Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **47**, 1 (1996), 77–89

On the continuity of Bessel potentials in Orlicz spaces

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Received July 3, 1995. Revised November 14, 1995

Abstract

It is shown that Bessel capacities in reflexive Orlicz spaces are non increasing under orthogonal projection of sets. This is used to get a continuity of potentials on some subspaces. The obtained results generalize those of Meyers and Reshetnyak in the case of Lebesgue classes.

Introduction

In [3, 4, 2] we have introduced a theory of capacities in Orlicz spaces and began to study potentials in these spaces.

In this paper we continue to study some properties of potentials in Orlicz spaces. Hence we prove in theorem 1 that the capacities in reflexive Orlicz spaces are non increasing under orthogonal projection of sets. This allows us to describe sets for which the potentials are continuous. In particular, for Bessel potentials, we have more information about their differentiability. On the other hand for some special Lipschitzian maps, T, we show the following:

$$B'_{m,A}[T(X)] \leq CB'_{m,A}(X) \text{ (for diam } X \leq \rho).$$

Hence the null sets (for $B'_{m,A}$) are conserved by T. We show also, as in the case of Lebesgue classes the following

$$B'_{m,A}(P_{GH}X) \leq CB'_{m,A}(X)$$

for X such that diam $P_{GH}X \leq \rho$.

Here we define P_{GH} as the projection of $\mathbb{R}^N$ onto G, parallel to H, when $\mathbb{R}^N$ is the affine direct sum of G and H.

All the results obtained generalize those of Meyers in [11] and Reshetnyak [13] in the case of Lebesgue spaces.

1. Preliminaries

Let a be a function defined in $[0,+\infty[$ and satisfying the following:

i) $a(0)=0,\ a(t)>0$ if $t>0$ and $\lim_{t\to\infty}\ a(t)=+\infty$

ii) a is right continuous on $[0,\ +\infty[$

iii) a is increasing in $[0,\ +\infty[$.

For $t\in\mathbb{R}$, set:

$$A(t)=\int_0^{|t|}a(x)dx.$$

Then A is called N-function. Define a^* in $[0,\ +\infty[$ as

$$a^*(s)=\sup\{t : a(t)\le s\}.$$

a^* verifies also i), ii) and iii). One associates to a^* the N-function A^* defined, for all $t\in\mathbb{R}$, by:

$$A^*(t)=\int_0^{|t|}a^*(x)dx.$$

A^* is called the N-function conjugate to A. Let A be an N-function and Ω an open set in $\mathbb{R}^N$. We note $\mathcal{L}_A(\Omega)$ the set of measurable functions f, on Ω, such that

$$\int_\Omega A(f(x))dx<\infty.$$

This set is called an Orlicz class. We put

$$\rho(f,A,\Omega)=\int_\Omega A(f(x))dx$$

and if $\Omega=\mathbb{R}^N$,

$$\rho(f,A)=\int_{\mathbb{R}^N}A(f(x))dx.$$

Let A and A^* be two conjugate N-functions and f a measurable function defined almost everywhere in Ω. The number $\|f\|_{A,\Omega}$, or $\|f\|_A$ if there is no confusion, defined by

$$\|f\|_A = \sup\left\{ \int_\Omega |f(x)g(x)|dx : g \in \mathcal{L}_{A^*}(\Omega) \text{ and } \rho(g, A^*, \Omega) \leq 1\right\}$$

is called the Orlicz norm of f.

The set $L_A(\Omega)$ of measurable functions f, such that $\|f\|_A < \infty$ is called an Orlicz space. When $\Omega = \mathbb{R}^N$, we set L_A in place of $L_A(\mathbb{R}^N)$.

The Luxemburg norm $|||f|||_{A,\Omega}$, or $|||f|||_A$ if there is no confusion, is defined in $L_A(\Omega)$ by:

$$|||f|||_A = \inf\left\{ s > 0 : \int_\Omega A\left[\frac{f(x)}{s}\right] dx \leq 1\right\}.$$

Let A be an N-function. We say that A verifies the Δ_2 condition if there exists a constant $C > 0$ such that $A(2t) \leq C\,A(t)$ for all $t \geq 0$.

