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Nilpotent elements and solvable actions (*)

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Abstract

In what follows we shall describe, in terms of some commutation properties, a method which gives nilpotent elements. Using this method we shall describe, as in [9] the irreducibility for Lie algebras which have Levi-Malçev decomposition property.

In 1935 N. Jacobson gave in [5] "a development by rational methods of that part of the theory of Lie algebras which centers around the Lie-Engel theorems". The proofs are based on nilpotent property for some elements and subalgebras of the enveloping of a Lie algebra.

We have proved in [7], [2], [8] an infinite-dimensional analogue of Lie Theorem concerning irreducible finite dimensional representations of solvable Lie algebra. The proofs of this results are based as in [5] on a method which gives nilpotent elements.

In the same time D. L. Gurarii and Ju. I. Lubich have proved in [4] a theorem analogous to Lie's weight theorem. Using this result and the ideas of [7], [2], [8] D. L. Gurarii and M. Sabac in [3], [9] gives some characterizations of irreducible representations on a Banach space for Lie algebras which have Levi-Malçev decomposition property (see [9] Definition 2).

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In what follows we shall describe, in terms of some commutation properties, a method which gives nilpotent elements. Using this method we shall describe, as in [9] the irreducibility for Lie algebras which have Levi-Malçev decomposition property.

The paper has two parts. The first part contains algebraic results (concerning Lie subalgebras of an associative algebra). In the second part we give analogous results for Lie subalgebra of a normed associative algebra respectively for Lie algebra of operators on some complex Banach space.

1. Nilpotency and solvability for Lie subalgebras of an associative algebra

In the following $\mathcal{A}$ will be an associative algebra over complex field $\mathbb{C}$ (or a field of characteristic 0).

We denote in a standard way,

$$\operatorname{ad} a(b) = [a, b] = ab - ba, \quad \text{for} \quad a, b \in \mathcal{A}$$

DEFINITION 1. For $a, b \in \mathcal{A}$ we say that b *polynomially* commutes with a if there exist a nonconstant polynomial $P \in \mathbb{C}[X]$ so that $P(b)$ commute with a i.e.

$$\big[P(b), a\big] = 0.$$

DEFINITION 2. For $b \in \mathcal{M} \subset \mathcal{A}$ we say that b is *polynomial central* in $\mathcal{M}$, if b polynomially commutes with a for every $a \in \mathcal{M}$.

Remark. If $q \in \mathcal{A}$ is nilpotent, then q is polynomial central in $\mathcal{A}$.

Lemma 1

Let $x, q, a \in \mathcal{A}$ so that $[x, q] = a$. If $[q, a] = 0$ and q polynomially commutes with x, then a is nilpotent.

Proof. We have by induction $[x, q^n] = nq^{n-1}a, n \in \mathbb{N}$ because $[q, a] = 0$. Hence we can write

$$[x, P(q)] = P'(q)a, \quad \text{for} \quad P \in \mathbb{C}[X] \tag{1}$$

where P' denotes the derivative of P.

But q polynomially commutes with x. Then, there exists $P \in \mathbb{C}[X]$, $[x, P(q)] = 0$ and by (1) we have,

$$P'(q)a = 0. \tag{2}$$

If $P_1 \in \mathbb{C}[X],\ k \geq 1$ and $P_1(q)a^k = 0$, we can write

$$\left[x, P_1(q)a^k\right] = [x, P_1(q)]a^k + P_1(q)[x, a^k]$$

$$0 = P_1'(q)a^{k+1} + P_1(q)xa^k\,.$$

We multiply at left the last relation by a^k and we have:

$$P_1(q)a^k = 0 \Rightarrow P_1'(q)a^{2k+1} = 0$$

because $[a, q] = 0$. Hence by (2) we deduce,

$$P'(q)a = 0 \Rightarrow P''(q)a^{2+1} = 0 \ldots \Rightarrow n!a^{\sum_{i=0}^{n-1} 2^i} = 0$$

where n is the degree of P. Lemma 1 is proved. □

Remark. If $\mathcal{A}$ has finite dimension or q is nilpotent, then q polynomially commutes with x for every $x \in \mathcal{A}$. In this case the hypothesis of Lemma 1 is more simple and in this form Lemma can be found in [5] and [8].

Theorem 1

Let $\mathcal{L} \subset \mathcal{A}$ be a Lie subalgebra of $\mathcal{A}, Z_{\mathcal{L}}$ the center of $\mathcal{L}$. If I is a Lie ideal of $\mathcal{L}, \dim\ I < +\infty$ and $\operatorname{ad} x|I$ is nilpotent for any $x \in \mathcal{L}$, then one of the following assertions is true:
(i) $[q, I] = 0$ for every polynomial central q in $\mathcal{L}$.
(ii) There exists $q \neq 0, q$ nilpotent, $q \in I \cap Z_{\mathcal{L}}$.

