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# On the $WM$ points of Orlicz function spaces endowed with Orlicz norm(*)

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Abstract

In this paper, we introduce the concept of $WM$ point and obtain the criterion of $WM$ points for Orlicz function spaces endowed with Orlicz norm and the criterion of $WM$ property for Orlicz space.

## § 1. Introduction

In 1975, while discussing the expansibility of local uniformly rotundity, B.B. Panda and O.P. Kappor [1] introduced the concept of $WM$ property. Afterwards, F. Sullivan [2] introduced the idea of local $k$ uniformly rotundity, and Nan Chaoxun and Wang Jianhua [3] introduced the notion of local $k$ rotundity. They both are related to local uniformly rotundity. By making use of $WM$ property, Wang Jianhua and Wang Musan [4] worked out the relationships among the local uniformly rotundity and obtained:

(1) Let $X$ be a Banach space. $X$ is local uniformly rotund if and only if $X$ is strictly convex local $k$ uniformly rotund and has $WM$ property.

(2) Let $X$ be a Banach space. $X$ is local uniformly rotund if and only if $X$ is local $k$ rotund and has $WM$ property.

$WM$ point is a kind of pointwise description of $WM$ property. Obviously, local uniformly convex points and weakly local uniformly convex points are all $WM$ points.

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For a Banach space $X$, let $B(X), S(X)$ be the unit ball and unit sphere of $X$, respectively $x \in S(X)$ is called a $WM$ point if $x_n \in B(X), \|x_n + x\| \to 2$ imply that there is a supporting functional $f$ of $x$, satisfying $f(x_n) \to 1$. If all points of $S(X)$ are $WM$ points, then $X$ has $WM$ property.

Let $M(u), N(v)$ be a pair of $N$-functions, $p_-(u)$ and $p(u)$ denote the derivatives of $M(u)$ from the left and from the right, respectively $[a,b]$ where $a < b$ is called an affine segment of $M(u)$, if $M(u)$ is linear in $[a,b]$, but not on $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for all $\varepsilon > 0$. We denote $S_M = R \backslash \cup_{i=1}^{\infty} [a_i, b_i]$ where $[a_i, b_i], i \in \mathbb{N}$ are the family of all affine segments of $M(u)$. $M(u)$ is said to satisfy $\Delta_2$-condition ($M \in \Delta_2$) if there is a constant $K$ such that $M(2u) \leq KM(u)$ for large $u, M \in \nabla_2$ if and only if $N \in \Delta_2$. Let $(G, \Sigma, \mu)$ be a nonatomic finite measurable space, and let $X$ be the set of all $\Sigma$-measurable real scalar functions defined over $G$. For $x \in X$, we denote the modular of $x$ by $\rho_M(x) = \int_G M(x(t)) d\mu$. The family

$$L_M = \left\{ x(t) \in X : \quad \text{for some} \quad \lambda > 0, \quad \rho_M(\lambda x) = \int_G M\big(\lambda x(t)\big)\, d\mu < \infty \right\}$$

is a linear set, and when it is endowed with Orlicz norm

$$\|x\|^\circ = \inf_{k>0} \frac{1}{k}\big(1 + \rho_M(kx)\big) = \sup_{\rho_N(y) \leq 1} \int_G x(t) y(t) d\mu$$

or Luxemburg norm

$$\|x\| = \inf \left\{ c > 0 : \rho_M\Big(\frac{x}{c}\Big) \leq 1 \right\}$$

forms a Banach space, called Orlicz space and denoted by $L_M^\circ, L_M$ respectively. We know that ([8]) for any $x \neq 0, \|x\|^\circ = \frac{1}{k}\big(1 + \rho_M(kx)\big)$ if and only if

$$k \in K(x) = [k_x^*, k_x^{**}], \quad \text{where} \quad k_x^* = \inf \left\{ k > 0 : \rho_N(p(kx)) \geq 1 \right\}$$
$$k_x^{**} = \sup \left\{ k > 0 : \rho_N(p(kx)) \leq 1 \right\}.$$

## § 2. Preparatory lemmas

For the sake of reading, we first give several Lemmas.

**Lemma 1** [9].

*For arbitrary $0 \le \lambda, \delta, \lambda' < 1$, there exists $0 < \delta' \le \delta$ such that for all $u, v > 0$ with $M(\lambda u + (1-\lambda)v) \le (1-\delta)(\lambda M(u) + (1-\lambda)M(v))$, we have that*

$$M(\lambda' u + (1-\lambda')v) \le (1-\delta')(\lambda' M(u) + (1-\lambda')M(v)).$$

**Lemma 2** [8].

*For $0 \ne x \in L_M^\circ, f = y + \phi$ is a supporting functional of $x$ if and only if for all (or some) $k \in K(x)$, where $y \in L_N^\circ, \phi$ is a singular function.*

*(i)* $\rho_N(y) + \|\phi\| = 1$

*(ii)* $\|\phi\| = \phi(x)$

*(iii)* $x(t)y(t) \ge 0$ *and* $p_-(k|x(t)|) \le |y(t)| \le p(k|x(t)|)$ $(\mu - a.e.)$.

**Lemma 3** [5].

*Assume $M \in \nabla_2$. If $[a,b]$ is an affine segment of $M(u)$, then for any $\varepsilon > 0$, and $\alpha \in \left(0, \frac{1}{2}\right)$, there is $\delta > 0$ such that if $\lambda \in [\alpha, 1-\alpha], v \in [a,b]$ and $\lambda M(u) + (1-\lambda)M(v) - M(\lambda u) - M\big((1-\lambda)v\big) < \delta$ then $u \in [a-\varepsilon, b+\varepsilon]$.*

**Lemma 4**

*Assume $M \in \nabla_2, x \in S(L_M^\circ)$ is a $WM$ point if and only if for $x_n \in B(L_M^\circ)$, if $\|x_n + x\|^\circ \to 2$, then there is a supporting functional $y \in L_N$ of $x$ satisfying $\langle x_n, y\rangle \to 1$.*

*Proof.* **Sufficiency** is trivial.

**Necessity.** For the cutting function $[x(t)]_n = x(t)$, if $|x(t)| \le n; = 0$ if $|x(t)| > n$, it holds that $x_n \xrightarrow{\mu} x$.

For $x_n \in B(L_M^\circ)$, $\|x_n + x\|^\circ \to 2$, take $[x_n]_{N_n}$ such that

$$\mu\big\{t : |x_n(t)| > N_n\big\} \to 0, \quad \|[x_n]_{N_n} + x\|^\circ \to 2.$$

Since $x$ is a $WM$ point, there is a supporting functional $f = y + \phi$ of $x$ satisfying $f([x_n]_{N_n}) = \langle [x_n]_{N_n}, y\rangle \to 1$. Hence $\|y\|_N \ge 1$. Moreover, by Lemma 2, we deduce that $\rho_N(y) \le 1$, so $\|y\| \le 1$. Thus $\|y\| = 1$. By $M \in \nabla_2$, it follows that $\lim\limits_n \langle x_n, y\rangle = \lim\limits_n \langle [x_n]_{N_n}, y\rangle = 1$. □

**Lemma 5**

*For $x, x_n \in S(L_M^\circ)$, if $\|x_n + x\|^\circ \to 2$, then for any $\eta > 0$*

$$\lim_{n\to\infty} \sup \rho_N\left(p\Big((1+\eta)\frac{kk_n}{k+k_n}(x+x_n)\Big)\right) \ge 1 \tag{1}$$

$$\lim_{n\to\infty} \inf \rho_N\left(p\Big((1-\eta)\frac{kk_n}{k+k_n}(x+x_n)\Big)\right) \le 1 \tag{2}$$

*where $k \in K(x), k_n \in K(x_n)$.*

*Proof.* By the definition of Orlicz norm and the convexity of $M(u)$,

$$\begin{aligned}0 &\leftarrow \|x\|^\circ + \|x_n\|^\circ - \|x_n + x\|^\circ \\ &\geq \frac{1}{k}\big(1+\rho_M(kx)\big) + \frac{1}{k_n}\big(1+\rho_M(k_n x_n)\big) - \frac{k+k_n}{kk_n}\left(1+\rho_M\Big(\frac{kk_n}{k+k_n}(x+x_n)\Big)\right) \\ &= \frac{k+k_n}{kk_n}\left[\frac{k_n}{k+k_n}\rho_M(kx) + \frac{k}{k+k_n}\rho_M(k_n x_n) - \rho_M\Big(\frac{kk_n}{k+k_n}(x+x_n)\Big)\right] \geq 0\,,\end{aligned}$$

i.e. $\frac{1}{h_n}\big(1+\rho_M(h_n(x+x_n))\big) - \|x_n + x\|^\circ \to 0$, where $h_n = \frac{kk_n}{k+k_n} \leq k\,$.