Recall that A verifies the Δ_2 condition if and only if $\mathcal{L}_A = L_A$. Moreover L_A is reflexive if and only if A and A^* satisfy the Δ_2 condition.

Let k be a positive and measurable function in $\mathbb{R}^N$, k is called a kernel. Let A be an N-function. For $X \subset \mathbb{R}^N$, we define

$$C_{k,A}(X) = \inf\{A(|||f|||_A) : f \in L_A^+ \text{ and } k * f \geq 1 \text{ on } X\}$$

$$C'_{k,A}(X) = \inf\{|||f|||_A : f \in L_A^+ \text{ and } k * f \geq 1 \text{ on } X\}$$

where $k * f$ is the usual convolution. The sign $+$ deals with positive elements in the considered space. Then $C'_{k,A}$ is a capacity in the ordinary sense and $C_{k,A} = A_0 C'_{k,A}$ is called A-capacity.

If a statement holds except on a set X where $C_{k,A}(X) = 0$, then we say that the statement holds $C_{k,A}$-quasi everywhere (abbreviated $C_{k,A}$-q.e or (k, A)-q.e if there is no confusion).

Let f and the elements of the sequence $(f_i)_i$ be real valued functions which are finite $C_{k,A}$-q.e. We say that $(f_i)_i$ converges $C_{k,A}$ quasi uniformly to f (in abbreviated $f_i \to f$ $C_{k,A}$-q.u) if:

$$\forall \varepsilon > 0,\ \exists X : C_{k,A}(X) < \varepsilon \ \text{ and } \ f_i \to f \ \text{ uniformly on } {}^cX.$$

We call a function f in L_A^+ such that $k*f \geq 1$ on X, a test function for $C'_{k,A}(X)$. Moreover, a test function, say f, for $C'_{k,A}(X)$ such that $C'_{k,A}(X) = |||f|||_A$ is called a $C'_{k,A}$-capacitary distribution of X and $k * f$ a $C'_{k,A}$-capacitary potential of X.

For the properties of $C'_{k,A}$ and $C_{k,A}$, see [3], and for the existence and uniqueness of a $C'_{k,A}$-capacitary distribution of a set, see [4].

M denotes the vector space of Radon measures. M_1 is the Banach space of measures, equipped with the norm $\|\mu\|$ = total variation of $\mu < \infty$.

F will stand for the σ-field of sets which are μ-measurable for all $\mu \in M_1^+$.

If $\mu \in M_1^+$, we say that μ is concentrated on X if $\mu(Y) = 0$ for all sets Y which are μ-measurable and such that $Y \subset^c X$.

Let A and A^* be two conjugate N-functions. For $X \in F$, we define

$$D_{k,A}(X) = \sup\{\|\mu\| : \mu \in M_1^+,\ \mu \text{ concentrated on } X \text{ and } \|k * \mu\|_{A^*} \leq 1\}$$

where $k * \mu$ is the convolution of k and μ defined by:

$$(k * \mu)(x) = \int k(x-y)\, d\mu(y).$$

A measure $\mu \in M_1^+$ such that μ is concentrated on X and $\|k * \mu\|_{A^*} \leq 1$ is called a test measure for $D_{k,A}(X)$. If in addition $D_{k,A}(X) = \|\mu\|$, we say that μ is a $D_{k,A}$-capacitary distribution and $k * \mu$ is called a $D_{k,A}$-capacitary potential for X.

For the properties of $D_{k,A}$, see [3,4].

Bessel kernel is of principal interest in this paper. As classical references, see [5,6,14].