Proof. Using the Engel's Theorem we can find $\{0\} = I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots \subset I_n = I$, I_k Lie ideal of $\mathcal{L}, \dim I_k = k$, $\operatorname{ad} x(I_k) \subset I_{k-1}$ for every $k = 1, \ldots n$ and $x \in \mathcal{L}$.
Obviously, $\operatorname{ad} x(I_1) = \{0\}$ for every $x \in \mathcal{L}$ and we have,
(i) $[q, I_1] = 0$ for every polynomial central q in $\mathcal{L}$
or
(ii) there exists $q \neq 0, q$ nilpotent, $q \in I_1 \cap Z_{\mathcal{L}}$.

Now, we suppose that (i) or (ii) holds for I_k i.e.

(i) $[q, I_k] = 0$ for every polynomial central q in $\mathcal{L}$

or

(ii) there exists $q \neq 0, q$ nilpotent, $q \in I_k \cap Z_{\mathcal{L}}$.

Obviously, if (ii) holds for I_k, then (ii) holds for I. It remains to study the case when (i) holds for I_k. In this case let a_{k+1} be so that $I_{k+1} = I_k + \mathbb{C}\, a_{k+1}$.

We can write

$$[q, a_{k+1}] = q_k \in I_k \quad \text{for every} \quad q \in \mathcal{L}$$

because $[q, I_{k+1}] \subset I_k$. If q is a polynomial central in $\mathcal{L}$, then $[q, q_k] = 0$ because I_k verifies (i).

Obviously, we have

(1) $\quad [q, a_{k+1}] = 0 \quad$ for every polynomial central q in $\mathcal{L}$

or

(2) $\quad$ there exists polynomial central q in $\mathcal{L}$ so that $[q, a_{k+1}] = q_k \neq 0$.

If we have (1), then (i) holds for I_{k+1}, because $I_{k+1} = I_k + \mathbb{C}\, a_{k+1}$ and I_k verifies (i). If we have (2), there exists polynomial central q in $\mathcal{L}$.

$$0 \neq [q, a_{k+1}] = q_k \in I_k, [q, q_k] = 0$$

and by Lemma 1 we deduce q_k nilpotent.

Hence, in the case (2) there exists $q_k \in I_k, q_k \neq 0, q_k$ nilpotent. For any $x \in \mathcal{L}$ we have $[x, q_k] = q_{k-1} \in I_{k-1}$ and one of the following is true:

α) $[x, q_k] = 0$ for every $x \in \mathcal{L}$ i.e. (ii) is true for I_k hence for I

β) there exists $x \in \mathcal{L}, [x, q_k] = q_{k-1} \neq 0$, q_k nilpotent.

I_k verifies (i), hence I_{k-1} verifies (i) because $I_{k-1} \subset I_k$.

We have $[q_k, q_{k-1}] = 0$, because q_k is nilpotent (hence polynomial central in $\mathcal{L}$). From β) and by Lemma 1 we deduce q_{k-1} nilpotent, $0 \neq q_{k-1} \in I_{k-1}$ and we can repeat the proof with α) and β) for q_{k-1} etc.

Clearly we obtain that (ii) holds for some $I_j, 1 \leq j \leq k$. Hence, in the case (2), (ii) holds for I. $\square$

Corollary

Let $\mathcal{L} \subset \mathcal{A}$ be a quasinilpotent Lie subalgebra of $\mathcal{A}$ (i.e. $\mathcal{L} = \sum I_\alpha, I_\alpha$ Lie finite dimensional nilpotent ideals of $\mathcal{L}$, see [8]). If $Z_{\mathcal{L}}$ (the center of $\mathcal{L}$) contains no nilpotent elements, then every polynomial central element in $\mathcal{L}$ is central in $\mathcal{L}$.

We can prove an analogue of Theorem 1 if the adjoint representation of $\mathcal{L}$ restricted to the ideal I of $\mathcal{L}$ is solvable. For this purpose, firstly we shall observe another nilpotent property in $\mathcal{A}$.

Lemma 2

Let $x, a \in \mathcal{A}$ so that $[x,a] = \lambda\, a,\ \lambda \neq 0,\ \lambda \in \mathbb{C}$.

If the set of all eigenvalues of $\operatorname{ad} x|\operatorname{span}\{a^n | n \in \mathbb{N}\}$ *is bounded, then a is nilpotent.*

Proof. We observe that $\lambda \neq 0$ and

$$[x, a^n] = n\lambda\ a^n \qquad n = 1, 2 \ldots \ \square$$

Definition 2. We say that $\mathcal{L} \subset \mathcal{A}$ *has the property* (m) if for every $x, a \in \mathcal{L}$ with $\operatorname{ad} x\big(\operatorname{span}\{a^n | n \in \mathbb{N}\}\big) \subset \operatorname{span}\ \{a^n | n \in \mathbb{N}\}$, the set of all eigenvalues of $\operatorname{ad} x|\operatorname{span}\{a^n | n \in \mathbb{N}\}$ is bounded.