If (1) fails, then for some $\eta_0 > 0$, and $\theta_0 > 0$, and for a subsequence of $(x_n)$, still denoted by $(x_n)$, $\rho_N\big(p((1+\eta_0)h_n(x+x_n))\big) \leq 1-\theta_0$ which leads to a contradiction:

$$\begin{aligned}0 &\leftarrow \frac{1}{h_n}\big(1+\rho_M(h_n(x+x_n))\big) - \|x_n+x\|^\circ \\ &\geq \frac{1}{h_n}\big(1+\rho_M(h_n(x+x_n))\big) - \frac{1}{(1+\eta_0)h_n}\big(1+\rho_M((1+\eta_0)h_n(x+x_n))\big) \\ &= \frac{\eta_0}{(1+\eta_0)h_n}\left\{1 - \frac{1+\eta_0}{\eta_0}\int_G \Big(\int_{h_n|x+x_n|}^{(1+\eta_0)h_n|x+x_n|} p(s)ds\Big)d\mu \right. \\ &\qquad \left. + \rho_M\big((1+\eta_0)h_n(x+x_n)\big)\right\} \\ &\geq \frac{\eta_0}{(1+\eta_0)h_n}\left\{1 - \frac{1+\eta_0}{\eta_0}\int_G p\big((1+\eta_0)h_n(x+x_n)\big)\eta_0 h_n|x+x_n|d\mu \right. \\ &\qquad \left. + \rho_M\big((1+\eta_0)h_n(x+x_n)\big)\right\} \\ &= \frac{\eta_0}{(1+\eta_0)h_n}\big\{1 - \rho_N\big((1+\eta_0)h_n(x+x_n)\big)\big\} \\ &\geq \frac{\eta_0}{(1+\eta_0)h_n}\big\{1-(1-\theta_0)\big\} \geq \frac{\eta_0\theta_0}{(1+\eta_0)k}\,.\end{aligned}$$

For (2), the argument is similar to that of (1). □

**Lemma 6**

*For* $1 = \|x\|^\circ = \frac{1}{k}\big(1+\rho_M(kx)\big); 1 = \|x_n\|^\circ = \frac{1}{k_n}\big(1+\rho_M(k_n x_n)\big)$ *if* $\overline{k} = \sup k_n < \infty, \|x+x_n\|^\circ \to 2$, *then* $k_n x_n - kx \xrightarrow{\mu} 0$ *over* $G_x$, *where* $G_x = \big\{t \in G : k|x(t)| \in S_M \backslash (\{a\} \cup \{b\})\big\}, \{a\}$ *and* $\{b\}$ *are the sets of left and right extreme points of affine segments of* $M(u)$, *respectively.*

*Proof.* Otherwise, for some $\varepsilon > 0$ and $\sigma > 0$, there exists a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, such that $\mu\big\{t \in G_x : |k_n x_n(t) - kx(t)| \geq \varepsilon\big\} \geq \sigma$.

From $\overline{k} \geq k_n > \rho_M(k_n x_n) \geq M(D)\mu\{t : |k_n x_n(t)| > D\}$, take $D$ large enough such that $\mu\{t : k|x(t)| > D\} < \frac{\sigma}{4}$ and $\mu\{t : k_n|x_n(t)| > D\} < \frac{\sigma}{4}$.

Denote $\{a\}$ and $\{b\}$ as $c_1, c_2, c_3, \dots$ Since $t \in G_x, kx(t) \neq c_1$ for all $i$, then we can take open segments $v_i \ni c_1$, so that $\mu\{t \in G_x; kx(t) \in v_i\} < \frac{\sigma}{4 \cdot 2^i}$. Hence

$$\mu\left\{d \in G_x : |kx(t)| \in \bigcup_{i=1}^{\infty} v_i\right\} \leq \frac{\sigma}{4}\,.$$

Denote

$$G_n = \left\{t \in G_x : |k_n x_n(t) - kx(t)| \geq \varepsilon; |kx(t)|, |k_n x_n(t)| \leq D, kx(t) \in S_M \Big\backslash \bigcup_{i=1}^{\infty} v_i\right\}.$$

Then $\mu G_n \geq \frac{\sigma}{4}$. For the bounded closed subset of $\mathbb{R}^3$

$$\left\{(u, v, \lambda) : |u - v| \geq \varepsilon, |u|, |v| \leq D, v \in S_M \Big\backslash \bigcup_{i=1}^{\infty} v_i, \lambda \in \left[\frac{1}{1+\bar{k}}, \frac{\bar{k}}{1+\bar{k}}\right]\right\},$$

because $M(u)$ is strictly convex on $S_M$, there is $\delta, 0 < \delta < 1$, such that for any element $(u, v, \lambda)$ of the above bounded closed subset,

$$M\big(\lambda u + (1-\lambda)v\big) \leq (1-\delta)(\lambda M(u) + (1-\lambda)M(v)\big)\,.$$

Thus, for $t \in G_n$,

$$M\left(\frac{kk_n}{k+k_n}\big(x(t) + x_n(t)\big)\right) \leq (1-\delta)\left(\frac{k_n}{k+k_n}M\big(kx(t)\big) + \frac{k}{k+k_n}M\big(k_n x_n(t)\big)\right).$$

It leads to a contradiction, since

$$\begin{aligned}
0 &\leftarrow \|x\|^\circ + \|x_n\|^\circ - \|x_n + x\|^\circ \\
&\geq \frac{k+k_n}{kk_n}\int_G \left[\frac{k_n}{k+k_n}M\big(kx(t)\big) + \frac{k}{k+k_n}M(k_n x_n(t)) - M\Big(\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)\Big)\right] d\mu \\
&\geq \frac{k+k_n}{kk_n}\int_{G_n} \left[\frac{k_n}{k+k_n}M\big(kx(t)\big) + \frac{k}{k+k_n}M(k_n x_n(t)) - M\Big(\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)\Big)\right] d\mu \\
&\geq \frac{k+k_n}{kk_n}\delta\int_{G_n} \left[\frac{k_n}{k+k_n}M\big(kx(t)\big) + \frac{k}{k+k_n}M(k_n x_n(t))\right] d\mu \\
&\geq \frac{1}{k}\delta M\Big(\frac{\varepsilon}{2}\Big)\frac{\sigma}{4}\,. \square \qquad (++)
\end{aligned}$$

In the following, we still use $\{a\}$ and $\{b\}$ to denote the subsets of left and right extreme points of affine segment $[a,b]$ of $M(u)$ for which $p_-(a) < p(a)$ and $p_-(b) < p(b)$.

