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# Dense barrelled subspace of Banach spaces

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Abstract

In this note, we give a gliding hump characterization of the dense barrelled solid subspaces of $\ell^1$ (recall that a sequence is *solid* if $\ell^\infty \cdot E = E$). Also, we present a sufficient condition for a dense subspace of $\ell^1$ to be barrelled, without assuming solidness, and generalizations to dense subspaces of arbitrary Banach spaces.

## 1. Introduction

The idea of barrelled locally convex topological vector spaces (LCTVS) originated with the desire to find an internal characterization of LCTVS for which the equicontinuity version of the Uniform Boundedness Principle (UBP) holds. It is noted in [4] that the Banach space $\ell^1$ has many "small" barrelled subspaces, and very often the proofs of barrelledness employ a "gliding hump" method, in the spirit of early proofs of the UBP (for example, see [5] and [3]).

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## 2. Main results

Let $\varphi$ denote the linear span of the canonical unit vectors $e^i$, $\ell^1$ the sequence space of all absolutely convergent series, and $\ell^\infty$ the sequence space of all bounded sequences.

To show that a dense subspace $E$ of $\ell^1$ is barrelled, it is enough to show that every pointwise bounded sequence $(y^n)$ in $\ell^\infty$ is uniformly bounded (that is, bounded in norm). We assume that $E$ contains $\varphi$; this causes no loss in generality since the dimension of $\varphi$ is countable and a subspace of countable codimension of an infinite-dimensional barrelled space is also barrelled (see [7] for this and other fundamental results on barrelled spaces).

In the following lemma and theorem, $E$ will denote a solid dense subspace of $\ell^1$ which contains $\varphi$.

**Lemma 2.1**

*If $(y^i) \subset \ell^\infty$ is pointwise bounded on $E$, then $(y^i)$ is uniformly bounded on the "sections" of the unit ball of $E$; that is, on*

$$S_n = \big\{(x_i) \in Ba(E) : x_i = 0 \quad for \quad i > n\big\}$$

*Proof.* $S_n$ is just the unit ball of $\mathbb{R}^n$ with the sup norm, so if $(y^i)$ is pointwise bounded on $E$ it is pointwise bounded on $S_n$, so $(y^i)$ is uniformly bounded on $S_n$ by the classical Uniform Boundedness Principle, since $\mathbb{R}^n$ is complete. □

In the following theorem, we will denote by $M_n = \sup_{x \in S_n, i} |y^i(x)|$ for a fixed sequence $(y^i)$ in $\ell^\infty$ that is pointwise bounded on $E$.

**Theorem 2.2**

*$E$ is barrelled $\Leftrightarrow$ there exists a sequence $(d_k)$ of real numbers that satisfy $(d_k) \in Ba(\ell^1), d_k > 0$ and decreasing, such that for every increasing sequence $(i_k)$ in $\mathbb{N}$, there is a further subsequence $(i_{k_\ell})$ and $x \in E$ such that $x_{i_{k_\ell}} = d_{k_\ell}$.*

*Proof.* ($\Leftarrow$) The proof is by contradiction. Assume that $(y^i) \subset \ell^\infty$ is pointwise bounded on $E$, but not uniformly bounded. Then there exists a positive integer $i_1$ such that $|y^{i_1}(e^{i_1})| > \frac{1}{d_1}$ and $|\,\|y^{i_1}\| - |y^{i_1}(e^{i_1})\|| < 1$, which is possible since $\|y^{i_1}\| = \sup_j |y^{i_1}(e^j)|$. Choose $i_2 > i_1$ such that $|y^{i_2}(e^{i_2})| > 2\big[\frac{1}{d_2}(\|y^{i_1}\| + 1)\big] + M_{i_1}\frac{1}{d_2}(\|y^{i_1}\| + 1)$ and $\cdot$ Let $N_{i_k} = \sup_{1 \le j \le k-1}(\|y^{i_1}\| + 1)$. We continue inductively, producing a sequence $(i_k)$ such that

