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On convex functions in $c_0(\omega_1)$

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Abstract

It is proved that no convex and Fréchet differentiable function on $c_0(\omega_1)$, whose derivative is locally uniformly continuous, attains its minimum at a unique point.

It is well-known that $c_0(\omega_1)$ admits an equivalent norm that is simultaneously LUR and Fréchet differentiable. In fact, norms sharing these properties form a residual set in the space of all equivalent norms on $c_0(\omega_1)$. There are also constructions of equivalent LUR norms on $c_0(\omega_1)$ that can be approximated by C^∞-smooth norms.

On the other hand, spaces that admit an equivalent LUR and C^2 Fréchet differentiable norm are automatically superreflexive. For these results and further information on these matters we refer the reader to [1].

It is therefore a natural question (posed e.g. in [1]) whether there exists an equivalent rotund norm on $c_0(\omega_1)$ having properties of some higher order of smoothness. We answer this question in the negative, by showing that there is no equivalent rotund and Fréchet differentiable norm on $c_0(\omega_1)$ whose derivative is locally uniformly continuous.

We denote the canonical norms on $c_0(\omega_1)$ and $l_1(\omega_1)$ respectively by $\|\cdot\|_\infty$ and $\|\cdot\|_1$. By e_λ and f_λ respectively we mean the λ-th unit vector in the canonical basis of $c_0(\omega_1)$ and $l_1(\omega_1)$. For $x \in c_0(\omega_1)$ or $l_1(\omega_1)$ we denote by $\mathrm{supp}(x)$ the support of x, that is for $x = \sum_{n=1}^\infty x_n e_{\lambda_n}$, $\mathrm{supp}(x) = \{\lambda_n\}_{n\in\mathbb{N}}$. Using the natural well-ordering of ω_1 we introduce the supremum sup and infimum inf of subsets of ω_1. For $x \in c_0(\omega_1)$ or $l_1(\omega_1)$ we denote by $\bar{x} = \sup \mathrm{supp}(x) + 1$. For $x_1, \ldots, x_N$ in $c_0(\omega_1)$ satisfying:

$$\bar{x}_1 < \inf \mathrm{supp}(x_2), \ldots, \bar{x}_{N-1} < \inf \mathrm{supp}(x_N)$$

we introduce a new vector $y = (x_1, \dots, x_N)$ in $c_0(\omega_1)$ as the vector satisfying $y_\lambda = (x_i)_\lambda$ for $\lambda \in \operatorname{supp}(x_i)$ and $y_\lambda = 0$ otherwise.

Whenever we write $(x_1, \dots x_N)$ we automatically assume that $x_1, \dots, x_N$ satisfy the above conditions.

A Fréchet differentiable real function f on $c_0(\omega_1)$ is said to have locally uniformly continuous derivative if every $x \in c_0(\omega_1)$ has a neighborhood where the Fréchet derivative f' is uniformly continuous.

Theorem

There is no convex and Fréchet differentiable function on $c_0(\omega_1)$ whose derivative is locally uniformly continuous and such that the function attains its minimum at a unique point.

Proof. We will proceed by contradiction. Let us suppose that f is convex, Fréchet differentiable function on $c_0(\omega_1)$ whose derivative is locally uniformly continuous and such that $f \geq 0$ and $f(x) = 0$ if and only if $x = 0$.

For arbitrary $1 > \varepsilon > 0$ we will find $\delta > 0$ such that for arbitrary $\tau > 0$ there exist $x_1, x_2 \in c_0(\omega_1)$ satisfying:

$$\|x_1\|_\infty < \varepsilon, \ \|x_1 - x_2\|_\infty < \tau \quad \text{and} \quad \|f'(x_1) - f'(x_2)\|_1 > \frac{\delta}{4}\,.$$

That is a contradiction that implies the statement of Theorem.

Step 1. The construction of an increasing transfinite sequence $S = \{s_\lambda\}_{\lambda \in \omega_1} \subset \omega_1$ such that:

$$f\big((y_1, y_2)\big) \geq f(y_1) \tag{1}$$

for arbitrary y_1, y_2 satisfying $\operatorname{supp}(y_1) \subset S$ and $\operatorname{supp}(y_2) \subset S$ $\big($and of course $\bar{y}_1 < \inf \operatorname{supp}(y_2)\big)$.