## § 3. The main result

**Theorem**

*For $x \in S(L_M^\circ), k \in K(x)$, let $G_a = \{t : k|x(t)| \in \{a\}\}, G_b = \{t : k|x(t)| \in \{b\}\}$. Then $x$ is a $WM$ point if and only if*

*(i)* $M \in \nabla_2$ ,

*(ii)* $\rho_N(p(kx)) \geq 1$ ,

*(iii)* $\rho_N(p(kx)) > 1 \Rightarrow \int_{G\setminus G_b} N\big(p(kx(t))\big)d\mu + \int_{G_b} N\big(p_-(kx(t))\big)d\mu \geq 1$ ,

$\rho_N(p_-(kx)) < 1 \Rightarrow \int_{G\setminus G_a} N\big(p_-(kx(t))\big)d\mu + \int_{G_a} N\big(p(kx(t))\big)d\mu \leq 1$ ,

*(iv)* $\rho_N\big(p(kx)\big) = 1 \Rightarrow \mu G_b = 0$ *or* $\rho_N\Big(p\big(\frac{kx}{1-\tau}\big)\Big) < \infty$ *for some* $\tau > 0$ ,

*(v) for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $y \in B(L_M^\circ)$ with $\|x+y\|^\circ > 2-\delta$ and for all $e \subset G$ with $\mu e < \delta$, we have $\rho_M(ky|e) < \varepsilon$, where $k \in K(y)$.*

*Proof.* **Necessity**. Without loss of generality, we can assume $x(t) \geq 0$.

(i) $M \in \nabla_2$.

We first take $y_n \in L_N,\ \rho_N(y_n) = 1$ and $\int_G x(t)y_n(t)d\mu > 1 - \dfrac{1}{n}$ and $d > 0$ so that $\mu E = \mu\{t \in G : kx(t) \leq d\} > 0$.

If $M \bar{\in} \nabla_2$, there exists $v_n \uparrow \infty$ with $N\Big(\frac{v_n}{1-1/n}\Big) > 2nN(v_n)$. Take $G_n \subset E$ so that $N(v_n)\mu G_n = \frac{1}{n}$. Define $z_n(t) = v_n|_{G_n}$. Since $\rho_N(z_n) = \frac{1}{n} < 1$ and $\rho_N\Big(\frac{z_n}{1-1/n}\Big) > 2nN(v_n)\mu G_n = 2,\ 1 \geq \|z_n\| \geq 1 - \frac{1}{n}$. By [8], there exists $x_n(t) = u_n|_{G_n}, \|x_n\|^\circ = 1$ such that $\langle x_n, z_n\rangle = u_n v_n G_n = \|z_n\| \geq 1 - \frac{1}{n}$. Set

$$g_n(t) = \Big(1 - \frac{1}{n}\Big)\big(y_n(t)|_{G\setminus G_n} + z_n(t)|_{G_n}\big)$$

then $\rho_N(g_n) \le \Big(1-\frac{1}{n}\Big)\Big(1+\frac{1}{n}\Big) = 1-\frac{1}{n^2} < 1$. Hence

$$\begin{aligned}
\|x+x_n\|^\circ &\ge \langle x+x_n, g_n\rangle \\
&= \Big(1-\frac{1}{n}\Big)\left(\int_{G\setminus G_n} x(t)y_n(t)d\mu + \int_{G_n} x(t)v_n d\mu + \int_{G_n} u_n v_n d\mu\right) \\
&= \Big(1-\frac{1}{n}\Big)\Big(\int_G x(t)y_n(t)d\mu - \int_{G_n} x(t)y_n(t)d\mu + \int_{G_n} x(t)v_n d\mu + u_n v_n \mu G_n\Big) \\
&\ge \Big(1-\frac{1}{n}\Big)\Big(1-\frac{1}{n} - \|x|_{G_n}\|^\circ - \|x|_{G_n}\|^\circ + 1 - \frac{1}{n}\Big) \to 2\,.
\end{aligned}$$

For any supporting functional $y \in L_N$ of $x$, noticing that $y(t) \le p(kx(t)) \le p(d)$ whenever $t \in G_n$, we deduce that

$$\langle x_n, y\rangle = \int_{G_n} u_n y(t) d\mu \le \|x_n\|^\circ \, \|y|_{G_n}\|_N \to 0$$

a contradiction with that $x$ is $WM$ point by Lemma 4.

(ii) $\rho_N\big(p(kx)\big) \ge 1$, otherwise, $\rho_N\big(p(kx)\big) < 1$. By Lemma 2, $x$ does not have any supporting functional in $L_N$, then by Lemma 2, we deduce that $x$ is not a $WM$ point, a contradiction.

$$\begin{aligned}
\text{(iii)}\quad &\rho_N\big(p(kx)\big) > 1 \Rightarrow \int_{G\setminus G_b} N\big(p(kx(t))\big)d\mu + \int_{G_b} N\big(p_-(kx(t))\big)d\mu \ge 1 \\
&\rho_N\big(p_-(kx)\big) < 1 \Rightarrow \int_{G\setminus G_a} N\big(p_-(kx(t))\big)d\mu + \int_{G_a} N\big(p(kx(t))\big)d\mu \le 1\,.
\end{aligned}$$

Denote $\{b\} = \{b_n\}_{n=1}^\infty, G_n = \{t \in G : kx(t) = b_n\}$, and assume $\mu G_n > 0$.

If the first statement is not true, then

$$\begin{aligned}
&\int_{G\setminus \bigcup_{n=1}^\infty G_n} N\big(p(kx(t))\big)d\mu + \int_{\bigcup_{n=1}^\infty G_n} N\big(p(kx(t))\big)d\mu \\
&\qquad\qquad = \int_{G\setminus \bigcup_{n=1}^\infty G_n} N\big(p(kx(t))\big)d\mu + \sum_{n=1}^\infty N\big(p(b_n)\big)\mu G_n > 1
\end{aligned}$$

and

$$\int_{G\setminus \bigcup_{n=1}^\infty G_n} N\big(p(kx(t))\big)d\mu + \sum_{n=1}^\infty N\big(p_-(b_n)\big)\mu G_n < 1\,.$$

For each $n$, take different subsets $G'_n \neq G''_n, G'_n, G''_n \subset G_n$ so that $\mu G'_n = \mu G''_n$ and

$$\int_{G\setminus\bigcup_{n=1}^{\infty}G_n} N\big(p(kx(t))\big)d\mu + \sum_{n=1}^{\infty}\big[N(p_-(b_n))\mu(G_n\setminus G'_n) + N(p(b_n))\mu G'_n\big] = 1, \quad (3)$$

$$\int_{G\setminus\bigcup_{n=1}^{\infty}G_n} N\big(p(kx(t))\big)d\mu + \sum_{n=1}^{\infty}\big[N(p_-(b_n))\mu(G_n\setminus G''_n) + N(p(b_n))\mu G''_n\big] = 1\,. \quad (4)$$

Take $c_n < b_n,\ p_-(c_n) = p(c_n) = p_-(b_n) < p(b_n)\,,$ and set

$$x_1(t) = x(t)\big|_{G\setminus\bigcup_{n=1}^{\infty}G_n} + \sum_{n=1}^{\infty}\Big(\frac{c_n}{k}|_{G_n\setminus G'_n} + \frac{b_n}{k}|_{G'_n}\Big)\,,$$

$$x_2(t) = x(t)\big|_{G\setminus\bigcup_{n=1}^{\infty}G_n} + \sum_{n=1}^{\infty}\Big(\frac{c_n}{k}|_{G_n\setminus G''_n} + \frac{b_n}{k}|_{G''_n}\Big)\,.$$

From (3) and (4), we get that $k \in K(x_1), k \in K(x_2)$. And by Lemma 2, we deduce that $x_1$ and $x_2$ have their unique supporting functionals:

$$y_1(t) = p\big(kx(t)\big)\big|_{G\setminus\bigcup_{n=1}^{\infty}G_n} + \sum_{n=1}^{\infty}\big[p_-(b_n)\big|_{G_n\setminus G'_n} + p(b_n)\big|_{G'_n}\big]\,,$$

$$y_2(t) = p\big(kx(t)\big)\big|_{G\setminus\bigcup_{n=1}^{\infty}G_n} + \sum_{n=1}^{\infty}\big[p_-(b_n)\big|_{G_n\setminus G''_n} + p(b_n)\big|_{G''_n}\big]\,,$$

i.e., $\big\langle \frac{x_1}{\|x_1\|^\circ}\,, y_1\big\rangle = 1\,,\ \big\langle \frac{x_2}{\|x_2\|^\circ}\,,\ y_2\big\rangle = 1$. By Lemma 2, it also follows that $y_1, y_2$ are supporting functionals of $x$, i.e, $\big\langle \frac{x}{\|x\|^\circ}, y_1\big\rangle = 1,\ \big\langle \frac{x}{\|x\|^\circ}\,, y_2\big\rangle = 1$. Thus $\big\langle x + \frac{x_1}{\|x_1\|^\circ}\,, y_1\big\rangle = 2\,,\ \big\langle x + \frac{x_2}{\|x_2\|^\circ}\,,\ y_2\big\rangle = 2$, so $\big\|x + \frac{x_1}{\|x_1\|^\circ}\big\|^\circ = 2$ and $\big\|x + \frac{x_2}{\|x_2\|^\circ}\big\|^\circ = 2$.

