+\beta_k-\beta_\ell\big) \equiv (-1)^{n+1}\nu(x,y)P(x) \\ \dot{y} = \nu(x,y)\sum_{j=1}^{n+1}(-1)^j\alpha_j\prod_{\substack{s=1\\ s\neq j}}^{n+1}(y-\alpha_s x-\beta_s)\prod_{\substack{1\le\ell<k\le n+1\\ \ell,k\neq j}}\big((\alpha_k-\alpha_\ell)x+\beta_k-\beta_\ell\big) \\ \equiv \nu(x,y)Q(x,y). \end{cases} \tag{4.2}$$

It is easy to prove that

$$\begin{cases} \max(\deg \mathrm{P(x)}) = \dfrac{n(n+1)}{2} \\ \max(\deg \mathrm{Q(x,y)}) = \dfrac{n(n+1)}{2}. \end{cases} \tag{4.3}$$

Let $L(n)$ be the maximum of invariant straight lines which admit a polynomial vector field of degree n.

We shall prove the following [4].

Proposition 4.1

$$L(n) \ge \begin{cases} 2n+1 & \text{if } n \text{ is even} \\ 2n+2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We shall prove this statement by constructing a polynomial vector of degree n with $2n+1$ invariant straight lines if n is even and $2n+2$ if n is odd [4]. If the given straight lines are such that

$$\alpha_1 \neq \alpha_2 = \alpha_3 = \ldots = \alpha_{n+1} = 0, \tag{4.4}$$

then by choosing

$$\nu(x,y) = \frac{-1}{\prod_{2\le\ell<k\le n+1}(\beta_k-\beta_\ell)},$$

it is easy to deduce that system (4.2) takes the form:

$$\begin{cases} \dot{x} = (\alpha_1 x+\beta_1-\beta_2)\ldots(\alpha_1 x+\beta_1-\beta_{n+1}) \\ \dot{y} = \alpha_1(y-\beta_2)\ldots(y-\beta_{n+1}). \end{cases} \tag{4.5}$$

Evidently these differential equations have the following $2n+1$ trajectories

$$\begin{aligned} l_1 : &\quad y = \alpha_1 x + \beta_1, \quad \alpha_1 \neq 0 \\ l_j : &\quad y = \beta_j, \quad j = 2, n+1 \\ l_k : &\quad x = \frac{\beta_j - \beta_1}{\alpha_1}, \quad k = n + j. \end{aligned}$$

If n is odd, the horizontal straight lines are equidistant and

$$\beta_j < \beta_{j+1}, \quad j = 2, \ldots, n.$$

Hence

$$\begin{aligned} l_{2n+2} : &\quad y = -\alpha_1 x + (n-1)\beta + 2\beta_2 - \beta_1, \\ &\quad \beta \equiv \beta_{j+1} - \beta_j, \quad j = 2, \ldots, n\,, \end{aligned}$$

is a trajectory of (4.5). So we show that for (4.5) it is possible to have $2n+1$ and $2n+2$ invariant straight lines if n is even and odd respectively. □

Is well known the following

Conjecture 1.

$$L(n) = \begin{cases} 2n+1 & \text{if } n \text{ is even} \\ 2n+2 & \text{if } n \text{ is odd.} \end{cases} \tag{4.6}$$

Based on Proposition 6.1 of section 6 this assertion can be substituted in particular case by the following geometrical conjecture.

Conjecture 2. Let $\infty > n > 1$ be an integer. The maxim number of straight lines which can be traced on the plane in such a way that

i) in every line there are at most n intersection points with the others lines,

ii) there is a direction in which exist n parallel straight lines,

is given by the formula (4.6) .

It should be pointed out that if we assume that there is a singular straight line, it is easy to construct examples of the polynomial vector field which have an infinite number straight lines.

To study this conjecture, the geometrical constructions given in [5] can be useful.

Now we will deal with Problem 2.

Firstly we will analyze the quadratic vector field. For $n = 2$ the system (4.2) takes the form

$$\begin{cases} \dot{x} = \nu(x,y)(e_1 x^3 + e_2 x^2 + e_3 x + e_4) \\ \dot{y} = \nu(x,y)\big(a_1 y^2 + (e_1 x^2 + a_2 x + a_3) y + \lambda_1 x^2 + \lambda_2 x + \lambda_3\big) \end{cases} \tag{4.7}$$

where e_j, a_j, λ_j are constants such that

$$
\begin{aligned}
a_1 &= \beta_1(\alpha_2-\alpha_3)+\beta_2(\alpha_3-\alpha_1)+\beta_3(\alpha_1-\alpha_2)\\
a_2 &= 2\big(\alpha_1\beta_1(\alpha_3-\alpha_2)+\alpha_2\beta_2(\alpha_1-\alpha_3)+\alpha_3\beta_3(\alpha_2-\alpha_1)\big)\\
a_3 &= \beta_1^2(\alpha_3-\alpha_2)+\beta_2^2(\alpha_1-\alpha_3)+\beta_3^2(\alpha_2-\alpha_1)\\
\lambda_1 &= \big(\beta_3\alpha_1\alpha_2(\alpha_1-\alpha_2)+\beta_1\alpha_2\alpha_3(\alpha_2-\alpha_3)+\beta_2\alpha_3\alpha_1(\alpha_3-\alpha_1)\big)\\
\lambda_2 &= 2\big(\beta_3\alpha_1\alpha_2(\beta_1-\beta_2)+\beta_1\alpha_2\alpha_3(\beta_2-\beta_3)+\beta_2\alpha_3\alpha_1(\beta_3-\beta_1)\big)\\
\lambda_3 &= \alpha_3\beta_1\beta_2(\beta_1-\beta_2)+\alpha_2\beta_3\beta_1(\beta_3-\beta_1)+\alpha_1\beta_2\beta_3(\beta_2-\beta_3)\\
e_1 &= (\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)\\
e_2 &= (\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\beta_3-\beta_1)+(\alpha_2-\alpha_3)(\alpha_1-\alpha_2)(\beta_3-\beta_1)\\
&\quad+(\alpha_3-\alpha_1)(\alpha_1-\alpha_2)(\beta_2-\beta_3)\\
e_3 &= (\alpha_1-\alpha_2)(\beta_2-\beta_3)(\beta_3-\beta_1)+(\alpha_2-\alpha_3)(\beta_1-\beta_2)(\beta_3-\beta_1)\\
&\quad+(\alpha_3-\alpha_1)(\beta_1-\beta_2)(\beta_2-\beta_3)\\
e_4 &= (\beta_1-\beta_2)(\beta_2-\beta_3)(\beta_3-\beta_1).
\end{aligned}
$$