For $m > 0$, the Bessel kernel, g_m, is most easily defined through its Fourier transform $F(g_m)$ as:

$$[F(g_m)](x) = (2\pi)^{-N/2}(1 + |x|^2)^{-m/2},$$

where

$$[F(f)](x) = (2\pi)^{-N/2} \int f(y)\, e^{-ixy}\, dy \quad \text{for } f \in L_1.$$

g_m is positive, in L_1 and verifies the equality

$$g_{r+s} = g_r * g_s.$$

In addition, we put

$$B_{m,A} = C_{g_{m,A}} \quad \text{and} \quad B'_{m,A} = C'_{g_{m,A}}.$$

If X is a locally compact set in $\mathbb{R}^N$ and $\tilde{X} = X \cup \{\tilde{x}\}$ its one point compactification, then we denote by $C_0(X)$ the Banach space of real continuous functions f on X normed by $\sup_x |f(x)|$, and such that $\lim_{x \to \tilde{x}} f(x) = 0$.

$C_c(X)$ will be the subspace of compact support functions in $C_0(X)$.

2. Continuity of potentials

Theorem 1

Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let k be a kernel on $\mathbb{R}^N$ which is spherically symmetric and non-increasing as $|x|$ increases. If S is an affine subspace of $\mathbb{R}^N$ and X a subspace of $\mathbb{R}^N$, then

$$C_{k,A}(P_S X) \leq C_{k,A}(X).$$

Proof. We begin by proving the theorem in the case when X is a compact set. Let ν be a test measure for $D_{k,A}(P_S X)$, ν is carried by $P_S\, X$. By the Hahn-Banach theorem there exists $\mu \in M^+(X)$ such that

$$P_S \mu = \nu.$$

Hence

$$\|\mu\| = \|\nu\|.$$

We must show that μ is a test measure for $D_{k,A}(X)$. From [8], if $f \in L_A$, then

$$\|f\|_A = \inf_{\beta > 0} \left\{ \beta^{-1} \left[1 + \int A(\beta f(x)) dx \right] \right\}.$$

Let $\beta > 0$ and ϕ_β the function defined on $[0, \infty]$ by: $\phi_\beta(x) = A^*(\beta x)$.
We remark that ϕ_β satisfies the conditions of [11, theorem 2]. Applying this theorem, we get

$$\int A^*[\beta(k * \mu)](x) dx \leq \int A^*[\beta(k * P_S \mu)](x) dx.$$

Hence

$$\forall \beta > 0,\ \beta^{-1} \left[1 + \int A^*[\beta(k * \mu)](x) dx \right] \leq \beta^{-1} \left[1 + \int A^*[\beta(k * P_S \mu)](x) dx \right].$$

This implies

$$\|k * \mu\|_{A^*} \leq \|k * P_S \mu\|_{A^*} \leq 1.$$

Then μ is a test measure for $D_{k,A}(X)$ and

$$\|\mu\| = \|\nu\| \leq D_{k,A}(X).$$

Hence

$$D_{k,A}(P_S X) \le D_{k,A}(X).$$

However, from [2], we have for analytic sets,

$$D_{k,A} = C'_{k,A}.$$

Whence

$$C_{k,A}(P_S X) \le C_{k,A}(X).$$

Now we consider the case when X is a countable union of compact sets. Then there exists an increasing sequence of compact sets, $(K_i)_i$ such that

$$X = \cup_i K_i.$$

Therefore

$$P_S X = \cup_i P_S K_i.$$

From [2] we deduce

$$\lim_i C_{k,A}(K_i) = C_{k,A}(X) \quad \text{and} \quad \lim_i C_{k,A}(P_S K_i) = C_{k,A}(P_S X).$$

Hence

$$C_{k,A}(P_S X) \le C_{k,A}(X).$$

Now we treat the general case. Let O be an open set containing X. Then

$$C_{k,A}(P_S X) \le C_{k,A}(P_S O) \le C_{k,A}(O).$$

Since $C_{k,A}$ is an outer capacity, (see [3, théorème 2]), we get

$$C_{k,A}(P_S X) \le C_{k,A}(X).$$

The proof is finished. □

Theorem 2

Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let k be a kernel on $\mathbb{R}^N$ which is spherically symmetric and non-increasing as $|x|$ increases. Further, suppose that k is locally Lebesgue integrable with $\lim_{|x|\to\infty} k(x) = 0$. Let S be an affine subspace of $\mathbb{R}^N$. Then