Define $z_n$ with $z_1 = z_3 = \ldots = \frac{x_1}{\|x_1\|^\circ}\,,\ z_2 = z_4 = \ldots = \frac{x_2}{\|x_2\|^\circ}\,,$ then $\|z_n + x\|^\circ = 2$. Since $x$ is a $WM$ point, there is a supporting functional $y \in L_N$ such that $\langle z_n, y\rangle \to 1$. Hence $\big\langle \frac{x_1}{\|x_1\|^\circ}\,, y\big\rangle = 1 = \big\langle \frac{x_1}{\|x_1\|^\circ}\,, y_1\big\rangle,\ \big\langle \frac{x_2}{\|x_2\|^\circ}\,, y\big\rangle = 1 = \big\langle \frac{x_2}{\|x_2\|^\circ}\,, y_2\big\rangle$. By the uniqueness of the supporting functionals of $x_1$ and $x_2$, it yields that $y_1 = y = y_2$, a contradiction.

We can make the similar argument for

$$\rho_N(p_-(kx)) < 1 \Rightarrow \int_{G\setminus G_a} N\big(p_-(kx(t))\big)d\mu + \int_{G_a} N\big(p(kx(t))\big)d\mu \le 1\,.$$

(iv) $\rho_N(p(kx)) = 1 \Rightarrow \mu G_b = 0$ or $\rho_N\Big(p\big(\frac{kx}{1-\tau}\big)\Big) < \infty$ for some $\tau > 0$. If (iv) fails, then for some $b \in \{b\}$, $\mu E = \mu\{t \in G : kx(t) = b\} > 0$ and for all $\varepsilon > 0$, $\rho_N\big(p((1+\varepsilon)kx)\big) = \infty\,.$

Take $c < b$, $p_-(c) = p(c) = p_-(b) < p(b)$. Set

$$x'(t) = x(t)\big|_{G\setminus E} + \frac{c}{k}\big|_E$$

then $\rho_N(p(kx')) < \rho_N\big(p(kx)\big) = 1$, and for all $\eta > 0$, $\rho_N\big(p((1+\eta)kx')\big) > \rho_N\big(p((1+\eta)kx|_{G\setminus E})\big) = \infty$. Thus $k \in K(x')$, so $k' = k\|x'\|^\circ \in K\Big(\frac{x'}{\|x\|^\circ}\Big)$. Clearly $k' \le k$. From

$$\rho_N\left(p\Big(\frac{kk'}{k+k'}\Big(x + \frac{x'}{\|x'\|^\circ}\Big)\Big)\right) = \rho_N\left(p\Big(\frac{k'}{k+k'}kx + \frac{k}{k+k'}kx\Big)\right) < \rho_N(p(kx)) = 1,$$
$$\rho_N\left(p\Big((1+\eta)\frac{kk'}{k+k'}\Big(x + \frac{x'}{\|x'\|^\circ}\Big)\Big)\right) > \rho_N\big(p((1+\eta)kx|_{G\setminus E})\big) = \infty$$

it follows that $\frac{kk'}{k+k'} \in K\Big(x + \frac{x'}{\|x'\|^\circ}\Big)$. Hence

$$\begin{aligned}
\Big\|x + \frac{x'}{\|x'\|^\circ}\Big\|^\circ &= \frac{k+k'}{kk'}\left(1 + \rho_M\Big(\frac{kk'}{k+k'}\Big(x + \frac{x'}{\|x'\|^\circ}\Big)\Big)\right)\\
&= \frac{k+k'}{kk'}\left[1 + \rho_M(kx|_{G\setminus E}) + M\Big(\frac{k'}{k+k'}b + \frac{k}{k+k'}c\Big)\mu E\right]\\
&= \frac{k+k'}{kk'}\left[1 + \rho_M(kx|_{G\setminus E}) + \Big(\frac{k'}{k+k'}M(b) + \frac{k}{k+k'}M(c)\Big)\mu E\right]\\
&= \frac{1}{k}\big(1 + \rho_M(kx)\big) + \frac{1}{k'}\left(1 + \rho_M\Big(k'\frac{x'}{\|x'\|^\circ}\Big)\right) = 2\,.
\end{aligned}$$

Since $\rho_N(p(kx)) = 1$, we get that $x$ has the unique supporting functional $y = p(k(x(t)))$ in $L_N$. From $cp(b) < M(c) + N(p(b))$,

$$\begin{aligned}
k'\Big\langle\frac{x'}{\|x'\|^\circ}, y\Big\rangle = \langle kx', y\rangle &= \int_{G\setminus E} kx(t)p(kx(t))d\mu + \int_E cp(b)d\mu\\
&< \rho_M(kx|_{G\setminus E}) + \rho_N(p(kx|_{G\setminus E})) + M(c)\mu E + N(p(b))\mu E\\
&= 1 + \rho_M(kx|_{G\setminus E}) + M(c)\mu E = 1 + \rho_M\Big(k'\frac{x'}{\|x\|^\circ}\Big) = k'
\end{aligned}$$

so $\big\langle\frac{x'}{\|x'\|^\circ}, y\big\rangle < 1$, a contradiction with $x$ is a$WM$ point.

(v) For any $\varepsilon > 0$, there is $\delta > 0$ such that for all $y \in B(L_M^\circ)$ with $\|x+y\|^\circ > 2-\delta$ and all $e \subset G$ with $\mu e < \delta$, we have $\rho_M(ky|_e) < \varepsilon$ where $k \in K(y)$.

If there exist $x_n \in B(L_M^\circ)$ and $e_n \subset G$, $\|x_n + x\|^\circ \to 2$, $\mu e_n \to 0$, but $\rho_M(k_n x_n|_{e_n}) \ge \varepsilon$ for some $\varepsilon > 0$.

Since $x$ is $WM$ point, and $M \in \nabla_2$, there is a supporting functional $y \in S(L_N) = S(E_N)$ such that $\langle x_n, y\rangle \to 1$. Hence

$$\begin{aligned}
1 \leftarrow \|x_n\|^\circ &= \frac{1}{k_n}\big(1+\rho_M(k_n x_n)\big) = \frac{1}{k_n}\big(\rho_N(y)+\rho_M(k_n x_n)\big)\\
&= \frac{1}{k_n}\big[\rho_N(y|_{G\setminus e_n}) + \rho_M(k_n x_n|_{G\setminus e_n} + \rho_N(y|_{e_n}) + \rho_M(k_n x_n|_{e_n})\big]\\
&\geq \frac{1}{k_n}\int_{G\setminus e_n} y(t)k_n x_n(t))d\mu + \frac{\varepsilon}{\overline{\overline{k}}}\\
&= \int_G y(t)x_n(t)d\mu - \int_{e_n} y(t)x_n(t)d\mu + \frac{\varepsilon}{\overline{\overline{k}}}\\
&\geq \langle x_n, y\rangle - \|x_n\|^\circ \|y|_{e_n}\|_N + \frac{\varepsilon}{\overline{\overline{k}}}\\
&\to 1 + \frac{\varepsilon}{\overline{\overline{k}}}\,,
\end{aligned}$$

a contradiction.