Remark. $\mathcal{A}$ has the property (m), particularly every $\mathcal{L} \subset \mathcal{A}$ has the property (m) in the following cases:

1. $\dim \mathcal{A} < \infty$
2. $\mathcal{A} = \mathcal{B}(X)$ the algebra of all bounded linear operators on some Banach space.

Corollary

Let $\mathcal{L} \subset \mathcal{A}$ be a Lie subalgebra of $\mathcal{A}$ and $\mathcal{L}$ has the property (m). The following implication is true:

$$x, a \in \mathcal{L}\,,\ [x,a] = \lambda\, a\,,\ \lambda \neq 0 \Rightarrow a \quad \textit{nilpotent}.$$

Theorem 2

Let $\mathcal{L} \subset \mathcal{A}$ be a Lie subalgebra with property (m). If $I_1 \subset I_2 \subset \ldots \subset I_n \subset I_{n+1} \subset \ldots \subset \mathcal{L}$ is an increasing sequence of Lie ideals in $\mathcal{L}$ and $\dim I_n = n$ for every $n = 1, 2 \ldots$ then one of the following assertions holds for every n:

(i) $[q, I_n] = 0$ for every polynomial central q in $\mathcal{L}$.

(ii) There exists $q \neq 0, q$ nilpotent, $q \in I_n \cap Z_{\mathcal{L}}$.

(iii) There exists an $\mathcal{N}, 0 \neq \mathcal{N} \subset [\mathcal{L}, I_n]$, a commutative ideal of $\mathcal{L}$ so that every $a \in \mathcal{N}$ is nilpotent.

Proof. Let us consider the following assertion:

(γ) ad $x|I_n$ is nilpotent for every $x \in \mathcal{L}$.

If one of (γ), (ii), (iii) is true for I_n, we have by Theorem 1 for I_n one of (i), (ii), (iii). We shall prove by induction that for every n, one of (γ), (ii), (iii) holds for I_n.

We have $I_1 = \mathbb{C}a_1,\ a_1 \in \mathcal{L}\,[x, a_1] = \lambda(x)a_1, \lambda(x) \in \mathbb{C}$.

If $\lambda(x) = 0$ for every $x \in \mathcal{L}$, then (γ) holds for I_1.

If $\lambda(x) \neq 0$ for some $x \in \mathcal{L}$, then by Corollary Lemma 2, $\mathcal{N} = I_1$ verifies (iii).

Now let us suppose that one of (γ), (ii), (iii) holds for I_n. It will be proved the same for I_{n+1}. Obviously (ii) or (iii) for I_n implies (ii) or (iii) for I_{n+1}. Hence it remains to be study the case when (γ) holds for I_n, (ii) and (iii) are not true for any $k \leq n$.

Let $a_{n+1} \in \mathcal{L}$ be so that $I_{n+1} = I_n + \mathbb{C}\,a_{n+1}$. For any $x \in \mathcal{L}$ we can write

$$[x, a_{n+1}] = b_n + \lambda_{n+1}(x)a_{n+1},\ b_n \in I_n,\ \lambda_{n+1}(x) \in \mathbb{C}\,.$$

From (γ) we have (ad $x)^m|I_n = 0$ and

$$(\mathrm{ad}\,x)^{m+1}(a_{n+1}) = \lambda_{n+1}(x)(\mathrm{ad}\,x)^m(a_{n+1})\,.$$

If for every $x \in \mathcal{L}$, we have $\lambda_{n+1}(x) = 0$ or (ad $x)^m(a_{n+1}) \in I_n$ then (γ) holds for I_{n+1}. The following case remains to be studied:

$$\text{there exists } x_0 \in \mathcal{L},\ 0 \neq \mathrm{ad}\,x_0\Big[\big(\mathrm{ad}\,x_0\big)^m\big(a_{n+1}\big)\Big] = \lambda_{n+1}(x_0)(\mathrm{ad}\,x_0)^m(a_{n+1}) \notin I_n\,.$$

Hence $a'_{n+1} = (\mathrm{ad}\,x_0)^m(a_{n+1}) \neq 0$ is nilpotent in virtue of Corollary Lemma 2 and $a'_{n+1} \notin I_n$. But (iii) is not true for I_n and ad $x|I_n$ is nilpotent for every $x \in \mathcal{L}$; by Theorem 1 we have

q polynomially central in $\mathcal{L} \Rightarrow q$ commutes with I_n.

