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Vector-valued transference and maximal ergodic theory in UMD-valued function spaces

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Abstract

We prove two transference theorems for maximal convolution operators on vector-valued L^p-spaces. We then present applications of these results towards ergodic theory. In particular, let R be a distributionally controlled representation of G, a locally compact abelian group, acting on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$ where X is a Banach space while $(\Omega, \mathcal{F}, \mu)$ is an abstract measure space. We show that, for $p \in [1, \infty)$, if the associated representation $R^{(p)}$ acting on $L^p(\Omega, X)$ is strongly continuous, then $R^{(p)}$ transfers strong-type and weak-type bounds for maximal convolution operators from $L^p(G, X)$ to $L^p(\Omega, X)$. The transference theorems hold for any Banach space X; however when seeking ergodic theorems related to singular integral kernels we need to require that X satisfy the UMD condition introduced by D. Burkholder.

1. Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be an abstract measure space. For each $p \in [1, \infty)$, the Banach space of scalar-valued measurable functions f on Ω satisfying $\int_\Omega |f(\omega)|^p \, d\mu(\omega) < \infty$ will be denoted $L^p(\Omega,\mu)$ with norm given by $\|f\|_p = \big(\int\limits_\Omega |f|^p\,d\mu\big)^{1/p}$. For $p=\infty$, $L^\infty(\Omega,\mu)$ is the Banach space of scalar-valued, measurable, essentially bounded functions, i.e. ess sup $|f(\omega)| < \infty$ with norm $\|f\|_\infty =$ ess sup $|f(\omega)|$. In all cases, functions differing only on a set of measure zero are identified.

In this paper, G will always denote a locally compact abelian group with the group operation written additively, and λ will denote a fixed Haar measure on G. For every $p \in [1,\infty]$, we will use $L^p(G)$ to denote $L^p(G,\lambda)$.

Let X be an arbitrary Banach space with norm denoted $\|\cdot\|_X$ or simply $\|\cdot\|$. A function $f:\Omega\to X$ is strongly measurable if there exists a sequence of X-valued simple functions on Ω which converge to f in X-norm μ-a.e. For each $p\in[1,\infty]$, we write $f\in L^p(\Omega,\mu,X)$ whenever $f:\Omega\to X$ is a strongly measurable function such that $\|f(\cdot)\|_X \in L^p(\Omega)$.

Identifying functions that are equal μ-almost everywhere, $L^p(\Omega,\mu,X)$ is a Banach space with norm $\|f\|_p = \big\|\|f(\cdot)\|_X\big\|_{L^p(\Omega)}$. For each $p\in[1,\infty)$, we will often denote $L^p(\Omega,\mu,X)$ as $L^p(\Omega,X)$ or simply E^p. As in the case of scalar-valued functions, $L^p(G,X)$ stands for $L^p(G,\lambda,X)$.

Throughout the sequel, if A is a set, $1_A(\cdot)$ will denote the characteristic function of A. The symbols $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ will denote the set of positive integers, the additive group of integers, and the additive group of real numbers, respectively. We will use $\mathbb{T}$ to denote the multiplicative group of the unit circle in the complex plane $\mathbb{C}$.

Our goal in this paper is to extend the basic transference methods of Calderón [12], and Coifman and Weiss [13] to the setting of vector-valued function spaces. As in the setting of [12] and [13], we will give applications to ergodic theory generalizing recent results in [1]. We will also present the transference of singular integrals on certain Banach-valued L^p-spaces. Similar ideas were used in [7] to give a simple proof of a result of Bourgain [9], based on the transference of the boundedness of the Hilbert transform on L^p-spaces of functions with values in UMD spaces.

This paper relies heavily on the study we initiated in [6]. For the reader's convenience, we recall in Section 2 various results from [6] that are needed in the sequel. In Section 3, we prove our central transference results. Section 4 contains the applications.

2. Distributionally controlled representations

Let Y be a vector space over $\mathbb{R}$ or $\mathbb{C}$. The group of bijective linear mappings of Y to itself with composition for the group operation is denoted $\mathrm{Aut}(Y)$. A representation

R of the locally compact abelian group G on Y is a group homomorphism $R : G \to \mathrm{Aut}(Y)$, and it is customary to write $R(u)$ as R_u.

If $f : \Omega \to X$, the distribution function of f is defined for all $y > 0$ as

$$\phi(f : y) = \mu\big(\{\omega \in \Omega : \|f(\omega)\| > y\}\big)\,.$$

The following definition is motivated by [4, Definition (2.1)] for representations acting on scalar-valued functions.

