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On the smoothness of Orlicz sequence spaces equipped with Orlicz norm*

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Abstract

We give a criterion of smoothness of Orlicz sequence spaces with Orlicz norm.

Let $x \neq 0$ be an element of a Banach space X, x is called a smooth point if it has the unique supporting functional $f \in X^*, \|f\| = 1$ and $f(x) = \|x\|$. X is smooth if and only if all elements ($\neq 0$) are smooth.

For Orlicz function spaces equipped with Orlicz norm and Luxemburg norm, and for Orlicz sequence spaces with Luxemburg norm, the criteria of smoothness were obtained by T. Wang, S. Chen, R. Grzasłewicz, H. Hudzik and others in [1-4] and [6-9]. But up to now no satisfied result has been seen for Orlicz sequence spaces equipped with Orlicz norm. Here we shall fill it.

In the sequel, M and N denote a pair of complemented N-functions, p and q their right-hand derivatives, respectively. $M \in \delta_2$ stands for that M satisfies δ_2-condition for small $u. N$ is strictly convex if $u \neq v$ implies

$$N\Big(\frac{u+v}{2}\Big) < \frac{N(u)+N(v)}{2}.$$

For a sequence $x = (x_j)$ we define a modular of x by $R_M(x) = \sum_{j=1}^{\infty} M(x_j)$. By an Orlicz sequence space l_M, generated by M, we understand

$$l_M = \big\{x = (x_j) : R_M(cx) < \infty \quad \text{for some} \quad c > 0\big\}$$

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and its subspace

$$h_M = \{x = (x_j) : R_M(cx) < \infty \quad \text{for all} \quad c > 0\}.$$

Both these spaces are equipped with Orlicz norm

$$\|x\| = \sup_{R_N(y)\le 1} \sum_{j=1}^{\infty} x_j y_j = \frac{1}{k}\left(1 + \sum_{j=1}^{\infty} M(kx_j)\right) \quad \text{for} \quad k \in K(x) = [k^*, k^{**}].$$

where $k^* = \inf\{k > 0 : R_N(p(kx)) \ge 1\}$, $k^{**} = \sup\{k > 0 : R_N(p(kx)) \le \}$ [1-5]. In the proof, we mainly use Theorem 2.55 of [5], as follows:

Lemma

Let $0 \ne x \in l_M$, x is a smooth point if and only if for any $k \in K(x)$, either A) $R_N(p_-(kx)) = 1$ or B-1) $\theta(kx) < 1$, and B-2) either $R_N(p(kx)) = 1$ or $\mu J_x \le 1$, where

$$J_x = \{j : p_-(k|x(j)|) < p(k|x(j)|)\},\ \theta(x) = \inf\left\{c > 0 : R_M\left(\frac{x}{c}\right) < \infty\right\}.$$

We denote $\pi_M = \inf\{c > 0 : N(p(c)) \ge \frac{1}{2}\}$. The main results are as follows:

Theorem 1

The following are equivalent:
(1) h_M is smooth;
(2) $p(u)$ is continuous over $[0, \pi_M)$, and $N(p_-(\pi_M)) = \frac{1}{2}$;
(3) $q(v)$ is strictly increasing on $[0, N^{-1}(\frac{1}{2}))$ (i.e, $N(v)$ is strictly convex on $[0, N^{-1}(\frac{1}{2})]$).

Proof. (1) $\Rightarrow$ (2).

Suppose that $p(u)$ has a discontinuous point in $(0, \pi_M)$, i.e, there is $0 < \alpha < \pi_M, p_-(\alpha) < p(\alpha)$. For $\varepsilon > 0$ small enough we have

$$2N(p_-(\alpha)) + (1+\varepsilon)(1 - 2N(p(\alpha))) < 1.$$

Denote

$$\beta = \inf\{b > 0 : N(p(b)) \ge (1+\varepsilon)(1 - 2N(p(\alpha)))\}.$$

set

$$x = (\alpha, \alpha, \beta, 0, 0, \ldots \quad \ldots).$$

By the right-continuity of $p(u)$ and $N\big(p(\alpha)\big) < \frac{1}{2}$, it follows that $R_N\big(p_-(x)\big) = 2N\big(p_-(\alpha)\big) + N\big(p_-(\beta)\big) \leq 2N\big(p_-(\alpha)\big) + (1+\varepsilon)\big(1 - 2N(p(\alpha))\big) < 1$, $R_N\big(p(x)\big) = 2N\big(p(\alpha)\big) + N\big(p(\beta)\big) \geq 2N\big(p(\alpha)\big) + (1+\varepsilon)\big(1 - 2N(p(\alpha))\big) > 1$, whence we get that $K(x) = \{1\}$. Since $\mu J_x \geq 2$, by the Lemma, x is not a smooth point.

Suppose now that $N\big(p_-(\pi_M)\big) < \frac{1}{2} \leq N\big(p(\pi_M)\big)$. Take $0 < s < \pi_M$ with

$$2N\big(p_-(\pi_M)\big) + N\big(p(s)\big) < 1$$

and set $x = (\pi_M, \pi_M, s, 0, 0, \ldots \ \ldots)$. Then $R_N\big(p_-(x)\big) = 2N\big(p_-(\pi_M)\big) + N\big(p_-(s)\big) < 1$, and

$$R_N\big(p(x)\big) = 2N\big(p(\pi_M)\big) + N\big(p(s)\big) \geq 1 + N\big(p(s)\big) > 1\,.$$

Hence $K(x) = \{1\}$. Since $\mu J_x \geq 2$, by the Lemma, x is not a smooth point.