Of course (4.7) is quadratic if and only if (we take $\nu = 1$)

$$e_1 = 0\,,$$

i.e., there must be at least 2 parallel straight lines. From Proposition 6.1 we show that in these lines there can be 2 critical points at most.

Let us suppose that the next condition holds

$$\alpha_1 \neq \alpha_2 = \alpha_3 \tag{4.8}$$

After some calculations it is easy to obtain that

$$\dot{x} = -\beta_{32}\big(\alpha_{31}x+\beta_{31}\big)\big(\alpha_{21}x+\beta_{21}\big)\,,$$

where $\beta_{ij} \equiv \beta_i - \beta_j,\ \alpha_{ij} \equiv \alpha_i - \alpha_j$. As a consequence, we obtain the straight lines

$$l_j: \quad y_j = \alpha_j x + \beta_j, \quad j = 1,2,3$$

which satisfy condition (4.8). System (4.7) has complementary trajectories

$$l_4: \quad x = -\frac{\beta_{21}}{\alpha_{21}}, \qquad l_5: \quad x = -\frac{\beta_{31}}{\alpha_{31}}.$$

From (4.6) we obtain that $L(2) = 5$. In such a way the stated problem is solved for $n = 2$.

Corollary 4.1.

The general integral of (4.7) is

$$y = \frac{y_1(x)\big(y_2(x)-y_1(x)\big)+C\big(y_1(x)-y_3(x)\big)y_2(x)}{y_2(x)-y_3(x)+C\big(y_1(x)-y_3(x)\big)}\,,$$

where C is an arbitrary constant.

Now we shall analyze the case $n \geq 3$.

It is clear that (4.2) is polynomial of degree n if there is a polynomial $\wedge(x)$ such that

$$\begin{cases} \deg\ \wedge(x) \leq \dfrac{n(n-1)}{2} \\ \wedge(x) = \displaystyle\prod_{1 \leq \ell < k \leq n} \big((\alpha_k - \alpha_\ell)x - \beta_\ell + \beta_k\big), \end{cases} \tag{4.9}$$

and

$$\begin{cases} P(x) = \wedge(x)\tilde{P}(x) \\ Q(x,y) = \wedge(x)\tilde{Q}(x,y) \end{cases}$$

where $\tilde{P}, \tilde{Q}$ are polynomials such that deg $\tilde{P} = n$ and deg $\tilde{Q} = n$.

Evidently, if

$$\deg\ \wedge(x) > 0,$$

the zeroes of $\wedge(x)$ must be zeroes of $Q(x,y)$, i.e.,

$$\wedge(x^*) = 0,$$

then

$$Q(x^*, y) = P(x^*) = 0\,.$$

The aim of the following assertions is to study the system

$$\begin{cases} \dot{x} = (-1)^{n+1}\tilde{P}(x) \\ \dot{y} = \tilde{Q}(x,y) \end{cases} \tag{4.10}$$

which can be obtained from (4.2) by choosing $\nu(x,\, y) = \wedge(x)^{-1}$. For the case when

$$\alpha_1 \neq \alpha_2 = \ldots = \alpha_n = \alpha_{n+1} = 0, \tag{4.11}$$

we obtain

$$\deg\ \wedge(x) = 0\,,$$

and the vector field (4.2) takes the expression (4.5).

Let us now study the different configurations of (4.11). We will analyze the case when $n = 3$.

1) Let us suppose that

$$\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4. \tag{4.12}$$

By choosing $\nu : \nu(x,\, y)(\beta_2 - \beta_1)\ (\alpha x + \beta_4 - \beta_2) = 1$, and requiring that

$$\beta_1 - \beta_2 = \beta_3 - \beta_4 \tag{4.13}$$

after some calculations it can be deduced that (4.2) for $n = 3$, is the following system

$$\begin{cases} \dot{x} = \beta_{21}(\alpha x + \beta_{41})\ (\alpha x + \beta_{32})\ (\alpha x + \beta_{31}) \\ \dot{y} = \beta_{21}\big(\alpha_1(y - y_3)(y - y_4)\big[2\,(y - \alpha_1 x) + \alpha x + \beta_{41} - 2\beta_2\big] - \\ \qquad -\alpha_4(y - y_1)(y - y_2)\big[2\,(y - \alpha_4 x) - \alpha x - \beta_{41} - 2\beta_3\big]\big), \end{cases} \tag{4.14}$$

where

$$\alpha \equiv \alpha_4 - \alpha_2 = \alpha_4 - \alpha_1 = \alpha_3 - \alpha_1,$$
$$\alpha_{ji} \equiv \alpha_j - \alpha_i,$$
$$\beta_{ji} \equiv \beta_j - \beta_i.$$

The system (4.14), with the given fundamental solution made up by four straight lines

$$\ell_j : \quad y = \alpha_j x + \beta_j, \quad j = 1, 2, 3, 4,$$

has the following complementary invariant straight lines as trajectories:

$$\ell_5 : \quad x = -\frac{\beta_{35}}{\alpha},$$
$$\ell_6 : \quad x = -\frac{\beta_{32}}{\alpha},$$
$$\ell_7 : \quad x = -\frac{\beta_{41}}{\alpha}.$$

It is easy to prove that the line which passes through the points of intersection of the lines ℓ_j, $j = 2, 3, 5$, and ℓ_i, $\quad i = 1, 4, 7$, and such that

$$\ell_8 : \quad y = \frac{\beta_{21}(\alpha_1 + \alpha_4)}{\beta_{41} - \beta_{32}}\, x + \frac{\beta_2\beta_4 - \beta_1\beta_3}{\beta_{41} - \beta_{32}}$$

is a trajectory of (4.14). With ℓ_8 we complete the maximum number of invariant straight lines which a cubic vector field, since $L(3) = 8$.