$$\big|y^{i_k}(e^{i_k})\big| > N_{i_k}\frac{1}{d_k}\big(k + M_{i_{k-1}}\big)$$

and $|\|y^{i_k}\| - |y^{i_2}(e^{i_k})\|| < 1$. By hypothesis there exists a subsequence $(i_{k_\ell})$ of $(i_k)$ and $x \in E$ such that $x_{i_{k_\ell}} = d_{k_\ell}$. We can assume, since $E$ is solid, that $x_j = 0$ for $j \neq i_{k_\ell}$. Define a sequence $z$ by

$$z_j = \begin{cases} s_\ell \dfrac{1}{N_{i_{k_\ell}}} x_{i_{k_\ell}} & \text{for} \quad j = i_{k_\ell} \\ 0 \quad \text{otherwise}, \end{cases}$$

where $s_\ell = \operatorname{sign}\big(y^{i_{k_\ell}}(e^{i_{k_\ell}})\big)$. Note that $z \in E$ by solidness, since $s_\ell \frac{1}{N_{i_{k_\ell}}}$ goes to 0 as $\ell \to \infty$. It is also easy to see that $z \in Ba(\ell^1)$, because $\big(s_\ell \frac{1}{N_{i_{k_\ell}}}\big) \in Ba(\ell^1)$. Now

$$\left|y^{i_{k_\ell}}(z)\right| = \left|\sum_{j<i_{k_\ell}} y_j^{i_{k_\ell}} z_j + y^{i_{k_\ell}}\left(z_{i_{k_\ell}}\right) + \sum_{j>i_{k_\ell}} y_j^{i_{k_\ell}} z_j\right|$$

and

$$\left|y^{i_{k_\ell}}(z)\right| \geq y^{i_{k_\ell}}\left(z_{i_{k_\ell}}\right) - \left|\sum_{j<i_{k_\ell}} y_j^{i_{k_\ell}} z_j\right| - \left|\sum_{j>i_{k_\ell}} y_j^{i_{k_\ell}} z_j\right| .$$

Using the definition of $z$ it follows that

$$\left|y^{i_{k_\ell}}(z)\right| \geq y^{i_{k_\ell}}\left(z_{i_{k_\ell}}\right) - M_{i_{k_{\ell-1}}} - \left|\left\|y^{i_{k_\ell}}\right\| \sum_{j>i_{k_\ell}} |z_j|\right| .$$

Since $\left\|y^{i_{k_\ell}}\right\| |z_j| < N_{i_{k_{\ell+1}}} |z_j| \leq |x_j|$ for $j > i_{k_\ell}$, and $y^{i_{k_\ell}}\left(z_{i_{k_\ell}}\right) \geq k_\ell + M_{i_{k_{\ell-1}}}$ we have

$$\left|y^{i_{k_\ell}}(z)\right| \geq k_\ell - \sum_{j>i_{k_\ell}} |x_j| \geq k_\ell - 1 .$$

This contradicts the pointwise boundedness of $(y^i)$, so $E$ is barrelled.

($\Rightarrow$) For the converse, we prove the contrapositive. Assume that for every $(d_k) \in Ba(\ell^1)$ that is positive and decreasing, there exists $(i_k)$ in $\mathbb{N}$ such that for no subsequence $(i_{k_\ell})$ of $(i_k)$ is there $x \in E$ for which $x_{i_{k_\ell}} = d_{k_\ell}$. We claim that this implies $\frac{1}{d_k} x_{i_k} \to 0$ for every $x \in E$. Suppose not. Then there is a subsequence $(i_{k_\ell})$ such that $\frac{1}{d_{k_\ell}} x_{i_{k_\ell}}$ is bounded away from 0. This means that $\big(\frac{d_{k_\ell}}{x_{i_{k_\ell}}}\big)$ is bounded, so the product $\big(\sum_{\ell=1}^\infty \frac{d_{k_\ell}}{x_{i_{k_\ell}}} e^{i_{k_\ell}}\big)x \in E$, which means that the sequence $\sum_{\ell=1}^\infty d_{k_\ell} e^{i_{k_\ell}} \in E$, contradicting the hypothesis. Define a sequence $(z^{i_k})$ in $\ell^\infty$ by

$$z_j^{i_k} = \begin{cases} \dfrac{1}{d_k} & \text{for} \quad j = i_k \\ 0 & \text{otherwise} \end{cases}$$