(2) $\Rightarrow$ (3).

For $0 \leq v_1 < v_2 < N^{-1}\big(\frac{1}{2}\big)$, take $v_1', v_2', v_1 < v_1' < v_2' < v_2$. Since $p(u)$ is continuous over $[0, \pi_M)$ and $N\big(p_-(\pi_M)\big) = \frac{1}{2}$, i.e, $p_-(\pi_M) = N^{-1}\big(\frac{1}{2}\big)$, there exist $u_1' < u_2'$ satisfying $p(u_1') = v_1', p(u_2') = v_2'$.

Hence

$$q(v_1) = \sup_{p(u) \leq v_1} u \leq u_1' < u_2' \leq \sup_{p(u) \leq v_2} u = q(v_2)$$

i.e, $q(v)$ is strictly increasing over $\big[0, N^{-1}(\frac{1}{2})\big)$.

(3) $\Rightarrow$ (1).

Let $x \in h_M$, $k \in K(x)$. If $R_N\big(p_-(kx)\big) = 1$, by the Lemma, x is a smooth point. If $R_N\big(p_-(kx)\big) < 1$, we see from

$$1 > R_N\big(p_-(kx)\big) = \sum_{j=1}^{\infty} N\big(p_-(kx(j))\big)$$

that there is at most one $'j'$ with $N\big(p_-(kx(j))\big) \geq \frac{1}{2}$ $\big($i.e, $p_-(k|x(j)|) \geq N^{-1}(\frac{1}{2})\big)$.

Since $q(v)$ is strictly increasing over $\big[0, N^{-1}(\frac{1}{2})\big)$, we know that $k|x(i)|$ is a continuous point of $p(u)$ for all i with $p_-\big(kx(i)\big) < N^{-1}\big(\frac{1}{2}\big)$, thus $p_-\big(k|x(i)|\big) = p\big(k|x(i)|\big)$. Hence $\mu J_x \leq 1$. Clearly $\theta(kx) = 0 < 1$, by the Lemma, x is a smooth point. $\square$

Theorem 2

The following are equivalent:

(1) l_M is smooth;

(2) $p(u)$ is continuous over $[0, \pi_M), N\big(p_-(\pi_M)\big) = \frac{1}{2}$ and $M \in \delta_2$;

(3) $q(v)$ is strictly increasing over $\big[0, N^{-1}(\frac{1}{2})\big)$ and $M \in \delta_2$.

Proof. It is enough to derive $M \in \delta_2$ from (1). Suppose that $M \notin \delta_2$. Then there exist $u_n \searrow 0$ such that

$$u_n p(u_n) < \frac{1}{2^n} \quad \text{and} \quad p\Big(\big(1+\frac{1}{n}\big)u_n\Big) > 2^n p(u_n) \ (n = 1, 2, \ldots)\,.$$

Choose natural numbers m_n satisfying $\frac{1}{2^{n+1}} \le m_n u_n p(u_n) < \frac{1}{2^n}(n = 1, 2, \ldots)$, and define

$$x = \Big(\ \overbrace{u_1, \ldots\, u_1}^{m_1}, \overbrace{u_2, \ldots\, u_2}^{m_2}, \overbrace{u_3, \ldots\, u_3}^{m_3}, \ldots \quad \ldots\Big)\,.$$

Then, we have

$$\langle x, p(x)\rangle = \sum_{n=1}^{\infty} m_n u_n p(u_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1\,,$$

whence by $\langle x, p(x)\rangle = R_M(x) + R_N(p(x))$ we get $R_N(p(x)) < 1$ and $R_M(x) < 1$. Moreover $R_N(p_-(x)) < 1$.

Notice that for any $\lambda > 0$,

$$\begin{aligned} R_N\big(p((1+\lambda)x)\big) + R_M(x) &\ge \langle x, p\big((1+\lambda)x\big)\rangle = \sum_{n=1}^{\infty} m_n u_n p\big((1+\lambda)u_n\big) \\ &\ge \sum_{n \ge \frac{1}{\lambda}} m_n u_n p\Big(\big(1+\frac{1}{n}\big)u_n\Big) \ge \sum_{n \ge \frac{1}{\lambda}} m_n u_n 2^n p(u_n) \ge \sum_{n \ge \frac{1}{\lambda}} \frac{1}{2} = \infty\,, \end{aligned}$$

so we derive that $R_N\big(p((1+\lambda)x)\big) = \infty$, which shows $K(x) = \{1\}$.

We also have that for any $\lambda > 0$,

$$\begin{aligned} R_M\big((1+2\lambda)x\big) &= \sum_{n=1}^{\infty} m_n M\big((1+2\lambda)u_n\big) \ge \sum_{n=1}^{\infty} m_n \int_{(1+\lambda)u_n}^{(1+2\lambda)u_n} p(s)\, ds \\ &\ge \sum_{n=1}^{\infty} m_n \lambda u_n p\big((1+\lambda)u_n\big) \ge \frac{\lambda}{1+\lambda} R_N\big(p(1+\lambda)x\big) = \infty\,. \end{aligned}$$

So $\theta(x) = 1$ and by the Lemma, x is not a smooth point. □

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