Definition 2.1. We say a representation $u \mapsto R_u$ of G on $E^1 \cap E^\infty$ is μ-*distributionally controlled* if there exist positive constants c and α such that for all $u \in G$

(2.1.1) $\phi(R_u f : y) \le c\,\phi(f : \alpha y)\,,\quad$ and

(2.1.2) $\phi(\min\{\|R_u f\|, \|R_u g\|\} : y) \le c\,\phi(\min\{\|f\|, \|g\|\} : \alpha y)$

for all $y > 0$ and for all $f, g \in E^1 \cap E^\infty$. When there is no ambiguity regarding the measure μ, we say simply that the representation is distributionally controlled.

We studied fundamental properties of μ-distributionally controlled representations in [6]. For the reader's convenience, we now recall those here that we will need for our transference proofs.

Theorem 2.2

Suppose R is a μ-distributionally controlled representation satisfying (2.1) with constants c and α. Then for each $p \in [1, \infty)$, there exists an extension of a R to a representation $R^{(p)}$ of G on E^p such that

(2.2.1)
$$\sup_{u \in G} \|R_u^{(p)}\| \le c^{1/p} \alpha^{-1}\,.$$

Moreover, for each $u \in G$,

(2.2.2)
$$\phi(R_u^{(p)} f : y) \le c\,\phi(f : \alpha y)$$

for all $y > 0$ and for all $f \in E^p$.

Remarks 2.3. (i) The existence of $R^{(p)}$ and (2.2.1) are proved in [6] as Theorem (2.3). The inequality (2.2.2) appears as [6, Proposition (2.7)], but can also be easily obtained using basic properties of the distribution function.

(ii) The following theorem, proved as Theorem (2.9) in [6], provides a characterization of distributionally controlled representations in terms of more conventional notions without reference to the distribution function. This characterization also helps clarify the place of our applications in Section 4 within ergodic theory. The papers [15] and [18] provide examples in ergodic theory where group actions satisfying hypotheses similar to (2.4.1) and (2.4.2) are considered.

Theorem 2.4

Let $p \in [1, \infty)$, and suppose $u \mapsto S_u$ is a representation of G on E^p. Then there is a μ-distributionally controlled representation R of G such that $S = R^{(p)}$ if and only if S consists of separation preserving operators and there exist constants $C_p, C_\infty > 0$ such that for all $u \in G$,

(2.4.1) $\|S_u f\|_p \le C_p \|f\|_p$ *for all* $f \in E^p$;

(2.4.2) $\|S_u f\|_\infty \le C_\infty \|f\|_\infty$ *for all* $f \in E^p \cap E^\infty$

Given a distributionally controlled representation R acting on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$, we will consider a related representation $\widetilde{R}^{(p)}$. Although (2.5.1)-(2.5.3) are all needed in the sequel, (2.5.3) is particularly important for the transference arguments.

Proposition 2.5

Let R be a μ-distributionally controlled representation of G satisfying (2.1) with constants c and α. Then, there exists a μ-distributionally controlled representation $u \mapsto \widetilde{R}_u$ of G on $L^1(\Omega) \cap L^\infty(\Omega)$, such that the following hold:

(2.5.1) *$\widetilde{R}$ satisfies (2.1) with constants $c' = c\,\alpha^{-2}$ and $\alpha' = 1$;*

(2.5.2) *Given $p \in [1, \infty)$, $\alpha^{-1}\widetilde{R}^{(p)}$ dominates $R^{(p)}$ in the sense that for each $u \in G$,*

$$\|R_u^{(p)} f(\cdot)\| \le \alpha^{-1} \widetilde{R}_u^{(p)}(\|f(\cdot)\|)$$

for all $f \in E^p$ and almost everywhere on Ω.

(2.5.3) *Given $\{g_j\}_{j=1}^N \subset L^p(\Omega, X)$, for every $u \in G$,*

$$\max_{1 \le j \le N} \|g(\cdot)\| \le \alpha^{-1} \left| \widetilde{R}_{-u}^{(p)}\big(\max_{1 \le j \le N} \|R_u^{(p)} g_j(\cdot)\|\big) \right|$$

almost everywhere on Ω.