**Sufficiency.** For $1 = \|x_n\|^\circ = \frac{1}{k_n}\big(1+\rho_M(k_n x_n)\big), \|x_n + x\|^\circ \to 2$, we shall consider three cases.

**(I).** $\rho_N\big(p_-(kx)\big) < 1 < \rho_N\big(p(kx)\big)$.

At first, by (v), we have

$$\lim_{\mu e\to 0}\ \sup_n\ \rho_M(k_n x_n|_e) = 0\,. \tag{$*$}$$

By (iii) it follows that

$$\begin{aligned}
&\int_{G\setminus(G_a\cup G_b)} N\big(p(kx(t))\big)d\mu + \int_{G_a} N\big(p(kx(t))\big)d\mu + \int_{G_b} N\big(p_-(kx(t))\big)d\mu \geq 1\,,\\
&\int_{G\setminus(G_a\cup G_b)} N\big(p_-(kx(t))\big)d\mu + \int_{G_a} N\big(p(kx(t))\big)d\mu + \int_{G_b} N\big(p_-(kx(t))\big)d\mu \leq 1\,.
\end{aligned}$$

Thus for $v(t)$ satisfying that $v(t) = p(kx(t))$ if $t \in G_a$ $v(t) = p_-(kx(t))$ if $t \in G_b; p_-(kx(t)) \leq v(t) \leq p(kx(t))$ if $t \in G\setminus(G_a \cup G_b)$ and $\rho_N(v) = 1$, clearly, $v$ is a supporting functional of $x$, we shall show $\langle x_n, v\rangle \to 1$.

Denote $E_i = \{t \in G : kx(t) \in [a_i, b_i]\}$ $(i = 1, 2, \ldots)$, and $E_0 = G\setminus \bigcup_{i=1}^{\infty} E_i$ where $[a_i, b_i], i = 1, 2, 3, \ldots$ is the set of affine segments of $M(\mu)$. For any $\varepsilon > 0$, by $(*)$, there exist $d > 0, e \subset G$ such that $\mu e < d$ and

$$\rho_M(k_n x_n|_e) < \varepsilon,\ \rho_M(kx|_e) < \varepsilon,\ \rho_N(v|_e) < \varepsilon\,.$$

Since $\sum_{i=1}^{\infty}\mu E_i \le \mu G < \infty$, choose $m$ so that $\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) < \frac{d}{3}$. Since $u \in [a_i, b_i], up(a_i) = M(u) + N(p(a_i))$, there is $\beta > 0$ such that for all $u \in [a_i - \beta, \quad b_i + \beta](i = 1, 2, \dots, m)$,

$$up(a_i) > M(u) + N(p(a_i)) - \varepsilon\,. \tag{5}$$

By Lemma 3, there is $\delta > 0$ such that for all $\lambda \in \Big[\frac{1}{1+\overline{k}}\,,\ \frac{k}{1+\overline{k}}\Big]$, $v \in [a_i, b_i]$, if $\lambda M(u)+(1-\lambda)M(v)-M(\lambda u+(1-\lambda)v) < \delta$, then $u \in [a_i-\beta, b_i+\beta]$ $(i = 1, 2, \dots, m)$.

As in (++), it is not difficult to prove that on $\bigcup_{i=1}^{m} E_i$

$$f_n(t) = \frac{k}{k+k_n}M(k_nx_n(t)) + \frac{k_n}{k+k_n}M(kx(t)) - M\Big(\frac{kk_n}{k+k_n}(x_n(t)+x(t))\Big) \xrightarrow{\mu} 0\,.$$

Denote $F_n = \Big\{t \in \bigcup_{i=1}^{m} E_i; f_n(t) \ge \delta\Big\}$, then for $n$ large enough, $\mu F_n < d$. Hence, for all $t \in \bigcup_{i=1}^{m} E_i \backslash F_n$, we have $kx(t) \in [a_i, b_i]$.

Combine with $\frac{1}{1+\overline{k}} \le \frac{k}{k+k_n}\,,\ \frac{k_n}{k+k_n} \le \frac{\overline{k}}{1+\overline{k}}\,,\ k_nx_n(t) \in [a_i - \beta, b_i + \beta]$. Thus, from (5), it yields

$$k_nx_n(t)p(a_i) > M(k_nx_n(t)) + N(p(a_i)) - \varepsilon\,. \tag{6}$$

Noticing that if $kx(t) = a_i; v(t) = p(kx(t)) = p(a_i)$; if $kx(t) = b_i\,,\ v(t) = p_-(kx(t)) = p_-(b_i) = p(a_i)$, if $a_i < kx(t) < b_i,\ v(t) = p(kx(t)) = p_-(kx(t)) = p(a_i)$, from (6), we get that for $t \in \bigcup_{i=1}^{m} E_i \backslash F_n$,

$$k_nx_n(t)v(t) > M(k_nx_n(t)) + N(v(t)) - \varepsilon\,. \tag{7}$$

By Lemma 6, it yields that $k_nx_n - kx \xrightarrow{\mu} 0$ on $E_0$, so there is $F_0 \subset E_0$ with $\mu F_0 < d$ such that for all $t \in E_0 \backslash F_0$,

$$\big|k_nx_n(t) - kx(t)\big| < \varepsilon\,, \quad \big|M(k_nx_n(t)) - M(kx(t))\big| < \varepsilon\,.$$

Thus for $t \in E_0 \backslash F_0$,

$$\begin{aligned} k_nx_n(t)v(t) &> (kx(t) - \varepsilon)v(t) = M(kx(t)) + N(v(t)) - \varepsilon v(t) \\ &> M(k_nv_n(t)) + N(v(t)) - \varepsilon - \varepsilon v(t) \end{aligned} \tag{8}$$

By the boundedness of $\{k_n\}$, (7) and (8), it follows that

$$\begin{aligned}
\langle k_n x_n, v\rangle &= \left(\int_{\bigcup_{i=1}^m E_i\setminus F_n} + \int_{E_0\setminus F_0} + \int_{F_n\cup\big(\cup_{i>m}E_i\big)\cup F_0}\right) k_n x_n(t)v(t)d\mu \\
&\geq \int_{\bigcup_{i=1}^m E_i\setminus F_n} \big(M(k_n x_n(t)) + N(v(t)) - \varepsilon\big)d\mu \\
&\quad + \int_{E_0\setminus F_0} \big(M(k_n x_n(t) + N(v(t)) - \varepsilon - \varepsilon\, v(t)\big)d\mu \\
&\quad - \int_{F_n\cup(\cup_{i>m}E_i)\cup F_0} \big(M(k_n x_n(t)) + N(v(t))\big)d\mu \\
&= \left(\int_{\bigcup_{i=1}^m E_i\setminus F_n} + \int_{E_0\setminus F_0}\right) \big[M(k_n x_n(t)) + N(v(t))\big]d\mu + 0(\varepsilon) \\
&= \int_G \big[M(k_n x_n(t)) + N(v(t))\big]d\mu + 0(\varepsilon) \\
&= 1 + \rho_M(k_n x_n) + 0(\varepsilon) = k_n + 0(\varepsilon)\,.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, we conclude that $\langle x_n, v\rangle \to 1$ and so $x$ is a $WM$ point.

**(II).** $\rho_N\big(p_-(kx)\big) < 1 = \rho_N\big(p(kx)\big)\,.$

In this case, $v(t) = p(kx(t))$ is the unique supporting functional of $x$ in $L_N$. By (iv), $\mu G_b = 0$ or $\rho_N\Big(p\big(\frac{kx}{1-\tau}\big)\Big) < \infty$ for some $\tau > 0$. (If $kx(t)$ is a left extreme point of an affine segment of $M(u)$, it is also a right extreme point of another segment. Let $t \in G_n$).