We proceed by transfinite induction. Put $s_1 = 1$. Inductive step: Suppose we have constructed $\{s_\lambda\}_{\lambda < \lambda_0}$ where $\lambda_0 < \omega_1$. Then s_{λ_0} is chosen to satisfy: $s_{\lambda_0} > \overline{f'(y)}$ for arbitrary y that is finitely supported by $\{e_{s_\lambda}\}_{\lambda < \lambda_0}$ with rational coordinates.

The existence of such $S = \{s_\lambda\}_{\lambda \in \omega_1}$ is clear. The validity of the desired inequality (1) is a result of the continuity of f'. Indeed, we have: $s_{\lambda_0} > \overline{f'(y)}$ for arbitrary $y \in c_0(\omega_1)$ where y_1 supported by $\{e_{s_\lambda}\}_{\lambda > \lambda_0}$. Consequently, $f\big((y_1, y_2)\big) - f(y_1) \geq f'(y_1)(y_2) = 0$ for y_2 supported by S.

From now on, all the vectors we are going to deal with are automatically assumed to be supported by S, so the inequality (1) holds true.

Step 2. The construction of functions $m_A(\alpha), m_A^O(\alpha)$.

We define for $A > 0,\ \alpha \in \omega_1$:

$$m_A = \sup_y f(y)$$

where $\alpha < \inf \operatorname{supp}(y), y$ is finitely supported and all for its nonzero coordinates are equivalent to A. The function $m_A(\alpha)$ is nonincreasing in α. It is well-known, that every nonincreasing real valued function on ω_1 is eventually constant. Therefore there exists $\alpha_A \in \omega_1$ such that $m_A(\alpha_A) = m_A(\beta)$ for $\beta > \alpha_A$.

Now suppose $O = \{o_\lambda\}_{\lambda\in\omega_1}$ is a transfinite sequence of nonempty finite subsets of S such that $\sup(o_\lambda) < \inf(o_\pi)$ for $\lambda < \pi$. We define a function:

$$m_A^O(\alpha) = \sup_y f(y)$$

where $\alpha < \inf \operatorname{supp}(y), y$ is finitely supported, all of its nonzero coordinates are equivalent to A and $\operatorname{supp}(y) \cap o_\lambda \neq 0$ implies $o_\lambda \subset \operatorname{supp}(y)$. Again, there exists an $\alpha_{O,A} \in \omega_1$ such that $m_A^O(\alpha_{O,A}) = m_A^O(\beta)$ for $\beta > \alpha_{O,A}$.

Step 3. Fix $a, 0 < a < \varepsilon$. Due to the uniqueness of the minimal point of f, we have $m_a(\alpha_a) > 0$. Choose a transfinite sequence $O = \{o_\lambda\}_{\lambda\in\omega_1}$ of finite subsets of ω_1 and $\{v_\lambda\}_{\lambda\in\omega_1}$ for vectors in $c_0(\omega_1)$ such that $\operatorname{supp}(v_\lambda) = o_\lambda$, all of the nonzero coordinates of v_λ are equivalent to a, and

$$\alpha_a < \inf(o_1)\,,$$

$$\sup(o_\lambda) < \inf(o_\pi) \quad \text{for} \quad \lambda < \pi\,,$$

$$\text{and} \quad f(v_\lambda) \geq \frac{m_a(\alpha_a)}{2} \quad \text{for} \quad \lambda \in \omega_1\,.$$

Due to the inequality (1), we also have:

$$f\big((v_{\lambda_1}, \dots, v_{\lambda_n})\big) \geq \frac{m_a(\alpha_a)}{2} \quad \text{for} \quad \lambda_1, \dots \lambda_n \in \omega_1\,.$$

Due to the convexity of f we have:

$$f'\big((v_{\lambda_1}, \dots, v_{\lambda_n})\big)\big((v_{\lambda_1}, \dots, v_{\lambda_n})\big) \geq f\big((v_{\lambda_1}, \dots, v_{\lambda_n})\big) \geq \frac{m_a(\alpha_a)}{2}\,.$$