$(z^{i-k})$ is bounded on $E$ because $z^{i_k}(x) = \frac{1}{d_k} x_{i_k} \to 0$ for any $x \in E$, but clearly, $\|z^{i_k}\| \to \infty$. □

The following definition is due to Bennett [1].

Definition 2.3. Let $r = (r_n)$ be a nondecreasing sequence of natural numbers such that $r_1 = 1$ and $\lim_n \frac{r_n}{n} = 0$, and for any sequence $x = (x_i)$, let $c_n(x)$ be the number of nonzero elements of $\{x_1, \ldots, x_n\}$. Then a *scarce copy* $\sum(E, r)$ of a sequence space $E$ is a the linear span of $\{x \in E : c_n(x) \leq r_n \text{ for all } n\}$.

There are many examples of "small" dense barrelled subspaces of $\ell^1$. For example, Bennett showed that scarce copies of $L = \bigcap_{0<p\leq 1} \ell^p$ are barrelled ([2]), and that such spaces are first category subspaces of $\ell^1$. The barrelledness of $L$ follows easily from the previous theorem.

**Corollary 2.4**

*Scarce copies of $L = \bigcap_{0<p\leq 1} \ell^p$ are barrelled in $\ell^1$.*

*Proof.* Let $d_k = 2^{-k}$. If $E = \sum(L, r)$ is a scarce copy of $L$ then for every subsequence $(e^{i_k})$ of $(e^i)$ there exists a further subsequence $\left(e^{i_{k_\ell}}\right)$ such that $x = \sum \frac{1}{2^{k_\ell}} e^{i_{k_\ell}} \in E$. To see this, note that $x \in L$ because $\sum_\ell \left(\frac{1}{2^{k_\ell}}\right)^p$ converges for all $p$. Also, it is always possible to choose $(i_{k_\ell})$ so that $c_n(x) \leq r_n$. Since $d_k$ satisfies the hypotheses of the theorem and $E$ is solid, the proof is complete. □

Another interesting corollary concerns the $\beta\varphi$ topology on a sequence space, as introduced by Ruckle (see [3]).

DEFINITION 2.5. Let $S$ be a real sequence space. A subset $B$ of $\varphi$ is *$S$-bounded* if

$$p_B(S) = \sup\left\{ \Big| \sum_j s_j t_j \Big| : t \in B \right\} < \infty$$

for each $s$ in $S$. The *$\beta\varphi$ topology* on $S$ is the locally convex topology determined by all seminorms of the form $p_B$ as $B$ ranges over all $S$-bounded subsets of $\varphi$.

It is noted [6] that any barrelled sequence space has a topology stronger than the $\beta\varphi$ topology, but that there are non-barrelled $\beta\varphi$ subspaces of $\ell^1$, answering a question raised in [4]. However, for solid spaces we have

**Corollary 2.6**

*If $E \supset \varphi$ is a solid subspace of $\ell^1$ with the $\beta\varphi$ topology, then $E$ is barrelled.*

*Proof.* The proof of the converse of Theorem 2.2 shows that if $E$ fails to be barrelled, then there is an $S$-bounded subset of $\varphi$ (namely $(z^{i_k})$) that fails to be $\beta\varphi$-bounded. So $E$ is not a $\beta\varphi$ space. □

There are examples of non-solid dense barrelled subspaces of $\ell^1$ (see [3], [4], and [5]). It would be highly desirable to have a gliding hump characterization of dense barrelled subspaces of $\ell^1$, similar in spirit to Theorem 2.2. We now present a fairly weak, sufficient condition for a dense subspace of $\ell^1$ to be barrelled without assuming solidness. A definition is convenient.