2) Now we shall study the following configuration: $\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4$.

With no loss of generality we choose the coordinate in such a way that

$$\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4 = 0. \tag{4.15}$$

Under this hypothesis, system (4.2) takes the form:

$$\begin{cases} \dot{x} = \nu(x)\beta_{43}(\beta_{42} - \alpha_2 x)(\beta_{41} - \alpha_1 x)(\beta_{32} - \alpha_2 x)(\beta_{31} - \alpha_1 x)(\beta_{21} + \alpha_{21} x) \\ \dot{y} = -\nu(x)\beta_{43}(y - \beta_3)(y - \beta_4)\big(\alpha_1(y - \alpha_2 x - \beta_2)(\beta_{42} - \alpha_2 x)(\beta_{32} - \alpha_2 x) - \\ \qquad -\alpha_2(y - \alpha_1 x - \beta_1)(\beta_{41} - \alpha_1 x)(\beta_{31} - \alpha_1 x)\big). \end{cases} \tag{4.16}$$

It is easy to observe that in this case polynomials $P(x)$ and $Q(x, y)$ are of degree 5, and so there must be a polynomial $\wedge$ of degree 2 which satisfies the conditions given above. Now

$$\wedge(x) = (\beta_{ij} + \alpha_{ij} x)(\beta_{ik} + \alpha_{ik} x)\,,$$

where $i, j, k = 1, 2, 3, 4$, $i \neq j \neq k$, $i + j \neq 7$, $i + k \neq 7$.

In order to illustrate these assertions we shall study the next particular cases:

$$\begin{aligned} \wedge(x) &= (\beta_{42} - \alpha_2 x)(\beta_{41} + \alpha_1 x) \\ \nu(x) &= \big(\beta_{43} \wedge (x)\big)^{-1} \end{aligned}$$

A polynomial vector field of degree 3 and with $L(3) = 8$ can be constructed in this case if the given straight lines satisfy the following conditions:

i) $\alpha_2 = -\alpha_1$, $\beta_4 - \beta_2 = \beta_1 - \beta_4$.

Equations (4.6) under these restrictions take the form:

$$\begin{cases} \dot{x} = (\beta_{32} + \alpha_1 x)(\beta_{31} - \alpha_1 x)(\beta_{21} - 2\alpha_1 x) \\ \dot{y} = 2\alpha_1(y - \beta_3)(y - \beta_4)(y - \beta_1 - \beta_2 + \beta_3) \end{cases} \tag{4.17}$$

The equations of the lines which are trajectories of the field (4.17) can easily be deduced. In particular the straight line ℓ_8 is such that $y = \beta_1 + \beta_2 - \beta_3$.

ii) $\alpha_2 = -\alpha_1$, $\beta_4 - \beta_2 = \beta_1 - \beta_3$.

The system (4.16) takes the form

$$\begin{cases} \dot{x} = (\beta_{32} + \alpha_1 x)(\beta_{31} - \alpha_1 x)(\beta_{21} - 2\alpha_1 x), \\ \dot{y} = \alpha_1(y - \beta_3)(y - \beta_4)(2y - \beta_1 - \beta_2). \end{cases} \tag{4.18}$$

It is easy to observe that this vector field has 8 invariant straight lines.

iii)

$$\alpha_1 = 2\alpha_2,\ \beta_4 - \beta_2 = \beta_2 - \beta_1.$$

Under these restrictions equations (4.16) takes the form

$$\begin{cases} \dot{x} = (\beta_{32} - \alpha_2 x)(\beta_{21} - \alpha_2 x)(\beta_{31} - 2\alpha_2 x) \\ \dot{y} = \alpha_2 (y - \beta_3)(y - \beta_4)(y - 3\alpha_2 x - \beta_1 - \beta_2 + \beta_3) \end{cases} \tag{4.19}$$

The straight line ℓ_8 in this case is the following $y = 2\alpha_2 x + 2\beta_2 - \beta_3$.

Likewise the other cases can be analyzed depending the form of the polynomial $\wedge$ but it is easy to prove that all these cases do not add new configurations to the solution of problem 2.

The configuration $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4$ does not hold for the cubic system with a maximum number of the invariant straight lines and with the restriction $g_l(x, y) = 0, \quad l = 1, 2$.

Now we shall analyze the case when $n = 4$.