If $\mu G_b = 0$. Using the argument similar to that of (I), it follows that $\langle x_n, v\rangle to 1$, i.e $x$ is a$WM$ point.

If $\mu G_b > 0$, and $\rho_N\Big(p\big(\frac{kx}{1-\tau}\big)\Big) < \infty$ for some $\tau > 0\,.$

We only need to show that $k_n x_n - kx \xrightarrow{\ \mu\ } 0$ on $G_b$. Then, similarly to (I), $\langle x_n, v\rangle \to 1$.

We first show that for some $0 < \theta < 1\,,$

$$\lim_{\mu e\to 0}\sup_n \rho_N\left(p\Big((1+\theta)\frac{kk_n}{k+k_n}(x + x_n|_e)\Big)\right) = 0\,. \qquad (**)$$

Observe that for $0 < \theta < 1\,,$

$$\begin{aligned}
M(u) > \int_{(1-\theta)u}^{u} p(t)d\mu \geq p\big((1-\theta)u\big)\theta u &= \frac{\theta}{1-\theta}(1-\theta)up\big((1-\theta)u\big) \\
&\geq \frac{\theta}{1-\theta} N\big(p((1-\theta)u)\big)
\end{aligned}$$

from $\lim_{\mu e\to 0}\sup_n \rho_M(k_n x_n)=0$, we get

$$\lim_{\mu e\to 0}\sup_n \rho_N\big(p(1+\theta)k_n x_n|_e)\big)=0\,.$$

On the other hand, for $\theta$ small enough, if $|k_n x_n(t)|\le \frac{kx(t)}{1-\tau/2}$, then

$$(1+\theta)\frac{kk_n}{k+k_n}|x(t)+x_n(t)|\le(1+\theta)\frac{kx(t)}{1-\tau/2}\le\frac{kx(t)}{1-\tau}\,;$$

and if $|k_n x_n(t)|>\frac{kx(t)}{1-\tau/2}$, then

$$(1+\theta)\frac{kk_n}{k+k_n}|x(t)+x_n(t)|\le(1+\theta)\left(1-\frac{\tau k_n}{\tau(k+k_n)}\right)k_n x_n\le(1-\theta')k_n x_n(t)\,.$$

From

$$\rho_N\left(p\Big((1+\theta)\frac{kk_n}{k+k_n}(x+x_n)|_e\Big)\right)\le\rho_N\left(p\Big(\frac{kx}{1-\tau}|_e\Big)\right)+\rho_N\big(p((1-\theta')k_n x_n|_e)\big)\to 0\,,$$

we conclude that $(**)$ holds.

Noticing that $b$ is the right extreme point of an affine segment of $M(u)$, not left one, analogously with the proof of Lemma 6, we can deduce that

$$\limsup_{n\to\infty} k_n x_n(t)\le kx(t)\quad\big(t\in G_b,\ \mu-a.e.\big)\,.$$

Hence

$$\limsup_{n\to\infty}\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)\le kx(t)\qquad\big(t\in G_b,\ \mu-a.e.\big)\,. \tag{9}$$

For any $\varepsilon>0$, by Lebesgue Theorem, there is $\eta>0$ so that $\rho_N\big(p((1+\eta)kx)\big)\le\rho_N(p(kx))+\varepsilon=1+\varepsilon$. For such $\varepsilon$ and $\eta$, by $(**)$, there is $d>0$, when $\mu e<d$,

$$\sup_n\rho_N\left(p\Big((1+\eta)\frac{kk_n}{k+k_n}(x+x_n)|_e\Big)\right)<\varepsilon\,. \tag{10}$$

Since $k_n x_n-kx\xrightarrow{\mu}0$ on $E_0$, $\frac{kk_n}{k+k_n}(x+x_n)-kx\xrightarrow{\mu}0$ on $E_0$.

Take $F_0\subset E_0$, $\mu F_0<d$ so that

$$\lim_{n\to\infty}\sup_{t\in E_0\setminus F_0}\left[\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)-kx(t)\right]=0\,. \tag{11}$$

Take $m$ so that $\mu\big(\bigcup_{i>m} E_i\big) < d$, and $h$ small enough

$$\mu\left\{\bigcup_{i=1}^{m}\big\{t\in E_i\backslash G_b : kx(t) > b_i - h\big\}\right\} < d\,.$$

Take $\varepsilon'$ small enough, so that $\frac{k}{k+k_n}(b_i+\varepsilon') + \frac{k_n}{k+k_n}(b_i - h) \le b_i - \frac{h}{2}$ $(i = 1, 2, \ldots, m)$.

By Lemma 3, it follows that there is $\delta > 0$ so that if $kx(t) \in E_i$ and

$$f_n(t) = \frac{k}{k+k_n}M(k_nx_n(t)) + \frac{k_n}{k+k_n}M(kx(t)) - M\left(\frac{kk_n}{k+k_n}(x_n(t)+x(t))\right) \ge \delta$$

then $k_nx_n(t) \in [a_i - \varepsilon', b_i + \varepsilon']$ $(i = 1, 2, 3, \ldots, m)$.

Since $f_n(t) \xrightarrow{\mu} 0$ on $\bigcup_{i=1}^{m} E_i$, for $n$ large enough, $\mu F_n = \mu\{t \in G : f_n(t) \ge \delta\} < d$. Denote $E_i' = \{t \in E_i\backslash G_b : kx(t) \ge b_i - h\}$ $(i = 1, 2, \ldots, m)$. Then for $n$ large enough, $k_nx_n(t) \le b_i + \varepsilon'$ whenever $t \in \bigcup_{i=1}^{m} E_i'\backslash F_n$ . Thus

$$\begin{aligned}\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) &\le \frac{k_n}{k+k_n}(b_i - h) + \frac{k}{k+k_n}(b_i+\varepsilon')\\ &\le b_i - \frac{h}{2}\ (i = 1, 2, \ldots, m,\ t \in E_i'\backslash F_n)\,.\end{aligned}$$

Take $\eta' \le \eta$ so that $(1+\eta')\Big(b_i - \frac{h}{2}\Big) \le b_i - \frac{h}{3}$ $(i = 1, 2, \ldots, m)$, then

$$(1+\eta')\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) \le b_i - \frac{h}{3}\ (i = 1, 2, \ldots, m,\ t \in E_i'\backslash F_n) \tag{12}$$

Combining the nondecrease and right-continuity of $p(u)$, from (9), (10) and (12), we have that for all $\overline{\eta} \le \eta'$,

$$\begin{aligned}&\limsup_{n\to\infty} N\left(p\Big((1+\overline{\eta})\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)\Big)\right)\\ &\le N\big(p\big((1+\overline{\eta})kx(t)\big)\big)\ \Big(t \in G_b \cup \big(E_0\backslash F_0 \cup \big(\bigcup_{i=1}^{m} E_i'\backslash F_n\big)\big)\Big)\,.\end{aligned}$$

Noticing that for $F_0, \cup_{i>m}E_i, F_n$ and $\bigcup_{i=1}^{m}(E_i\backslash G_b\backslash E_i')$, their measures can be arbitrarily small, by (11), we have that for all $\overline{\eta} \le \eta'$, and all $T_n \subset G$,

$$\limsup_{n\to\infty} \rho_N\left(p\Big((1+\overline{\eta})\frac{kk_n}{k+k_n}(x+x_n)|_{T_n}\Big)\right) \le \limsup_{n\to\infty} \rho_N\Big(\big((1+\overline{\eta})kx|_{T_n}\big)\Big) + 4\varepsilon\,. \tag{13}$$

If $k_nx_n - kx \overset{\mu}{\not\longrightarrow} 0$ on $G_b$, then $\frac{kk_n}{k+k_n}(x+x_n) - kx \overset{\mu}{\not\longrightarrow} 0$ over $G_b$ . From (9), for some $\theta > 0, \sigma > 0$ and a right extreme point $b$ ,

$$\mu H_n = \mu \left\{ t \in G_b : kx(t) = b\,,\ \frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) \le b - \theta \right\} \ge \sigma\,.$$