Particularly,

(*) $q \in \mathcal{L},\ q$ nilpotent $\Rightarrow q$ commutes with I_n.

Hence a'_{n+1} commutes with I_n and a'_{n+1} commutes with I_{n+1}, because $a'_{n+1} \notin I_n, a'_{n+1} \in I_{n+1}$ implies $I_{n+1} = I_n + \mathbb{C}\,a'_{n+1}$. We can write,

$q \in I_{n+1}$, q nilpotent $\Rightarrow q$ commutes with a'_{n+1}.

By (*) we deduce,

$q \in I_{n+1}$, q nilpotent $\Rightarrow q$ commutes with I_{n+1}.

Hence,

$$0 \neq a'_{n+1} \in \mathcal{N} = \big\{q | q \in [\mathcal{L}, I_{n+1}],\ q \text{ nilpotent}\big\} \subset Z_{I_{n+1}}\,.$$

If $n \in \mathcal{N}, x \in \mathcal{L}$ we have $[x, n] = n' \in I_{n+1}$ and $[n, n'] = 0$. By Lemma 1 n' is nilpotent. We deduce that $\mathcal{N}$ is a commutative Lie ideal of $\mathcal{L}$ because $[\mathcal{L}, I_{n+1}]$ is a Lie ideal of $\mathcal{L}$. Hence $\mathcal{N}$ is a commutative Lie ideal of $\mathcal{L}$, $0 \neq \mathcal{N} \subset [\mathcal{L}, I_{n+1}]$, consisting of nilpotent elements and (iii) holds for I_{n+1}. □

A nil-ideal is an ideal consisting of nilpotent elements. We remember also (see [9]) that $\mathcal{R}$ is quasisolvable Lie algebra, if $\mathcal{R} = \sum_{\alpha\in\Lambda} I_\alpha, I_\alpha$ finite dimensional solvable Lie ideals of $\mathcal{R}$. We have the following corollary.

Corollary

Let $\mathcal{R} \subset \mathcal{A}$ be a Lie subalgebra with property (m) and quasisolvable. If $\mathcal{R}$ contains no non-zero Lie nil-ideal then every polynomial central element in $\mathcal{R}$ is central.

We can prove as in [9] an analogue of this result for Lie subalgebra $\mathcal{U} \subset \mathcal{A}$ with property (m) and LM-decomposable.

We remember (see [9]) that a LM-decomposable Lie algebra is a Lie algebra $\mathcal{U} = \mathcal{R} + \mathcal{J}$, $\mathcal{R} = \sum_{\alpha\in\Lambda} I_\alpha$, I_α finite dimensional ideals of $\mathcal{U}$ and $\mathcal{J}$ is a Lie algebra so that every ideal of $\mathcal{J}$ is primitive. An ideal $\mathcal{P}$ of $\mathcal{J}$ is primitive if $\mathcal{P}$ contains any ideal I of $\mathcal{J}$ with $[I, I] \subset \mathcal{P}$.

Every finite dimensional Lie algebra is LM-decomposable (Levi-Malçev Theorem). Every ideally-finite Lie algebra ([11]) is LM-decomposable ([9]).

Now let us consider $\mathcal{U} \subset \mathcal{A}$ a Lie subalgebra of $\mathcal{A}$ with property (m) and LM-decomposable; $\mathcal{U} = \mathcal{R} + \mathcal{J}$, $\mathcal{R} = \sum_{\alpha\in\Lambda} I_\alpha$ as before. We denote

$$\mathcal{U}_\alpha = \operatorname{ad} \mathcal{U}\,|I_\alpha = \left\{\operatorname{ad} x|I_\alpha;\ x \in \mathcal{U}\right\},\ \mathcal{R}_\alpha = \operatorname{ad} \mathcal{R}\,|I_\alpha\,.$$

Because $\dim\, \mathcal{U}_\alpha < \infty, \dim \mathcal{R}_\alpha < \infty$ and $\mathcal{R}_\alpha$ is a solvable ideal of $\mathcal{U}_\alpha$, it is well-known (see [6], Corollary 2 Theorem 8 sect. 5 Chap II) that the nil-radical of the associative algebra generated by $\mathcal{U}_\alpha$ contains $[\mathcal{U}_\alpha, \mathcal{R}_\alpha]$. Then $[I_\alpha, \mathcal{U}]$ is finite dimensional nilpotent ideal of $\mathcal{U}$. We have,

$$\mathcal{B}_\alpha = [I_\alpha, \mathcal{U}] \subset I_\alpha$$

and the representation of $\mathcal{J}$ $g \longmapsto \operatorname{ad} g|\mathcal{B}_\alpha$ is semisimple, because every ideal of $\mathcal{J}$ is primitive.