1. Let us suppose that $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4 = \alpha_5$. With no loss of generality we require that

$$\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4 = \alpha_5 = 0. \tag{4.20}$$

System (4.2), under the given restrictions, can be rewritten as follows

$$\begin{cases} \dot{x} = -\nu(x, y)\beta_{54}\beta_{53}(\beta_{52} - \alpha_1 x)(\beta_{51} - \alpha_1 x)\beta_{43}(\beta_{42} - \alpha_1 x)\times \\ \qquad \times(\beta_{41} - \alpha_1 x)(\beta_{32} - \alpha_1 x)(\beta_{31} - \alpha_1 x)\beta_{21} \equiv \nu(x, y)P(x) \\ \dot{y} = \nu(x, y)(y - \beta_3)(y - \beta_4)(y - \beta_5)\beta_{54}\beta_{53}\beta_{43}\alpha_1 \times \\ \qquad \times\big((\beta_{51} - \alpha_1 x)(\beta_{41} - \alpha_1 x)(\beta_{31} - \alpha_1 x)(y - \alpha_1 x - \beta_1) - \\ \qquad -(\beta_{52} - \alpha_1 x)(\beta_{42} - \alpha_1 x)(\beta_{32} - \alpha_1 x)(y - \alpha_1 x - \beta_2)\big) \equiv \\ \qquad \equiv \nu(x, y)Q(x, y), \end{cases} \tag{4.21}$$

The polynomials P, Q are of degree 6 and 7 respectively. By choosing $\wedge$ as a product of any pair of binomials of $P(x)$ we deduce that system (4.21) is a quartic vector field with 9 invariant straight lines as trajectories ($L(4) = 9$).

It is interesting to observe that in this case the configuration of the straight line is independent of $\wedge$. So it is sufficient to analyze, for example, the following case

$$\wedge(x) = (\beta_{52} - \alpha_1 x)(\beta_{51} - \alpha_1 x). \tag{4.22}$$

It is easy to prove that in this case the relation $Q(x^*, y^*) = 0$ holds if the following conditions take place i)

$$\begin{cases} \beta_1 - \beta_4 = \beta_2 - \beta_5 \\ \beta_1 - \beta_5 = \beta_2 - \beta_3 \,, \end{cases} \tag{4.23}$$

ii)

$$\begin{cases} \beta_1 - \beta_5 = \beta_2 - \beta_4 \\ \beta_1 - \beta_3 = \beta_2 - \beta_5 \,. \end{cases}$$

From the conditions in (4.23)-i) we deduce that (similarly for (ii))

$$\begin{cases} \beta_{21} = \beta_{35} = \beta_{54} \\ \beta_{41} = \beta_{52} \\ \beta_{51} = \beta_{32} \\ \beta_{43} = 2\,\beta_{12} \\ \beta_{42} + \beta_{13} = 3\,\beta_{12} \,. \end{cases}$$

By putting $Q(x, y), P(x)$ and by choosing ν:

$$\nu(x) = \frac{1}{2\beta_{21}^3\,(\alpha x + \beta_{52})\,(\alpha x + \beta_{51})},$$

we obtain the quartic vector field with 9 invariant straight lines as trajectories

$$\begin{cases} \dot{x} = -\beta_{21}(\beta_{42} - \alpha_1 x)(\beta_{41} - \alpha_1\, x)(\beta_{32} - \alpha_1 x)(\beta_{31} - \alpha_1 x) \\ \dot{y} = \alpha_1(y - \beta_3)(y - \beta_4)(y - \beta_5)\big(3\beta_{21}(y - \alpha_1 x) - \\ \qquad - \beta_{21}\alpha_1 x + \beta_2\beta_{42} - \beta_1\beta_{31}\big). \end{cases} \tag{4.24}$$

2. Likewise we can study the case when [5] $\alpha_1 \neq \alpha_2 \neq \alpha_3 = \alpha_4 = \alpha_5 = 0$ and $0 \neq \alpha_1 = \alpha_2 \neq \alpha_3 \neq \alpha_4 = \alpha_5 = 0$.

Analogously we can study the case for $n > 4$. In this way it is possible to construct the polynomial vector field of degree n from given $n + 1$ straight lines.

5. Construction of the analytical vector fields in the plane from given solutions

In this section we study the problem of constructing an analytical vector field in a certain region $G \subset \mathbb{R}^2$:

$$\begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases} \tag{5.1}$$

by using a finite number of given solutions.

Firstly, let us introduce some necessary notations and concepts.

Let $(Z, Z_1(t), \ldots, Z_m(t), F) \in \mathbb{C}$ be complex such that

$$\begin{cases} Z = x + iy \ , \ \overline{Z} = x - iy \ , \ i = \sqrt{-1} \\ Z_j(t) = x_j(t) + iy_j(t) \ , \ j = 1, 2, .., m \\ F(Z, \overline{Z}) = P(x, y) + iQ(x, y), \end{cases}$$

where $Z_j : I \subset \mathbb{R} \longrightarrow \mathbb{C}$, are curves of class $C^r(I), r \geq 1$.

By $\mathcal{D}$ we denote the matrix:

$$\mathcal{D}(Z, \overline{Z}, t) = \begin{bmatrix} f_1(Z, \overline{Z}) & f_1\big(Z_1(t), \overline{Z}_1(t)\big) & \ldots & f_1\big(Z_m(t), \overline{Z}_m(t)\big) \\ f_2(Z, \overline{Z}) & f_2\big(Z_1(t), \overline{Z}_1(t)\big) & \ldots & f_2\big(Z_m(t), \overline{Z}_m(t)\big) \\ \vdots & & & \\ f_m(Z, \overline{Z}) & \ldots\ldots\ldots\ldots & \ldots & f_m\big(Z_m(t), \overline{Z}_m(t)\big) \\ \dot{Z} & \dot{Z}_1(t) & \ldots & \dot{Z}_m(t) \end{bmatrix} \tag{5.2}$$

where $f_1, \ldots, f_m$ are certain analytical and independent functions on G.

DEFINITION 5. The arbitrary analytical function Φ^* on G such that

$$\Phi^*\big(Z_j(t), \overline{Z}_j(t), t\big) \equiv 0, \qquad j = 1, 2, .., m$$

will be called the Eruguin function.