DEFINITION 2.7. A sequence $(d_k) \in Ba(\ell^1)$ barrels to zero if $d_k \neq 0$ and

$$\lim_k \left( \frac{1}{|d_k|} \sum_{j>k} |d_j| \right) = 0, .$$

For example, the sequence $\frac{1}{e}\left(\frac{1}{k!}\right)$ barrels to zero. Note that every subsequence of such a sequence also barrels to zero.

**Theorem 2.8**

*Let $\varphi \subset E \subset \ell^1 . E$ is barrelled if there exist a sequence $(d_k)$ that barrels to zero, and positive constants $C_1$ and $C_2$ such that for every increasing sequence $(i_k)$ in $\mathbb{N}$ there exist a subsequence $(i_{k_\ell})$ and $x \in E$ such that $x = \sum_\ell x_{i_{k_\ell}} e^{i_{k_\ell}}$ and $C_1 < \left|\frac{x_{i_{k_\ell}}}{d_{k_\ell}}\right| < C_2$.*

*Proof.* If $E$ is not barrelled, there is $(y^n) \subset \ell^\infty$ that is pointwise bounded on $E$ but not uniformly bounded. Choose $i_1$ (possibly after some rearrangement of $(y^n)$) such that $|y^{i_1}(e^{i_1})| > \frac{1}{|d_1|}$ and $|\,\|y^{i_1}\| - |y^{i_1}(e^{i_1})\|| < 1$. As in Theorem 2.2, we can continue inductively, producing a sequence $(i_k)$ such that $|y^{i_k}(e^{i_k})| > k\frac{1}{|d_k|}\frac{C_2}{C_1} M_{i_{k-1}}$ and $|\,\|y^{i_k}\| - |\,y^{i_k}(e^{i_k})\|| < 1$.

By hypothesis, there is a subsequence $(i_{k_\ell})$ and $x \in E$ satisfying the conditions stated above. Now

$$\left|y^{i_{k_\ell}}(x)\right| = \left| \sum_{j<i_{k_\ell}} y^{i_{k_\ell}}(x_j) + y^{i_{k_\ell}}(x_{i_{k_\ell}}) + \sum_{j>i_{k_\ell}} y^{i_{k_\ell}}(x_j) \right| .$$

It follows that

$$\begin{aligned} \left|y^{i_{k_\ell}}(x)\right| &\geq \left|y^{i_{k_\ell}}(x_{i_{k_\ell}})\right| - \Big| \sum_{j<i_{k_\ell}} y^{i_{k_\ell}}(x_j) \Big| - \Big| \sum_{j>i_{k_\ell}} y^{i_{k_\ell}}(x_j) \Big| \\ &\geq \Big( \|y^{i_{k_\ell}}\| - 1 \Big) |x_{i_{k_\ell}}| - C_2 M_{i_{k_{\ell-1}}} - \|y^{i_{k_\ell}}\| \sum_{j>i_{k_\ell}} |x_j| \\ &\geq \Big( \|y^{i_{k_\ell}}\| - 1 \Big) C_1 |d_{k_\ell}| - C_2 M_{i_{k_{\ell-1}}} - \|y^{i_{k_\ell}}\| C_2 \sum_{j>\ell} |d_{k_j}| \\ &\geq \|y^{i_{k_\ell}}\| |d_{k_\ell}| \left( C_1 - C_2 \Big[ \frac{1}{|d_{k_\ell}|} \sum_{j>\ell} |d_{k_j}| \Big] \right) - C_2 M_{i_{k_{\ell-1}}} . \end{aligned}$$