1) For $f \in L_A$ and $\varepsilon > 0$, there exists a closed set $F \subset S$ such that

$$C_{k,A}(S - F) < \varepsilon \quad \text{and} \quad k * f \in C_0(F + S^{\perp}).$$

Hence

$$k * f \in C_0(x + S^{\perp})\ C_{k,A}\text{-q.e in } S.$$

2) Let $(f_i)_i$ be a sequence convergent to f in L_A. Then there is a subsequence $(f_{i'})_{i'}$, such that given $\varepsilon > 0$, there exists a closed set $F \subset S$ with the property

$$C_{k,A}(S - F) < \varepsilon \quad \text{and} \quad k * f_{i'} \to k * f \text{ in } C_0(F + S^{\perp}).$$

Hence

$$k * f_{i'} \to k * f \text{ in } C_0(x + S^{\perp}) C_{k,A}\text{-q.e in } S.$$

Proof. 1) There exists a sequence $(f_i)_i$ in $C_c(\mathbb{R}^N)$ such that $f_i \to f$ in L_A.
It is clear that

$$k * f_i \in C_0(\mathbb{R}^N).$$

From [3, théorème 4] there is a subsequence $(f_{i'})_{i'}$ such that

$$k * f_{i'} \to k * f\ C_{k,A}\text{-q.u.}$$

Since $C_{k,A}$ is an outer capacity, (see [3, théorème 2]), there is an open set O such that

$$C_{k,A}(O) < \varepsilon \quad \text{and} \quad k * f_{i'} \to k * f \text{ uniformly on } {}^cO.$$

We define

$$F = S - P_S O.$$

Then 1) follows from theorem 1.

Part 2) is proved by practically the same argument. □

We apply theorem 2 to Bessel kernels and more precisely to their derivatives.

Lemma 1

Let A be an N-function and r, s two real numbers such that $0 < r < s$. Then for all X,

$$B_{r,A}(X) \le B_{s,A}(X).$$

Proof. Let f be a test function for $B_{s,A}(X)$. Then

$$g_s * f \geq 1 \text{ on } X.$$

This implies

$$g_{r^*}(g_{s-r} * f) \geq 1 \text{ on } X.$$

From [7 or 12] we have

$$\|\!|g_{s-r} * f|\!\|_A \leq \|\!|f|\!\|_A \ \|g_{s-r}\|_1.$$

Hence $g_{s-r} * f$ is a test function for $B_{r,A}(X)$ and since $\|g_{s-r}\|_1 = 1$, we get

$$B_{r,A}(X) \leq A(\|\!|f|\!\|_A).$$

Whence

$$B_{r,A}(X) \leq B_{s,A}(X).$$

The proof is finished. □

Theorem 3

Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let S be an affine subspace of $\mathbb{R}^N$ and $0 \leq s < m$. Then

1) For $f \in L_A$ and $\varepsilon > 0$, there exists a closed set $F \subset S$ such that

$$B_{m-s,A}(S - F) < \varepsilon$$

and

$$D^j(g_m * f) \in C_0(F + S^\perp) \text{ for all } j, |j| \leq s.$$

Hence for such j,

$$D^j(g_m * f) \in C_0(x + S^\perp) \ B_{m-s,A}\text{-q.e in } S.$$

2) Let $(f_i)_i$ be a sequence convergent to f in L_A. Then there is a subsequence $(f_{i'})_{i'}$ such that given $\varepsilon > 0$, there exists a closed set $F \subset S$ with the property

$$B_{m-s,A}(S - F) < \varepsilon$$

and

$$D^j(g_m * f_{i'}) \to D^j(g_m * f) \text{ in } C_0(F + S^\perp) \text{ for all } j, |j| \leq s.$$

Hence for such j,

$$D^j(g_m * f_{i'}) \to D^j(g_m * f) \text{ in } C_0(x + S^{\perp})B_{m-s,A}\text{-q.e in } S.$$