Take $\eta'' \le \eta'$ so that $(1+\eta'')(b-\theta) \le b - \frac{\theta}{2}$ . From (13) and Lemma 5

$$\begin{aligned}
1 &\le \limsup_{n\to\infty} \rho_N\left(p\Big((1+\eta'')\frac{kk_n}{k+k_n}(x+x_n)\Big)\right)\\
&\le \limsup_{n\to\infty}\left[\int_{G\setminus H_n} \rho_N\Big(p\big((1+\eta'')kx(t)\big)\Big)d\mu + N\Big(p\big(b-\frac{\theta}{2}\big)\Big)\mu H_n\right] + 4\varepsilon\\
&= \limsup_{\infty}\left[\int_{G\setminus H_n} \rho_N\Big(p\big((1+\eta'')kx(t)\big)\Big)d\mu + \big(p(b)\big)\mu H_n\right.\\
&\quad \left. - \Big(N\big(p(b)\big) - N\big(p_-(b)\big)\Big)\mu H_n\right] + 4\varepsilon\\
&\le \rho_N\Big(p\big((1+\eta'')kx\big)\Big) - \Big(N\big(p(b)\big) - N\big(p_-(b)\big)\Big)\sigma + 4\varepsilon\\
&\le 1 - \Big(N\big(p(b)\big) - N\big(p_-(b)\big)\Big)\sigma + 5\varepsilon\,.
\end{aligned}$$

Since $\varepsilon$ is arbitrary, it leads to a contradiction, which show $k_nx_n - kx \overset{\mu}{\longrightarrow} 0$ on $G_b$.

**(III).** $\rho_N\big(p_-(kx)\big) = 1$ .

In this case, $v(t) = p_-\big(kx(t)\big)$ is the unique supporting functional of $x$ in $L_N$. We only need to show that $k_nx_n - kx \overset{\mu}{\longrightarrow} 0$ on $G_a$, then similar to (I), the proof is completed.

Since a is the left extreme point of an affine segment of $M(u)$, not right one, similarly to Lemma 6, we can deduce that

$$\liminf_{n\to\infty} k_nx_n(t) \ge kx(t) \qquad (t \in G_a,\ \mu - a.e.)\,.$$

Thus

$$\liminf_{n\to\infty} \frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) \ge kx(t), \qquad \big(t \in G_a, \mu - a.e.). \tag{14}$$

For any $\varepsilon > 0$, take $d > 0$ so that for all $e \subset G$ with $\mu e < d$, then

$$\rho_N\big(p_-(kx|_e)\big) < \varepsilon\,. \tag{15}$$

Take $\eta > 0$ so that $\rho_N\big(p_-\big((1-\eta)kx\big)\big)\rho_N\big(p_-(kx)\big) - \varepsilon = 1-\varepsilon$. Since $k_n x_n - kx \xrightarrow{\mu} 0$ on $E_0$, $\frac{kk_n}{k+k_n}(x+x_n) - kx \xrightarrow{\mu} 0$ on $E_0$.

Choose $F_0 \subset E_0, \mu F_0 < d$ so that

$$\lim_{n\to\infty} \sup_{t\in E_0\backslash F_0} \left[\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) - kx(t)\right] = 0\,. \tag{16}$$

Take $m$ so that $\mu\Big(\bigcup_{i=1}^m E_i\Big) < d$, and $h$ small enough, then

$$\mu\bigcup_{i=1}^m \big\{t \in E_i\backslash G_n : kx(t) < a_i + h\big\} < d\,.$$

Take $\varepsilon' > 0$ small enough so that $\frac{k}{k+k_n}(a_i - \varepsilon') + \frac{k_n}{k+k_n}(a_i + h) \geq a_i + \frac{h}{2}$ $(i = 1, 2, \ldots, m)$.

By Lemma 3, there is $\delta > 0$ so that if $f_n(t) < \delta$ and $kx(t) \in E_i$, then $k_n x_n(t) \in [a_i - \varepsilon', b_i + \varepsilon']$ $(i = 1, 2, \ldots, m)$.

Since $f_n(t) \xrightarrow{\mu} 0$, for $n$ large enough, we get that $\mu F_n = \mu\{t \in G : f_n(t) \geq \delta\} < d$. Denote $E_i' = \{t \in E_i\backslash G_n : kx(t) \geq a_i + h\}$ $(i = 1, 2, \ldots, m)$. Then for $n$ large enough, $k_n x_n(t) \geq a_i + \varepsilon'$ whenever $t \in \bigcup_{i=1}^m E_i'\backslash F_n$.

Take $\eta' \leq \eta$ so that $(1-\eta')\Big(a_i + \frac{h}{2}\Big) \geq a_i + \frac{h}{3}$ $(i = 1, 2, \ldots, m)$, then

$$(1-\eta')\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) \geq a_i + \frac{h}{3}\,,\ (t \in E_i'\backslash F_n)\,. \tag{17}$$

Combining the nondecrease and left continuity of $p_-(u)$, from (14), (16) and (17), we deduce that for all $\overline{\eta} \leq \eta'$,

$$\limsup_{n\to\infty} N\left(p_-(1-\overline{\eta})\frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big)\right) \geq N\big(p_-\big((1-\overline{\eta})kx(t)\big)\big)$$
$$\left(t \in G_n \cup \Big(E_0\backslash F_0 \cup \big(\bigcup_{i=1}^m E_i'\backslash F_n\big)\Big)\right).$$

Similarly to the above, from (15), we can show that for all $T_n \subset G$,

$$\begin{aligned}&\liminf_{n\to\infty} \rho_N\left(p_-\Big((1-\overline{\eta})\frac{kk_n}{k+k_n}(x+x_n)|_{T_n}\Big)\right)\\ &\geq \liminf_{n\to\infty} \rho_N\big(p_-\big((1-\overline{\eta})kx|_{T_n}\big)\big) - 4\varepsilon\,.\end{aligned} \tag{18}$$

If suppose that $k_nx_n - kx \overset{\mu}{\not\longrightarrow} 0$ on $G_a$, then$\frac{kk_n}{k+k_n}(x+x_n) - kx \overset{\mu}{\not\longrightarrow} 0$ on $G_a$.

From (14), it follows that for some $\theta > 0,\ \sigma > 0$ and a left extreme point of an affine segment of $M(u)$,

$$\mu H_n = \mu\left\{t \in G_a : kx(t) = a,\ \frac{kk_n}{k+k_n}\big(x(t)+x_n(t)\big) \geq a+\theta\right\} \geq \sigma\,.$$

Take $\eta'' \leq \eta'$ so that $(1-\eta'')(a+\theta) \geq a + \frac{\theta}{2}$. From (18) and Lemma 5 we have

$$\begin{aligned}
1 &\geq \liminf_{n\to\infty} \rho_N\left(p_-\Big((1-\eta'')\frac{kk_n}{k+k_n}(x+x_n)\Big)\right)\\
&\geq \liminf_{n\to\infty}\left[\int_{G\setminus H_n} N\Big(p_-\big((1-\eta'')kx(t)\big)\Big)d\mu + N\Big(p_-\big(a+\frac{\theta}{2}\big)\Big)\mu H_n\right] - 4\varepsilon\\
&\geq \liminf_{n\to\infty}\left[\int_{G\setminus H_n} N\Big(p_-\big((1-\eta'')kx(t)\big)\Big)d\mu + N\big(p(a)\big)\mu H_n\right] - 4\varepsilon\\
&= \liminf_{n\to\infty}\Bigg[\int_{G\setminus H_n} N\Big(p_-\big((1-\eta'')kx(t)\big)\Big)d\mu + N\big(p_-(a)\big)\mu H_n\\
&\qquad + \Big(N\big(p(a)\big) - N\big(p_-(a)\big)\Big)\mu H_n\Bigg] - 4\varepsilon\\
&\geq \rho_N\Big(p_-\big((1-\eta'')kx\big)\Big) + \Big(N\big(p(a)\big) - N\big(p_-(a)\big)\Big)\sigma - 4\varepsilon\\
&> 1 - \Big(N\big(p(a)\big) - N\big(p_-(a)\big)\Big)\sigma - 5\varepsilon\,.
\end{aligned}$$

By the arbitrariness of $\varepsilon$, it leads a contradiction, which shows $k_nx_n - kx \overset{\mu}{\longrightarrow} 0$ on $G_a$. □

**Corollary**

*$L_M^\circ$ possesses a $WM$ property if and only if $M \in \Delta_2 \cap \nabla_2$ and all extreme points of the affine segments of $M(u)$ are continuous points of $p(u)$.*

*Proof.* **Sufficiency**. Let $x \in S(L_M^\circ)$.