Proposition 5.1

The differential equations of the first order have the system of curves:

$$\omega = \big(Z_1(t), \ldots, Z_m(t)\big) \tag{5.3}$$

as solutions such that

$$\Delta(\omega) = \begin{vmatrix} f_1\big(Z_1(t), \overline{Z}_1(t)\big) & \ldots\ldots & f_1\big(Z_m(t), \overline{Z}_m(t)\big) \\ \vdots & & \\ \vdots & & \\ f_m\big(Z_1(t), \overline{Z}_1(t)\big) & \ldots\ldots & f_m\big(Z_m(t), \overline{Z}_m(t)\big) \end{vmatrix} \neq 0 \tag{5.4}$$

$\forall t \in I^* \subset I$, can be represented as follows

$$\det \mathcal{D} = \Phi^*(Z, \overline{Z}, t) \tag{5.5}$$

or, expressed in another way,

$$\dot{Z} = \sum_{j=1}^{m} (-1)^{m+1} \frac{\Delta_j(\omega)}{\Delta(\omega)} f_j(Z, \overline{Z}) + \Phi(Z, \overline{Z}, t) \tag{5.6}$$

where $\Delta_1(\omega), \ldots, \Delta_m(\omega), \Delta_{m+1}(\omega)$ are the adjoints of matrix $\mathcal{D}$ which correspond to the elements in the first column: $f_1(Z, \overline{Z}), f_2(Z, \overline{Z}), \ldots, f_m(Z, \overline{Z}), \dot{Z}$,

$$\Phi(Z, \overline{Z}, t) = \frac{(-1)^m \Phi^*(Z, \overline{Z}, t)}{\Delta(\omega)},$$
$$\Delta_{m+1}(\omega) = (-1)^m \Delta(\omega).$$

In fact, it is evident that all the functions $Z_j(t)$, $j = 1, .., m$ are solutions of (5.5). If we develop $\det D(Z, \overline{Z}, t)$ with respect to the first column, and considering (5.5), we get

$$\sum_{j=i}^{m} \Delta_j(t) f_j(Z, \overline{Z}) + \Delta_{m+1}(\omega)\, \dot{Z} = \Phi^*(Z, \overline{Z}, t)$$

By using condition (5.4) we finally obtain

$$\dot{Z} = -\sum_{j} \frac{\Delta_j(t) f_j(Z, \overline{Z})}{\Delta_{m+1}(\omega)} + \Phi(Z, \overline{Z}, t) \equiv G(Z, \overline{Z}, t) + \Phi(Z, \overline{Z}, t)\,.$$

We should now like to prove that the above equation is the most general first order differential equation which has system of curves as solutions.

In fact, let $\dot{Z} = F(Z, \overline{Z}, t)$ be an arbitrary differential equation which has system (5.3) as solutions, i.e., $F(Z_j, \overline{Z_j}, t) = \dot{Z}_j$, $j = 1, .., m$.

This equation can be rewritten in the form (5.5) if we define Φ^* as follows

$$\Phi^*(Z, \overline{Z}, t) = (-1)^m \Delta_m(\omega) \Phi(Z, \overline{Z}, t) = (-1)^m \Delta_m(\omega)[F(Z, \overline{Z}, t) - G(Z, \overline{Z}, t)]$$

which is evidently an Eruguin function.

Definition 6. The system of curves (5.3) which satisfies (5.4) will be called the fundamental solution of (5.5). Similar to definition 2 of section 1, the system of curves

$$W_j : I \subset \mathbb{R} \longrightarrow \mathbb{C} \quad \text{de clase} \quad C^r(I), r \geq 1, j = \overline{1, m}$$
$$\Omega = \big(W_1(t), \ldots, W_m(t)\big),$$

which in $I^* \subset I$ satisfies the condition

$$\begin{cases} \Delta(\Omega) \neq 0 \\ \dfrac{\Delta_j(\omega)}{\Delta(\omega)} = \dfrac{\Delta_j(\Omega)}{\Delta(\Omega)} \ , \quad j = \overline{1, m}, \end{cases} \tag{5.7}$$

will be called equivalent to ω and then we write

$$\omega \approx \Omega.$$

We can observe from (5.6) that any first order differential equations for which a finite number of solutions is known can be constructed with an arbitrariness in the Eruguin function.

It is evident that if the following relation holds

$$\dot{Z}_j(t) = 0 \ , \quad j = \overline{1, m}$$

then

$$\dot{Z} = \Phi(Z, \overline{Z}, t). \tag{5.8}$$

The natural arbitrariness with which we construct the above equations can be restricted by using complementary conditions.

In the following assertions we will construct the polynomial vector field from given solutions. We shall also propose a formal method to construct the Eruguin function Φ.

Firstly the Lie algebra of the matrix of order S is denoted by

$$\mathfrak{M}(\mathbb{C}, S) \equiv \mathfrak{M}$$

Let $B_1, \ldots, B_s \in \mathfrak{M}$ be the arbitrary matrix such that

$$H_0(Z, \overline{Z}) = \sum_{k=1}^{s} B_k f_k(Z, \overline{Z}) \ , \quad m \leq s < \infty.$$

First we introduce matrices $H_1(Z, \overline{Z}), \ldots, H_m(Z, \overline{Z})$ such that

$$\begin{aligned} H_j(Z, \overline{Z}) = \big[H_{j-1}(Z, \overline{Z}), H_{j-1}(Z_j, \overline{Z}_j)\big] \equiv \\ \equiv H_{j-1}(Z, \overline{Z}) H_{j-1}(Z_j, \overline{Z}_j) - H_{j-1}(Z_j, \overline{Z}_j) H_{j-1}(Z, \overline{Z}), \quad j = 1, .., m. \end{aligned}$$

Of course $H_j(Z_k, \overline{Z}_k) \equiv 0, \quad j = 1, .., m, k = 1, .., j$. By ξ we denote the function:

$$\xi(Z, \overline{Z}, t) = \sum_{k,j} \eta_{kj}(\beta) H_m^{kj}(Z, \overline{Z}) \tag{5.9}$$

where $\eta(\beta) = \big(\eta_{kj}(\beta)\big)$ is a matrix which depend on a finite number of the arbitrary functions. It is clear that

$$\xi\big(Z_j(t), \overline{Z}_j(t), t\big) \equiv 0 \ , \quad j = 1, ..., m. \tag{5.10}$$

Corollary 5.1

The Eruguin function can be represented as follows

$$\Phi(Z, \overline{Z}, t) = \xi(Z, \overline{Z}, t), \tag{5.11}$$

where ξ is defined by the formula (5.9).