By Weyl's Theorem $\mathcal{B}_\alpha$ splits into the direct sum

$$\mathcal{B}_\alpha = \mathcal{B}_\alpha^0 + \mathcal{B}_\alpha^1$$

$\mathcal{B}_\alpha^0, \mathcal{B}_\alpha^1$ vector spaces which are invariant to ad $\mathcal{J}|\mathcal{B}_\alpha^0$ and $\cap_{g\in\mathcal{J}} \ker(\operatorname{ad} g|\mathcal{B}_\alpha^1) = \{0\}$.

If Z_α is the center of $\mathcal{B}_\alpha$, by Theorem 1 we deduce that one of the following assertions holds:

1. $q \in Z_\alpha$ for every polynomial central $q \in \mathcal{B}_\alpha$.
2. There exists q nilpotent, $0 \neq q \in Z_\alpha$.

We discuss the following cases: $\mathcal{B}_\alpha^1 \neq \{0\}$, $\mathcal{B}_\alpha^1 = \{0\}$.

If $\mathcal{B}_\alpha^1 \neq \{0\}$, by Corollary Lemma 2 we deduce that $\mathcal{B}_\alpha^1$ contains nonzero nilpotent because $\mathcal{U}$ has property (m). In this case 1 or 2 proves that there exist nonzero nilpotent elements in Z_α the center of $\mathcal{B}_\alpha$. Then

$$N_\alpha = \big\{q \in Z_\alpha | q \text{ nilpotent}\big\} \neq 0$$

is an ideal in $\mathcal{U}$. This is an easy consequence of Lemma 1, because Z_α is an ideal in $\mathcal{U}$.

Therefore $\mathcal{B}_\alpha^1 \neq \{0\}$ implies the following assertion:

(N) there exists a finite dimensional commutative nonzero Lie ideal of $\mathcal{U}$ consisting of nilpotent elements.

If $\mathcal{B}_\alpha^1 = \{0\}$, we have $\mathcal{B}_\alpha = \mathcal{B}_\alpha^0$ and

$$\big[\mathcal{J}, [\mathcal{U}, I_\alpha]\big] = 0\,.$$

Since $\mathcal{R}$ is quasisolvable and $\mathcal{U}$ has the property (m), by the proof of Theorem 2 one of the following assertions is true for I_α:

a) ad $x|I_\alpha$ is nilpotent for every $x \in \mathcal{R}$

b) there exists $\mathcal{N}$ a commutative nil-ideal of $\mathcal{R}$, $0 \neq \mathcal{N} \subset [\mathcal{R}, I_\alpha]$

c) there exists $q \neq 0$, q nilpotent, $q \in Z_{\mathcal{R}} \cap I_\alpha$.

By Theorem 1 we deduce in the case (a) one of the following assertions:

(a1) $[q, I_\alpha] = 0$ for every polynomial central q in $\mathcal{R}$.

(a2) there exists $q \neq 0$, q nilpotent, $q \in Z_{\mathcal{R}} \cap I_\alpha$.

Therefore, if $\mathcal{B}_\alpha^1 = \{0\}$ one of (a1), (b), (c) is true. Obviously (b) $\Rightarrow$ (N), because $\mathcal{B}_\alpha^1 = \{0\}$ shows that $\mathcal{N}$ is an ideal of $\mathcal{U}$.

Also we have (c)$\Rightarrow$ (N), because

$$0 \neq \{q | q \text{ nilpotent}, \ q \in Z_{I_\alpha}\}$$

is a finite dimensional Lie ideal in $\mathcal{U}$, in virtue of Lemma 1.

Hence there exists two possibilities:

A) (N) is true.

B) $\big[\mathcal{J}, [\mathcal{U}, I_\alpha]\big] = 0$, ad $x|I_\alpha$ nilpotent for every $x \in \mathcal{R}$, $\mathcal{R}$ verifies (a1) for every $\alpha \in \Lambda$.
In the case B) we can eliminate the situation when $\mathcal{R}$ contains nilpotent elements. Indeed, by (a1) for any $\alpha \in \Lambda$ we can find $I = \sum_{k=1}^{p} I_{\alpha_k}$ a finite-dimensional ideal of $\mathcal{U}$ so that $Z_{\mathcal{R}} \cap I$ contains non zero nilpotent elements and by Lemma 1,

$$0 \neq \{q|q \;\text{ nilpotent},\;\; q \in Z_I\} = \mathcal{N}$$

verifies (N).