First, (i) holds. Then by $M \in \Delta_2 \cap \nabla_2$, (ii) holds. Since there is not any extreme point of affine segment of $M(u)$ which is not any continuous point of $p(u)$, it follows that $\mu G_a = 0$ and $\mu G_b = 0$, thus (iii) and (iv) hold.

If (v) fails, then there exist $1 = \|x_n\|^\circ = \frac{1}{k_n}\big(1+\rho_M(k_nx_n)\big)$ $\|x+x_n\|^\circ \to 2$, and $e_n \subset G,\ \mu e_n \to 0$, but $\rho_M\big(k_nx_n|_{e_n}\big) \geq \sigma$ for some $\sigma > 0$.

For any $\varepsilon > 0$, since $M \in \nabla_2$ and

$$\rho_M\left(\frac{kk_n}{k+k_n}(x+x_n)\right) \le \frac{k}{k+k_n}\rho_M(kx) + \frac{k_n}{k+k_n}\rho_M(kx) = \overline{k}$$

for $n$ large enough,

$$\rho_M\left(\frac{kk_n}{k+k_n}(x+x_n)|_{e_n}\right) \le \rho_M\left(\frac{kk_n}{k+k_n}x_n|_{e_n}\right) + \varepsilon$$

and for some $0 < \delta < 1$, it holds

$$M\left(\frac{kk_n}{k+k_n}u\right) \le (1-\delta)\frac{k}{k+k_n}M(k_n u)\,.$$

Hence

$$\begin{aligned}\rho_M\left(\frac{kk_n}{k+k_n}(x+x_n)|_{e_n}\right) &\le (1-\delta)\frac{k}{k+k_n}\rho_M(k_n x_n|_{e_n}) + \varepsilon \\ &\le \frac{k}{k+k_n}\rho_M(k_n x_n|_{e_n}) - \frac{\delta\sigma}{1+\overline{k}} + \varepsilon\,.\end{aligned}$$

which leads to a contradiction:

$$\begin{aligned}2 &= \lim_{n\to\infty}\|x+x_n\|^\circ = \lim_{n\to\infty}\frac{k+k_n}{kk_n}\left(1+\rho_M\Big(\frac{kk_n}{k+k_n}(x+x_n)\Big)\right) \\ &= \lim_{n\to\infty}\frac{k+k_n}{kk_n}\left[1+\rho_M\Big(\frac{kk_n}{k+k_n}(x+x_n|_{G\setminus e_n}\Big)+\rho_M\Big(\frac{kk_n}{k+k_n}(x+x_n)|_{e_n}\Big)\right] \\ &\le \lim_{n\to\infty}\frac{k+k_n}{kk_n}\left[1+\frac{k_n}{k+k_n}\rho_M\big(kx|_{G\setminus e_n}\big)+\frac{k}{k+k_n}\rho_M\big(k_n x_n|_{G\setminus e_n}\big)\right. \\ &\qquad \left.+\frac{k}{k+k_n}\rho_M\big(k_n x_n|_{e_n}\big)-\frac{\delta\sigma}{1+\overline{k}}+2\varepsilon\right] \\ &\le \lim_{n\to\infty}\frac{1}{k_n}\big(1+\rho_M(k_n x_n)\big)+\frac{1}{k}\big(1+\rho_M(kx)\big)-\frac{\delta\sigma}{1+\overline{k}}+2\varepsilon \\ &= 2-\frac{\delta\sigma}{1+\overline{k}}+2\varepsilon\,.\end{aligned}$$

The contradiction shows (v) holds. So $x$ is a $WM$ point.

**Necessity.** Clearly $M \in \nabla_2$. If $M \bar{\in} \Delta_2$, by [7], there is $x \in S(L_M^\circ)$ with $\rho_N\big(p(kx)\big) < 1$, (ii) fails, which contradicts that $x$ is a $WM$ point.

If for some $[a,b]$ with $p_-(b) < p(b)$ and $[a,b]$ is an affine segment of $M(u)$, then take disjoint subsets $E, D \subset G,\ a < c < b$ and $d > 0$ such that

$$\frac{N\big(p(c)\big) + N\big(p(b)\big)}{2}\mu E + N\big(p(d)\big)\mu D = 1\,.$$

Set

$$x = \frac{b|_E + d|_D}{k} \quad \text{where} \quad k = \|b|_E + c|_D\|^\circ\,.$$

Take disjoint $A, B \subset E$ with $\mu A = \mu B = \frac{1}{2}\mu E$, set

$$v = p(b)|_A + p(c)|_B + p(d)|_D\,,$$

then $\rho_N(v) = 1$. Hence

$$\begin{aligned}
1 \geq \langle x, v\rangle &= \frac{1}{k}\big(bp(b)\mu A + bp(c)\mu B + dp(d)\mu D\big)\\
&= \frac{1}{k}\left(M(b)\mu E + \frac{N\big(p(c)\big) + N\big(p(b)\big)}{2}\mu E + M(d)\mu D + N\big(p(d)\big)\mu D\right)\\
&= \frac{1}{k}\big(1 + \rho_M(kx)\big) \geq \|x\|^\circ = 1
\end{aligned}$$

and $k \in K(x)$. But on the other hand,

$$\rho_N\big(p(kx)\big) = N\big(p(b)\big)\mu E + N\big(p(d)\big)\mu D > 1$$

and

$$N\big(p_-(b)\big)\mu E + N\big(p(d)\big)\mu D = N\big(p(c)\big)\mu E + N\big(p(d)\big)\mu D < 1$$

which shows that (iii) fails. So $x$ is not a $WM$ point, a contradiction.

Analogously, we can show that all $a \in \{a\}$ of left extreme points of affine segments of $M(u)$ are continuous points of $p(u)$. $\square$

## References

1. B.B. Panda and O.P. Kapoor, A generalization of local uniform convexity of the norm, *J. Math. Anal. Appl.* **52** (1975), 300–308.
2. F. Sullivan, A generalization of uniformly rotund Banach spaces, *Canad. J. Math.* **31** (1979), 628–646.
3. Nan Chaoxun and Wang Jianhua, On the $LKUR$ and $LKR$ spaces, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 521–526.
4. Wang Jianhua and Wang Musan, Fully $k$ convexity and locally fully $k$ convexity, to appear.
5. Shutao Chen and Yanzheng Duan, $WM$ property of Orlicz spaces, *Northeast. Math. J.* **8**(4) (1992), 498–502.
6. Shutao Chen, Yanming Lu and Baoxiang Wang, $WM$ property and $LKR$ of Orlicz sequence spaces, *Fasc. Math.* **4** (1991), 19–25.
7. Yanzheng Duan, *Several Structural Problems of Orlicz spaces,* Ph.D Dissertation, HIT, 1990.
8. C. Wu and T. Wang, *Orlicz spaces and its applications,* Heilongjiang Sci. & Tech. Press, Harbin, 1983.
9. T. Wang and Z. Shi, On the locally uniformly weak star Rotundity of Orlicz space, *Rev. Mat. Univ. Complut. Madrid* **7** (1994), 79–98.