Proof. Let $S(\mathbb{R}^N)$ be the Schwartz space of rapidly decreasing C^{∞} functions and $f \in S(\mathbb{R}^N)$. For j such that $|j| < m$, we have

$$F[D^j(g_m * f)] = Cy^j(1 + |y|^2)^{-|j|/2}(1 + |y|^2)^{-(m-|j|)/2}F(f)$$

where C is a constant. We define

$$f^j = D^j(g_{|j|} * f).$$

From [7], the map $\phi_j : g \to g^j$ is continuous from L_A into L_A because A and A^* verify the Δ_2 condition. Furthermore (see [7])

$$D^j(g_m * f) = g_{m-|j|} * f^j.$$

Now, let $f \in L_A$. Then there is a sequence $(f_i)_i$ in $S(\mathbb{R}^N)$ such that

$$|f_i - f| \to 0 \text{ in } L_A.$$

Theorem 4 in [3] gives

$$g_m * |f_i - f| \to 0 \ B_{m,A}\text{-q.e.}$$

And from lemma 1, we get

$$g_m * |f_i - f| \to 0 \ B_{m-|j|,A}\text{-q.e.}$$

Another application of [3, théorème 4] gives

$$g_{m-|j|} * |f_i - f| \to 0 \ B_{m-|j|,A}\text{-q.e.}$$

On the other hand, from [5 or 6], there are constants C_1 and C_2 such that

$$|D^j[g_m * (|f_i - f|)]| \le |D^j g_m| * |f_i - f| \le C_1(g_m * |f_i - f|) + C_2(g_{m-|j|} * |f_i - f|).$$

Hence

$$D^j[g_m * (|f_i - f|)] \to 0 \ B_{m-|j|,A}\text{-q.e.}$$

Theorem 4 in [3] and the continuity of ϕ^j imply

$$g_{m-|j|} * f_i^j \to g_{m-|j|} * f^j \ B_{m-|j|,A}\text{-q.e.}$$

But

$$g_{m-|j|} * f_i^j = D^j(g_m * f_i) \to D^j(g_m * f) \ B_{m-|j|,A}\text{-q.e.}$$

Hence

$$D^j(g_m * f) = g_{m-|j|} * f^j \ B_{m-|j|,A}\text{-q.e.}$$

From lemma 1 we get that if $|j| \le s$, then

$$D^j(g_m * f) = g_{m-|j|} * f^j \ B_{m-s,A}\text{-q.e.}$$

The theorem follows by an application of theorem 2 to the kernel $g_{m-|j|}$. □

Theorem 4

Let A be an N-function and T be a one to one map of $\mathbb{R}^N$ onto itself. Suppose that T and its inverse T^{-1} satisfy a Lipschitz condition. Let ρ, $0 < \rho < \infty$, and $X \subset \mathbb{R}^N$ such that diam $X \le \rho$. Then there exists a constant C, independent of X such that

$$B'_{m,A}[T(X)] \le C\, B'_{m,A}(X).$$

Proof. Let x_0 be a fixed point in X. Since the capacity is invariant under translation, we can take $x_0 = 0$. Hence

$$X \subset B(0,\rho).$$

Let f be a test function for $B'_{m,A}(X)$ such that

$$|||f|||_A \le 2\, B'_{m,A}[B(0,\rho)].$$

Let θ be such that $\theta > 1$. θ will be fixed later. We pose

$$E(\theta,\rho) = \{y : |y| \ge \theta\rho\} \quad \text{and} \quad g_m(\,.\,,x) = g_m(x - .).$$

Hölder inequality in Orlicz spaces gives

$$\int_{E(\theta-1,\rho)} g_m(x-y) f(y) dy \le 2|||f|||_A \ |||g_m(\,.\,,x)|||_{A^*,E(\theta-1,\rho)}.$$