If $\mathcal{R}$ contains no non zero nilpotent elements, we obtain $[\mathcal{J}, I_\alpha] = 0$ for any $\alpha \in \Lambda$. Indeed, we can write by virtue of the semisimplicity of the representation ad $\mathcal{J}\ |I_\alpha, I_\alpha = I_\alpha^0 + I_\alpha^1$ ad $\mathcal{J}\ |I_\alpha^0 = 0$ and $\cap_{g\in\mathcal{J}} \ker(\text{ad } g|I_\alpha^1) = 0$. By Lemma 1 we deduce $I_\alpha^1 = 0$ so $[\mathcal{J}, I_\alpha] = 0$.

We have proved the following theorem:

Theorem 3

Let $\mathcal{U} \subset \mathcal{A}$ be a LM-*decomposable Lie subalgebra which has the property* (m). *If $\mathcal{U} = \mathcal{R} + \mathcal{J}$ is the decomposition of $\mathcal{U}$, then one of the following statements is true:*
(I) there exists $\mathcal{N}$ a finite dimensional commutative non zero ideal of $\mathcal{U}$, consisting of nilpotent elements.
(II) $[\mathcal{J}, \mathcal{R}] = 0$, $\mathcal{R}$ is a quasi nilpotent Lie algebra, $\mathcal{R}$ contains no non zero nilpotent elements and every polynomial central element in $\mathcal{R}$ is central in $\mathcal{R}$.

Corollary

Let $\mathcal{U} \subset \mathcal{A}$ be a finite dimensional Lie subalgebra and $\mathcal{U} = \mathcal{R} + \mathcal{J}$ its Levi-Malçev decomposition. If $\mathcal{U}$ contains no non zero commutative nil-ideals, then $[\mathcal{J}, \mathcal{R}] = 0$, $\mathcal{R}$ is a nilpotent Lie algebra, $\mathcal{R}$ contains no non zero nilpotent elements and every polynomial central element in $\mathcal{R}$ is central in $\mathcal{R}$.

2. Nilpotent and solvable Lie algebra of bounded operators on a complex Banach space

In that which follows $\mathcal{X}$ will be a complex Banach space, $\mathcal{B}(\mathcal{X})$ the algebra of all linear bounded operators on $\mathcal{X}$ with standard structure of Lie algebra given by

$$\text{ad } A(B) = [A, B] = AB - BA \quad \text{for} \quad A, B \in \mathcal{B}(\mathcal{X}).$$

For $T \in \mathcal{B}(\mathcal{X})$, $\sigma(T)$ will be the spectrum of T, $f(T)$ denotes the value of analytic functional calculus of T for $f \in \mathcal{O}\big(\sigma(T)\big)$. For every spectral set σ of T we denote $E(\sigma)$ the spectral projection of T associated to σ by analytic functional calculus of T.

Firstly we prove a useful formula for ad x, $x \in \mathcal{B}(\mathcal{X})$.

Lemma

Let $X, Q, A \in \mathcal{B}(\mathcal{X})$. If $[X,Q] = A$ and $[Q,A] = 0$, then $[X, f(Q)] = f'(Q)A$ for every $f \in \mathcal{O}\big(\sigma(Q)\big)$, where f' is the derivative of f.

Proof. Let be $f \in \mathcal{O}\big(\sigma(Q)\big)$ belong to analytic functional calculus of Q and γ an admissible curve for f such that

$$f(Q) = \frac{1}{2\pi i}\int_\gamma f(\lambda)R(\lambda,Q)d\lambda, \quad R(\lambda,Q) = (\lambda I - Q)^{-1}.$$

Obviously we have

$$[X, f(Q)] = \frac{1}{2\pi i}\int_\gamma f(\lambda)[X, R(\lambda,Q)]d\lambda$$
$$[X, R(\lambda,Q)] = -R(\lambda,Q)[X, \lambda I - Q]\ R(\lambda,Q) = R\,(\lambda,Q)\,[X,Q]\,R(\lambda,Q).$$

But Q commutes with A, hence we have

$$[X, R(\lambda,Q)] = [R(\lambda,Q)]^2 A$$

and

$$[X, f(Q)] = \frac{1}{2\pi i}\Big[\int_\gamma f(\lambda)[R(\lambda,Q)]^2 d\lambda\Big]A = \frac{1}{2\pi i}\int_\gamma f'(\lambda)\,R(\lambda,Q)d\lambda A = f'(Q)A$$

because $\frac{d}{d\lambda}R(\lambda,A) = -[R(\lambda,A)]^2$. $\square$

We can prove an analogue of Lemma 1 §1.

Lemma 1

Let $X, Q, A \in \mathcal{B}(\mathcal{X})$ be so that $[X,Q] = A$, $[Q,A] = 0$. If there exists an open set $V \supset \sigma(Q)$ and $f \in \mathcal{O}\ (V)$ so that $[X, f(Q)] = 0$, then we have:
$A|E(\sigma)\mathcal{X}$ is nilpotent for every spectral set σ in $\sigma(Q)$, $\sigma \subset W$, where W is a connected component of V so that $f|_W$ is nonconstant.