EXAMPLE1: Let us suppose that $m = 2$, $s = 6$; $Z_1(t), Z_2(t)$ y $f_1, \dots, f_6$ are such that

$$\omega = \big(0, Z_2(t)\big),$$

$$\vec{L}(Z, \overline{Z}) = \big(f_1(Z, \overline{Z}), \dots, f_6(Z, \overline{Z})\big) = \big(1, Z, \overline{Z}, Z^2, |Z|^2, \overline{Z}^2\big).$$

After some calculations it is easy to prove that

$$\xi(Z, \overline{Z}, t) = \Big(\vec{L}\big(Z_2(t), \overline{Z}_2(t)\big), B(\beta)\vec{L}(Z, \overline{Z})\Big), \tag{5.12}$$

where

$$B(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ 0 & -\beta_1 & 0 & \beta_5 & \beta_6 & \beta_7 \\ 0 & -\beta_2 & -\beta_5 & 0 & \beta_8 & \beta_9 \\ 0 & -\beta_3 & -\beta_6 & -\beta_8 & 0 & \beta_{10} \\ 0 & -\beta_4 & -\beta_7 & -\beta_9 & -\beta_{10} & 0 \end{pmatrix}. \tag{5.13}$$

6. Stationary polynomial vector fields with a finite number of given solutions

The aim of this section is to construct polynomial vector fields

$$\begin{cases} \dot{x} = \displaystyle\sum_{k+j=0}^{n} a_{k_j} x^k y^j \\ \dot{y} = \displaystyle\sum_{k+j=0}^{n} b_{k_j} x^k y^j \end{cases} \tag{6.1}$$

from a finite number of given solutions.

Let $f_1, \ldots, f_s$ be the functions such that

$$\begin{cases} f_1(Z,\overline{Z}) = 1 \\ \vdots \\ f_{\frac{(k+1)k}{2}+(j+1)}(Z,\overline{Z}) = Z^{k-j}\overline{Z}^j, \quad 0 \le j \le k \le n \\ \vdots \\ f_s(Z,\overline{Z}) = \overline{Z}^n. \end{cases}$$

It is easy to deduce that

$$s = 1 + 2 + ... + (n+1) = \frac{(n+1)(n+2)}{2}.$$

If $\vec{q} = (q_1, \ldots, q_s) \in \mathbb{C}^s$ is constant, then (6.1) can be written in the equivalent form:

$$\dot{Z} = \left(\vec{q}, \vec{L}(Z,\overline{Z})\right) \tag{6.2}$$

where

$$\vec{L}(Z,\overline{Z}) = \left(f_1(Z,\overline{Z}), \ldots, f_s(Z,\overline{Z})\right) =$$
$$= \left(1, Z, \overline{Z}, Z^2, Z\overline{Z}, \overline{Z}^2, \ldots, Z^n, Z^{n-1}\overline{Z}, \ldots, Z\overline{Z}^{n-1}, \overline{Z}^n\right).$$

Problem 3.

Let us give the system of functions

$$\omega = \left(Z_1(t), \ldots, Z_m(t)\right)$$

of class $C^r(I), r \ge 1, I \subset \mathbb{R}$, such that

$$\Delta(\omega) = \begin{vmatrix} 1 & \ldots\ldots & 1 \\ Z_1(t) & \ldots\ldots & Z_m(t) \\ \overline{Z}_1(t) & \ldots\ldots & \overline{Z}_m(t) \\ \vdots & & \vdots \\ \overline{Z}_1^p(t) & \ldots\ldots & \overline{Z}_m^p(t) \end{vmatrix} \ne 0 \tag{6.3}$$

$\forall t \in I^* \subset I$, where $\dfrac{(p+1)(p+2)}{2} = m$.

We want to construct the vector field of degree n in the variables $Z, \overline{Z}$, where $n \geq p$, in such a way that ω is a fundamental solution.

Solution. Firstly it can be seen that (6.3) excludes, from what has been described in the previous section, all the solutions for which the following relations hold:

$$|Z_1(t)| = \ldots = |Z_m(t)|.$$

By choosing the Eruguin function Φ (see for instance (5.9), (5.11))

$$\Phi(Z, \overline{Z}, t) = \xi(Z, \overline{Z}, t) = \sum \eta_{kj}(\beta) H_m^{kj}(Z, \overline{Z}),$$

we can easily deduce that

$$\dot{Z} = \left(\vec{K}(t), \vec{L}(Z, \overline{Z})\right) + \Phi(Z, \overline{Z}, t), \tag{6.4}$$

where

$$\vec{K}(t) = (k_1(t), ..., k_m(t), 0, ..., 0),$$
$$k_j(t) = (-1)^m \frac{\Delta_j(\omega)}{\Delta(\omega)}.$$

From the representation

$$\xi(Z, \overline{Z}, t) = \left(\vec{h}(t), \vec{L}(Z, \overline{Z})\right),$$

we deduce the following expression for (6.4):

$$\dot{Z} = \left(\vec{K}(t) + \vec{h}(t), \vec{L}(Z, \overline{Z})\right). \tag{6.5}$$

Corollary 6.1

The dynamic system (6.5) is stationary if and only if

$$\frac{d}{dt}\left(\vec{K}(t)\right) = -\frac{d}{dt}\left(\vec{h}(t)\right).$$

This assertion is illustrated in the following example.