The expression in square brackets goes to 0 as $\ell \to \infty$, so the expression in parentheses will eventually be larger than $\frac{C_1}{2}$, so we have

$$\begin{aligned} \left|y^{i_{k_\ell}}(x)\right| &\geq \left\|y^{i_{k_\ell}}\right\| |d_{k_\ell}| \frac{C_1}{2} - C_2 M_{i_{k_{\ell-1}}} \\ &> \Big(k_\ell \frac{1}{|d_{k_\ell}|} \frac{C_2}{C_1} M_{i_{k_{\ell-1}}} - 1\Big) |d_{k_\ell}| \frac{C_1}{2} - C_2 M_{i_{k_{\ell-1}}} . \end{aligned}$$

Simplifying, we get

$$|y^{i_{k_\ell}}(x)| \geq C_2 \Big(\frac{k_\ell}{2} - 1\Big) M_{i_{k_{\ell-1}}} - \frac{C_1}{2} |d_{k_\ell}|$$

which goes to $\infty$ as $\ell \to \infty$, contradicting the pointwise boundedness of $(y^n)$. This establishes the result. □

*Remarks.* The sequence $x$ in the previous theorem need only satisfy the stated conditions for all $\ell > N$, for $N$ some integer. Also, with the additional assumption that $(d_k)$ is decreasing, we can replace $(d_{k_\ell})$ in the theorem's statement with $(d_\ell)$.

It is possible to find sufficient conditions for a dense subspace of an arbitrary Banach space $X$ to be barrelled, similar to Theorem 2.2 and 2.8. We begin with a definition.

DEFINITION 2.9. A subset $S$ of the unit sphere of $X, sph(X)$, is *norming* if $\|f\| = \sup\{|f(x)| : x \in S\}$ for all $f \in X'$.

It is easy to see that any dense subset of $sph(X)$ is norming. The sequence $(e^i)_{i \in I}$ in $\ell^1(I)$ is an example of a nondense norming set, for any set $I$. It is also easy to see that if $S$ is norming, then the span of $S$ is a dense subspace of $X$ (if $\overline{sp(S)} \neq X$, then there exists $f \in X'$ such that $\|f\| = 1$ and $f(x) = 0$ for all $x \in \overline{sp(S)}$ by the Hahn-Banach theorem).

For BK-AK spaces (Banach sequence spaces for which $(e^i)$ forms a Schauder basis), the set of all unit vectors with finite support forms a norming set. For the space $c_0$ of all null sequences, it is possible to use a significantly smaller set. Since $\|y\|_1 = \sup\left\{|\sum_k y_k x_k| : x \in \varphi \text{ and } x_i = \pm 1\right\}$, it is possible to restrict the norming set to sequences in $\varphi$ with values $\pm 1$.

The following two theorems generalize Theorems 2.2 (sufficiency) and 2.8 to an arbitrary Banach space $X$. The proofs are virtually the same, with sequences in the norming set $S$ playing the role of $(e^i)$.

**Theorem 2.10**

*Let $X$ be a Banach space and $E$ a dense subspace of $X$ containing a norming set $S$. Assume that there exists $(d_k) \in Ba(\ell^1), d_k > 0$ and decreasing, such that for every sequence $(x_k) \subset S$ there is a subsequence $(x_{k_\ell})$ such that $\sum_\ell d_{k_\ell} x_{k_\ell} \in E$ and the series is bounded-multiplier convergent in $E$. Then $E$ is barrelled.*

*Proof.* Proceed as in Theorem 2.2, constructing a sequence $(x_k)$ from $S$ that plays the part of $(e^{i_k})$. □

**Theorem 2.11**

*Let $X$ be a Banach space and $E$ a dense subspace of $X$ containing a norming set $S$. Assume that there is a sequence $(d_k)$ that barrels to zero and positive constants $C_1$ and $C_2$ such that for every sequence $(x_k)$ in $S$ there is a subsequence $(x_{k_\ell})$ such that $\sum_\ell t_\ell x_{k_\ell} \in E$ and $C_1 \leq \left|\frac{t_\ell}{d_{k_\ell}}\right| \leq C_2$. Then $E$ is barrelled.*

*Proof.* Proceed as in Theorem 2.8, with a sequence $(x_k)$ in $S$ playing the role of $(e^{i_k})$. □

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