Proof. Let σ be a spectral set for Q. By the preceding Lemma we have,

$$[X, E(\sigma)] = e'_\sigma(Q)A = 0$$

where $e_\sigma \in \mathcal{O}\big(\sigma(Q)\big)$ so that $E(\sigma) = e_\sigma(Q)$.

It is well known that $E(\sigma)$ commutes with A and with every value $\varphi(Q)$ of analytic functional calculus of Q. Therefore $E(\sigma)\mathcal{X} = \mathcal{X}_\sigma$ is invariant to $X, A, \varphi(Q)$ for every $\varphi \in \mathcal{O}\big(\sigma(Q)\big)$.

Now we consider $V \supset \sigma(Q)$, $f \in \mathcal{O}(V)$, $[X, f(Q)] = 0$ and a non empty $\sigma = \sigma(Q) \cap W$, where W is a connected component of V so that $f|_W$ is non constant.

Let W_1 be a relative compact open set so that

$$W \supset W_1 \supset \sigma = \sigma(Q) \cap W.$$

Obviously f' has in W_1 a finite number of zeros because $f|_W$ is nonconstant. Therefore $f'|_{W_1} = Pg$ where $g \in \mathcal{O}(W_1)$, $g(z) \neq 0$ for every $z \in W_1$ and $P \in \mathbb{C}[X]$ is a polynomial.

We have by the preceding lemma,

$$0 = [X,\ f(Q)] = f'(Q)A.$$

It is well known that

$$f'(Q)|\mathcal{X}_\sigma = f'(Q|\mathcal{X}_\sigma) = g(Q|\mathcal{X}_\sigma)\ P(Q|\mathcal{X}_\sigma).$$

Hence

$$0 = f'(Q)A|\mathcal{X}_\sigma = g(Q|\mathcal{X}_\sigma)\ P(Q|\mathcal{X}_\sigma)A|\mathcal{X}_\sigma.$$

But $g(Q|\mathcal{X}_\sigma)$ is invertible because $g(z) \neq 0$ for $z \in \sigma$. Therefore $P(Q|\mathcal{X}_\sigma)\ A|\mathcal{X}_\sigma = 0$, $[X|\mathcal{X}_\sigma,\ Q|\mathcal{X}_\sigma] = 0$ and $[A|\mathcal{X}_\sigma,\ Q|\mathcal{X}_\sigma] = 0$. As in the last part of the proof of Lemma 1 §1 we deduce that $A|\mathcal{X}_\sigma$ is nilpotent. □

Corollary

Let $X, Q, A \in \mathcal{B}(\mathcal{X})$, $[X, Q] = A$, $[Q, A] = 0$. *If there exists an open set* $V \supset \sigma(Q)$, $f \in \mathcal{O}(V)$ *so that* $[X, f(Q)] = 0$ *and* f *is non constant on every* W *connected component of* V *with property* $W \cap \sigma(Q) \neq \emptyset$, *then* A *is nilpotent.*

We put the following definition

Definition 1. For $A, B \in \mathcal{B}(\mathcal{X})$ we say that B *analytically commutes with* A if there exists $f \in \mathcal{O}\big(\sigma(B)\big)$, f nonconstant on every connected open set D with $D \cap \sigma(B) \neq \emptyset$, so that $[A,\ f(B)] = 0$.

Obviously, the above Corollary may be rewritten as follows.

Corollary 1

Let $X, Q, A \in \mathcal{B}(\mathcal{X})$ so that $[X, Q] = A$, $[Q, A] = 0$. If Q analytically commutes with X, then A is nilpotent.

Remarks. 1) B polynomially commutes with $A \Rightarrow B$ analytically commutes with A.
2) $Q \in \mathcal{B}(\mathcal{X}),\ Q$ nilpotent $\Rightarrow Q$ analytically commutes with every $A \in \mathcal{B}(\mathcal{X})$.
3) Corollary 1 is an extension of Lemma 1 §1 when $\mathcal{A} = \mathcal{B}(\mathcal{X})$.

Definition 2. We say that $B \in \mathcal{M} \subset \mathcal{B}(\mathcal{X})$ is *analytic central in $\mathcal{M}$* if B analytically commutes with every $A \in \mathcal{M}$.

Remarks. 1) B polynomial central in $\mathcal{M} \Rightarrow B$ analytic central in $\mathcal{M}$.
2) $Q \in \mathcal{B}(\mathcal{X}),\ Q$ nilpotent $\Rightarrow Q$ analytic central in every $\mathcal{M}$ which contains Q.