Let $S = 8\, B'_{m,A}[B(0,\rho)]$ and estimate the integral

$$I = \int_{E(\theta-1,\rho)} A^*[S g_m(x-y)] dy, \quad \text{for } x \in X.$$

We remark that

$$I \leq \int_{E(\theta,\rho)} A^*[Sg_m(y)]\, dy.$$

Using the following behavior of g_m near ∞, (see [14]),

$$Sg_m(t) \sim C'|t|^{(m-N-1)/2}\ e^{-|t|},$$

we can find y_0 such that for $|y| \geq y_0$,

$$Sg_m(y) \leq |y|^{-N-1}.$$

On the other hand, since $\lim_{t\to 0}\ A^*(t)/t = 0$, there is y_1 such that for $|y| \geq y_1$,

$$A^*(y^{-N-1}) \leq |y|^{-N-1}.$$

Choose $\theta > 1$ such that $\theta\rho \geq \sup(y_0, y_1)$. Then

$$I \leq C^m \int_{\theta\rho}^{\infty} A^*[Sg_m(y)]y^{N-1}dy \leq C'' \int_{\theta\rho}^{\infty} y^{-2}dy = C''(\theta\rho)^{-1}.$$

If we take $\theta\rho \geq C''$, we get

$$I \leq 1.$$

So

$$\|g_m(\,.\,,x)\|_{A^*,E(\theta-1,\rho)} \leq S^{-1}.$$

This implies

$$\int_{E(\theta-1,\rho)} g_m(x-y)f(y)dy \leq 1/2 \quad \text{for } x \in X.$$

Hence

$$\int_{B(0,(\theta+1)\rho)} g_m(x-y)2f(y)dy \leq 1 \quad \text{for } x \in X.$$

Set $h = 2\chi f$, where χ is the indicator function of $B(0,(\theta+1)\rho)$. Then

$$g_m * h \geq 1 \text{ on } X,$$

and thus

$$g_m * h(T^{-1}(x)) \geq 1 \text{ for } x \in T(X).$$

By a change of variables in the convolution we get

$$\int g_m[(L_T)^{-1}(x-z)]\ h(T^{-1}z)\ J_{T^{-1}}(z)dz \geq 1 \text{ for } x \in T(X)$$

where L_T is the Lipschitz constant for T and $J_{T^{-1}}$ is the Jacobean of T^{-1}.

Since $h(T^{-1}z) = 0$ for $z \notin B(0, (\theta+1)\rho)$, we will consider, for $x \in T(X)$, for the above integral, only points z such that

$$|x - z| \leq L_T(\theta + 2)\rho.$$

An asymptotic behavior of g_m in the neighborhood of zero (see [5] section 2) gives

$$g_m[(L_T)^{-1}x] \leq \kappa g_m(x) \quad \text{for } |x| \leq L_T(\theta+2)\rho,$$

where κ is a constant independent of x.

This implies that the function

$$z \to \kappa h(T^{-1}z)J_{T^{-1}}(z)$$

is a test function for $B'_{m,A}[T(X)]$ and we get

$$B'_{m,A}[T(X)] \leq \kappa' \|\|f\|\|_A$$

where κ' is a constant independent of x.

Hence there exists a constant C, independent of X such that

$$B'_{m,A}[T(X)] \leq C\, B'_{m,A}(X).$$

The theorem is proved. □

If $\mathbb{R}^N$ is the affine direct sum of G and H, we define P_{GH} as the projection of $\mathbb{R}^N$ onto G, parallel to H.

Theorem 5

Let A be an N-function such that A and A^* satisfy the Δ_2 condition. Let ρ, $0 < \rho < \infty$ and $X \subset \mathbb{R}^N$ such that diam $P_{GH}X \leq \rho$. Then there exists a constant C, independent of X such that

$$B'_{m,A}(P_{GH}X) \leq C\, B'_{m,A}(X).$$

Proof. The proof es identical with the Meyers one in [11]. For completeness, we give it.

The projection $P_{H^\perp}$ restricted to G has an affine extension to T, mapping $\mathbb{R}^N$ onto itself. Hence $P_{GH} = T\,P_{H^\perp}$. Applying successively theorem 4 and theorem 1, we obtain the result. □

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