EXAMPLE2: A quadratic vector field on the Z y $\overline{Z}$ variables with the fundamental solution

$$\omega = \big(0, Z_2(t)\big).$$

The solution to this problem can be easily deduced from the above results.

In fact, in this case from (6.4) we get

$$\dot{Z} = \Big(\vec{K}(t) + \vec{L}(Z_2(t), \overline{Z}_2(t))B(\beta), \vec{L}(Z, \overline{Z})\Big)$$

or, expressed in a different way,

$$\dot{Z} = \frac{\dot{Z}_2(t)}{Z_2(t)} Z + \Big(\vec{L}\big(Z_2^{(t)}, \overline{Z}_2(t)\big)B(\beta), \vec{L}(Z, \overline{Z})\Big), \tag{6.6}$$

where $\vec{K}(t) = (0, \frac{\dot{Z}_2}{Z_2}, 0, 0, 0, 0)$, L and B are the matrix defined in the previous section. So, the vector field is stationary if and only if

$$\vec{K}(t) + \vec{L}\big(Z_2(t), \overline{Z}_2(t)\big)B(\beta) = (c_1, c_2, ..., c_6),$$

where c_j, $j = 1, ..., 6$ are constants.

In particular, if the arbitrary functions $\beta_1, \ldots, \beta_{10}$ in the formulas (5.12), (5.13) and Z_2 are such that

$$\beta_j = \begin{cases} 0 & \text{if } j \neq 4, \\ a + ib & \text{if } j = 4, \end{cases}$$

$Z_2 = \varepsilon \in \mathbb{R}$, then (6.6) take the form

$$\dot{Z} = -\beta_4(\varepsilon^2 Z - \varepsilon \overline{Z}^2),$$

or, what is the same,

$$\begin{cases} \dot{x} = a^*(\varepsilon x + y^2 - x^2) - b^* \partial_y H, \\ \dot{y} = a^*(\varepsilon y + 2xy) + b^* \partial_x H, \end{cases}$$

where $H = \varepsilon/2(x^2 + y^2) + xy^2 - x^3/3$, $a^* = -\varepsilon a$, $b^* = -\varepsilon b$.

By continuing the study of (6.4) we will prove the following proposition

Proposition 6.1

Let us give the fundamental solution ω of the polynomial vector field of degree n (6.4), such that

$$\omega = \big(\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, Z_{n+2}(t), \ldots, Z_m\big)$$

where $\arg \alpha_j = a$, $j = \overline{1, n+1}$, *and m satisfies the inequality*

$$n + 1 \leq m \leq \frac{(n+2)(n+1)}{2} .$$

Then straight line ℓ where the points $\alpha_1, \ldots, \alpha_{n+1}$ lie, is a singular straight line of the vector field.

In fact, with no loss in generality we shall suppose that $\arg \alpha_j = 0$ and $\alpha_{n+1} = 0$, i.e. $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{R}$ and the point α_{n+1} coincides with the origin. So the line ℓ coincides with the real axes

$$Z - \overline{Z} = 0. \tag{6.7}$$

After some calculations it can be proved that the component F_0 of (6.4)

$$F_0 \equiv \Big(\vec{K}(t), \vec{L}(Z, \overline{Z})\Big)$$

can be represented as

$$F_0(Z, \overline{Z}, t) = (Z - \overline{Z})\phi_1(Z, \overline{Z}, t),$$

which evidently vanishes along (6.7).

On the other hand, if (6.7) holds, then all the elements of the matrix $H_j(Z, \overline{Z})$ are real.

By considering that

$$H_n(Z_k, \overline{Z}_k) \equiv H_n(\alpha_k, \alpha_k) \equiv 0,$$

$\forall k \leq n$, we have the representation

$$H_n(Z, \overline{Z}) = H(t)\,(Z - \alpha_1)(Z - \alpha_2)...(Z - \alpha_n),$$

where H is an arbitrary matrix which depends only on t, because the vector field is a polynomial in Z, $\overline{Z}$ of degree n.

From here we can easily deduce that

$$H_{n+1}(Z,\,\overline{Z}) = \Big[H_n(Z,\overline{Z}),\; H_n(Z_n(t),\overline{Z}_n(t))\Big] \equiv 0.$$

Evidently $H_j(Z,\overline{Z}) \equiv 0 \;\forall j = (n+1), ..., m$, so

$$\Phi(Z,\overline{Z},t) \equiv 0$$

if $Z = \overline{Z}$. As a consequence we have the following representation for the Eruguin function

$$\Phi(Z,\overline{Z},t) = (Z-\overline{Z})\phi_2(Z,\overline{Z},t),$$

$$\dot{Z} = (Z-\overline{Z})\Big(\phi_1(Z,\overline{Z},t)+\phi_2(Z,\overline{Z},t)\Big),$$

or, what is the same,

$$\begin{cases} \dot{x} = 2y\,\mathrm{Im}\Big(\phi_1(Z,\overline{Z},t)+\phi_2(Z,\overline{Z},t)\Big), \\ \dot{y} = 2y\,\mathrm{Re}\Big(\phi_1(Z,\overline{Z},t)+\phi_2(Z,\overline{Z},t)\Big). \end{cases}$$

Hence we obtain that $y = 0$ is the singular straight line of vector field (6.4). In other words, any polynomial vector field of degree n with a finite number of critical points can have at most n critical points in a straight line.