Therefore:

$$\sum_{i\in\bigcup_{i=1}^n o_{\lambda_i}} \Big(f'\big((v_{\lambda_1}, \dots, v_{\lambda_n})\big)\Big)_i \geq \frac{m_a(\alpha_a)}{2a}$$

for arbitrary choice of $\lambda_1, \dots, \lambda_n$. We put $\delta = \frac{m_a(\alpha_a)}{2a}$. Choose $b, a < b < a(1+\tau) < 1$, and $\rho, \rho < \frac{(b-a)\delta}{8}$. According to Step 2, there exists $\alpha_{O,b} \in \omega_1$ such that $m_b^O(\beta) = m_b^O(\alpha_{O,b})$ for $\beta > \alpha_{O,b}$. Again, there exist a transfinite sequence $P = \{p_\psi\}_{\psi\in\omega_1}$ of finite nonempty subsets of ω_1 and a transfinite sequence of vectors $\{u_\psi\}_{\psi\in\omega_1}$ in $c_0(\omega_1)$ such that: $\mathrm{supp}(u_\psi) = p_\psi$, all nonzero coordinates of u_ψ are equivalent to b,

$$\alpha_{O,b} < \inf(p_1)\,,$$

$$\sup(p_\lambda) < \inf(p_\pi) \quad \text{for} \quad \lambda < \pi\,,$$

$$f(u_\psi) \geq m_b^O(\alpha_{O,b}) - a\rho\,,$$

and each p_ψ is a union of finitely elements of O.

Consider the vector u_{ω_0}, where ω_0 is the first infinite ordinal There exists a finite ordinal n such that:

$$\sum_{i\in p_n} \big|\big(f'(u_{\omega_0})\big)_i\big| < \rho\,.$$

Thus

$$f\Big(\big(\frac{a}{b}u_n, u_{\omega_0}\big)\Big) \geq f(u_{\omega_0}) + f'(u_{\omega_0})\Big(\frac{a}{b}u_n\Big) \geq f(u_{\omega_0}) - a\rho\,.$$

Also,

$$f\Big(\big(u_n, \frac{a}{b}u_{\omega_0}\big)\Big) \geq f(u_n) \geq f(u_{\omega_0}) - a\rho\,.$$

Put $y = \big(\frac{a}{b}u_n, \frac{a}{b}u_{\omega_0}\big)$. Then $\sum_{i\in p_n\cup p_{\omega_0}} \big(f'(y)\big)_i \geq \delta$. Thus either $\sum_{i\in p_n} \big(f'(y)\big)_i \geq \frac{\delta}{2}$ or $\sum_{i\in p_{\omega_0}} \big(f'(y)\big)_i \geq \frac{\delta}{2}$. Since the rest of the proof is the same in either of the cases, let us suppose the former case is true.

Thus $g = f'\big((\frac{a}{b}\, u_n, u_{\omega_0})\big)$. We have

$$\begin{aligned} f\Big(\big(\frac{a}{b}\, u_n, u_{\omega_0}\big)\Big) &\geq f(u_{\omega_0}) - a\rho \\ &\geq m_b^O(\alpha_{O,b}) - 2\rho \geq \Big(\big(\frac{a}{b}\, u_n, u_{\omega_0}\big)\Big) + g\Big(\frac{b-a}{b}\, u_n\Big) - 2\rho \\ &= f\Big(\big(\frac{a}{b}\, u_n, u_{\omega_0}\big)\Big) + (b=a)\sum_{i\in p_n} g_i - 2\rho\,. \end{aligned}$$

Thus $\sum_{i\in p_n} g_i \leq \frac{2\rho}{b-a} < \frac{\delta}{4}$. Altogether:

$$\|f'(y) - g\|_1 \geq \sum_{i\in p_n} |(f'(y) - g)_i| \geq \sum_{i\in p_n} \big(f'(y)\big)_i - \sum_{i\in p_n} g_i > \frac{\delta}{4}\,.$$

But

$$\left\| \left(\frac{a}{b}\, u_n, u_{\omega_0} \right) - \left(\frac{a}{b}\, u_n,\, \frac{a}{b}\, u_{\omega_0} \right) \right\|_\infty = (b-a) < \left(\frac{b-a}{a} \right) < \tau\,.$$

Putting $x_1 = \left(\frac{a}{b}\, u_n,\, \frac{a}{b}\, u_{\omega_0}\right), x_2 = \left(\frac{a}{b}\, u_n, u_{\omega_0}\right)$ finishes the proof. □

Corollary

There is no equivalent rotund and Fréchet differentiable norm on $c_0(\omega_1)$ *whose derivative is locally uniformly continuous.*

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References

1. R. Deville, G. Godefroy, V. Zizler, *Smoothness and Renormings in Banach Spaces,* Pitman Monographs and Surveys in Pure and Applied Mathematics **64** 1993.