The results of §1 can be extended for Lie subalgebras in $\mathcal{B}(\mathcal{X})$ if we change "polynomial central" by "analytic central" by using Corollary 1 §2 instead of Lemma 1 §1, and the following consequence of the property (m) of $\mathcal{B}(\mathcal{X})$, instead of Corollary Lemma 2 §1.

Corollary 2

If $X, A \in \mathcal{B}(\mathcal{X})$ and $[X, A] = \lambda A, \lambda \in \mathbb{C},\ \lambda \neq 0$, then A is nilpotent.

$\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ is topologically irreducible if there are no closed invariant to $\mathcal{L}$ subspaces different from 0 and $\mathcal{X}$. Also we mention the possibility to eliminate a part of conclusions of these results when the Lie subalgebras are topologically irreducible.

Theorem 1

Let $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ be a Lie subalgebra of $\mathcal{B}(\mathcal{X})$, $Z_{\mathcal{L}}$ the centre of $\mathcal{L}$. If I is a Lie ideal of $\mathcal{L}$, $\dim I < +\infty$ and ad $x|I$ is nilpotent for any $x \in \mathcal{L}$, then one of the following assertions is true:
(i) $[Q, I] = 0$ for every analytic central Q in $\mathcal{L}$.
(ii) There exists $Q \neq 0,\ Q$ nilpotent, $Q \in I \cap Z_{\mathcal{L}}$.

Obviously, if $0 \neq Q \in I \cap Z_{\mathcal{L}},\ Q$ nilpotent, then $0 \neq \ker Q \neq X$ is a closed subspace which is invariant to $\mathcal{L}$.

Corollary

Let $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ be a quasinilpotent Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $\mathcal{L}$ is topologically irreducible then every analytic central element of $\mathcal{L}$ is central in $\mathcal{L}$.

Theorem 2

Let $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ be a quasisolvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $I_1 \subset I_2 \subset \dots I_n \subset I_{n+1} \subset \dots \subset \mathcal{L}$ is an increasing sequence of Lie ideals in $\mathcal{L}$ and $\dim I_n = n$ for every $n = 1, 2, \dots,$ then one of the following statements holds for every n:
(i) $[Q, I_n] = 0$ for every Q analytic central in $\mathcal{L}$.
(ii) There exists $Q \neq 0,$ Q nilpotent, $Q \in I_n \cap Z_{\mathcal{L}}$.
(iii) There exists an $\mathcal{N},$ $0 \neq \mathcal{N} \subset [\mathcal{L}, I_n]$ a commutative Lie ideal of $\mathcal{L}$ so that every $Q \in \mathcal{N}$, is nilpotent.

An irreducible variant of this result is the following

Corollary

Let $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ be a quasisolvable Lie subalgebra of $\mathcal{B}(\mathcal{X})$. If $\mathcal{L}$ is topologically irreducible then the following statements are equivalent:
(ac) *Q is analytic central in $\mathcal{L}$.*
(c) *Q is central in $\mathcal{L}$.*

Proof. $\mathcal{L}$ quasisolvable means that $\mathcal{L} = \sum_{\alpha \in \Lambda} I_\alpha,$ I_α finite dimensional solvable ideals of $\mathcal{L}$. By the irreducibility of $\mathcal{L}$ we can eliminate in conclusion of Theorem 2 (ii) and (iii) because $\ker Q$ respectively $\cap_{Q \in \mathcal{N}} \ker Q$ are closed invariant subspaces to $\mathcal{L}$. □

Theorem 3

Let $\mathcal{U} \subset \mathcal{B}(\mathcal{X})$ be a LM-decomposable Lie subalgebra of $\mathcal{B}(\mathcal{X})$,$\mathcal{U} = \mathcal{R} + \mathcal{J}$ the decomposition of $\mathcal{U}$. One of the following statements holds:
(I) there exists $\mathcal{N}$ a finite dimensional commutative non zero Lie ideal of $\mathcal{U}$, consisting of nilpotent elements.
(II) $[\mathcal{J}, \mathcal{R}] = 0, \mathcal{R}$ is a quasinilpotent Lie algebra $\mathcal{R}$ contains no nonzero nilpotent operators and every analytic central element in $\mathcal{R}$ is central in $\mathcal{R}$.

Corollary

Let $\mathcal{U} \subset \mathcal{B}(\mathcal{X})$ be a finite dimensional Lie sub algebra of $\mathcal{B}(\mathcal{X})$. If $\mathcal{U}$ is topologically irreducible and $\mathcal{U} = \mathcal{R} + \mathcal{J}$ is the Levi-Malçev decomposition of $\mathcal{U}$, then $[\mathcal{J}, \mathcal{R}] = 0,$ $\mathcal{R}$ is a nilpotent Lie algebra, $\mathcal{R}$ contains no non zero nilpotent operators and every analytic central element in $\mathcal{R}$ is central in $\mathcal{R}$.

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