7. Study of the polynomial vector field of degree n with n^2 critical points

Using the results in section 6 we will construct the polynomial vector field which has the following critical points:

$$\begin{cases} \quad (a_j, b_k)\;, \quad j=\overline{1,n}, \;\; k=\overline{1,n}, \\ a_1 < a_2 < \ldots < a_n, \\ b_1 < b_2 < \ldots < b_n. \end{cases}$$

It is easy to prove that the polynomial vector field of degree n which generates the differential equations

$$\begin{cases} \dot{x} = \alpha \prod_{j=1}^{n}(x-a_j) - \beta \prod_{k=1}^{n}(y-b_k) \\ \dot{y} = \beta \prod_{j=1}^{n}(x-a_j) + \alpha \prod_{k=1}^{n}(y-b_k) \end{cases} \tag{7.1}$$

has the given configuration (under some restrictions with respect to location of the points (a_j, b_k)), where α, β are the parameters such that

$$\alpha^2 + \beta^2 = 1.$$

It is interesting to observe that on the plane of the parameters (α, β), the dynamic systems which correspond to the points $(\pm 1, 0), (0, \pm 1)$ are the systems which are diametrically opposed to the structural stability.

So, for $\alpha = 1$, $\beta = 0$ the system (see for instance the formula (4.5))

$$\begin{cases} \dot{x} = \prod(x - a_j) \\ \dot{y} = \prod(y - b_j) \end{cases} \tag{7.2}$$

is structurally stable. The singular trajectories are:

i) stable saddles and nodal points,

ii) separatrices of the saddle points which go towards the nodal points.

For $\alpha = 0$, $\beta = 1$, we have the following Hamiltonian system

$$\begin{cases} \dot{x} = -\partial_y H \\ \dot{y} = \partial_x H \end{cases} \tag{7.3}$$

where H :

$$H(x, y) = \int \left(\prod_{j=1}^{n} (y - b_j) dy + \prod_{j=1}^{n} (x - a_j) dx \right). \tag{7.4}$$

It is well-known [2] that system (7.3) is structurally unstable to an infinite degree.

The following assertion can be made [5]:

Proposition 7.1

Let $C(n)$ be the number of centers in (7.3).
The following equalities hold

$$C(n) = \begin{cases} \dfrac{n^2}{2} & \text{if } n \text{ is even,} \\ \dfrac{n^2 + 1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{7.5}$$

Proof. By translating the origin to point (a_ℓ, b_m), we obtain the following expression for the Hamiltonian H:

$$H_{\ell m}(x, y) = \prod_{j \neq m} (b_m - b_j) \frac{y^2}{2} + \prod_{j \neq \ell} (a_\ell - a_j) \frac{x^2}{2} + V^*(x, y)$$

where V^* is a polynomial in x and y of degree n

$$V^*(x,y) = \sum_{j=3}^{n} v_j(x,y)$$

v_j are the homogeneous forms of degree j.

By considering that $H_{\ell m}$ is an analytical integral positive definite iff

$$\prod_{j\neq m}(b_m - b_j) \cdot \prod_{j\neq \ell}(a_\ell - a_j) > 0 \ , \quad j = \overline{1,n},$$

from here we can easily deduce the veracity of (7.5). □

For $n = 2,\ n = 3$ the coordinates of the centers are:

$$(a_1, b_1), \quad (a_2, b_2) \quad \text{si } n = 2$$

$$(a_1, b_1), \quad (a_1, b_3), \quad (a_2, b_2), \quad (a_3, b_1), \quad (a_3, b_3) \quad \text{si } n = 3.$$

By considering the properties of the structurally stable and unstable dynamic systems on the plane [2] we can affirm that the analytical vector field with a maximum number of centers must be structurally unstable to an infinite degree. Hence we can say that the maximum number of the centers for a polynomial vector field of degree n is (7.5).

Let us give the dynamic system δ-close to the system (7.3) [2]:

$$\begin{cases} \dot{x} = -\prod(y - b_j) + \mu\left(\sum_{k+j=0}^{n} a_{kj}\ x^k y^j\right) \equiv \sum_{k+j=0}^{n} A_{kj}\ x^k y^j \\ \dot{y} = \prod(x - a_j) + \mu\left(\sum_{k+j=0}^{n} b_{kj}\ x^k y^j\right) \equiv \sum_{k+j=0}^{n} B_{kj}\ x^k y^j, \end{cases} \tag{7.6}$$

where μ is the small parameter.

From the above results we deduce that system (7.14) can be constructed if we know $(n+1)$ trajectories which satisfy (2.2) or $1/2(n+1)(n+2)$ solutions which satisfies (6.3).

We denote by $B^0(n)$ and $H^0(n)$ the maximum number of limit cycles which can be generated by perturbations of one of the centers (7.3) and the maximum number of limit cycles which the system (7.6) can have, respectively.

Conjecture 3.

$$B^0(n) \leq \frac{1}{2}(n+1)(n+2)$$

Conjecture 4.

$$H^0(n) \leq \begin{cases} \dfrac{n^2}{2}(n+1) & \text{if } n \text{ is even} \\ \dfrac{n^2+1}{2}(n+1) & \text{if } n \text{ is odd.} \end{cases}$$

References

1. N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type (R), *Mat. Sb.* **30** (72) (1952), 181–196. *Amer. Math. Soc. Transl.* **100** (1954).
2. N. Bautin and E. Leontovich, *Metodi i priomi kachestvienogo isledovanyia dinamicheskix sistem na ploskosti,* Ed. "Nauka", Moscow, 1976 (in Russian).
3. A.S. Galiullin, *Inverse Problems of Dynamics,* Ed. "Mir", Moscow, 1984.
4. R. Ramírez and N. Sadovskaia, Construcción de campos vectoriales en base a soluciones dadas, Preprint, Universitat Politècnica de Catalunya (1992).
5. R. Ramiírez and N. Sadovskaia Ecuaciones diferenciales con soluciones dadas, Preprint, Universitat Politècnica de Catalunya (1995).