Proof. The details of the construction of $\widetilde{R}$ and the verification of (2.5.1) and (2.5.2) are carried out in several stages in section 2 of [6]. We prove the inequality (2.5.3) here. First observe that by (2.5.2) we have

$$\max_{1\le j\le N} \|g(\cdot)\| \le \alpha^{-1} \max_{1\le j\le N} \widetilde{R}^{(p)}_{-u}\big(\|R^{(p)}_u g_j(\cdot)\|\big)$$

for each $u \in G$. Now, (2.5.3) follows from [5, Theorem (2.19)] since the representation consists of separation preserving operators by Theorem (2.4). □

Remarks 2.6. (i) We will now provide an example of the construction described above. Let $G = \mathbb{Z}$ and let $\Omega = \mathbb{R}$ endowed with Lebesgue measure. For the Banach space, we take $X = L^q(\mathbb{T})$, $1 < q < \infty$, and define $T : X \to X$ by $Tg = \widetilde{g} + i\,\hat{g}(0)$ where $\hat{g}$ and $\widetilde{g}$ denote the Fourier transform and the harmonic conjugate of g respectively. Note that T is bounded on X by the M. Riesz Theorem, and we use $\|T\|_{q\to q}$ to denote the operator norm. We also define $\tau : \mathbb{R} \to \mathbb{R}$ by $\tau(t) = -2^{\mathrm{sgn}(t)}\,t$ for $t \ne 0$ and $\tau(0) = 0$. We now define an action of $G = \mathbb{Z}$ on $E^p = L^p(\mathbb{R}, X) = L^p(\mathbb{R}, L^q(\mathbb{T}))$ for $p \in [1,\infty)$ by letting $R^{(p)}_1(f)(t) = T\big(f(\tau(t))\big)$ almost everywhere on $\mathbb{R}$. More generally, we will let $R^{(p)}_j = \big(R^{(p)}_1\big)^j$ for all $j \in \mathbb{Z}$. Straightforward calculations show that for all $f \in E^p$, $R^{(p)}_2(f) = -f$ and $R^{(p)}_4(f) = f$. One can check that the representation satisfies (2.4.1) and (2.4.2) with $C_\infty = \|T\|_{q\to q}$ and $C_p = 2\|T\|^p_{q\to q}$. Since the representation is separation preserving, the representation $R^{(p)}\big|_{L^1(\mathbb{R},X)\cap L^\infty(\mathbb{R},X)}$ is distributionally controlled by (2.4). Using τ_j to denote the j-fold composition of τ, the corresponding representation $\widetilde{R}$ on $L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ would satisfy

$$\widetilde{R}_j\Big(\sum_{l=1}^m \gamma_l 1_{\delta_l}\Big) = \sum_{l=1}^m \gamma_l 1_{\tau_j(\delta_l)}$$

for each $j \in \mathbb{Z}$ whenever $\sum\limits_{l=1}^m \gamma_l 1_{\delta_l}$ is a scalar-valued simple function on $\mathbb{R}$.

(ii) We now consider the concept of strong continuity for a representation. Suppose $u \mapsto S_u$ is a representation of the locally compact abelian group G on $L^p(\Omega, X)$ for some $p \in [1,\infty)$. We say that S is strongly continuous if for every $f \in L^p(\Omega, X)$, the mapping $u \mapsto S_u f$ defines a continuous function of G into $L^p(\Omega, X)$. Theorem (3.10) in [6] shows that if $u \mapsto R_u$ is a μ-distributionally controlled representation of G on $E^1 \cap E^\infty$ such that $R^{(p_0)}$ is strongly continuous on E^{p_0} for some $p_0 \in [1,\infty)$, then $R^{(p)}$ is strongly continuous on E^p for every $p \in [1,\infty)$. Therefore, we can make the following definition.

Definition 2.7. Let $u \mapsto R_u$ be a distributionally controlled representation of G on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. Such a representation will be called *strongly continuous* provided $R^{(p)}$ is strongly continuous on E^p for some $p \in [1,\infty)$ (hence, for every $p \in [1,\infty)$).

3. The weak-type and strong-type bounds

Let $p \in [1,\infty)$ be fixed throughout this section. Suppose R is a strongly continuous, distributionally controlled representation of G on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. Given $k \in L^1(G)$, we have that straightforward estimates show that the following expression:

$$H_k^{(p)}(f) = \int_G k(u) R_{-u}^{(p)} f \, d\lambda(u) \tag{3.1}$$

defines a bounded operator on $L^p(\Omega, X)$. In fact, if $N_p(k)$ denotes the norm on $L^p(G, X)$ of the mapping $f \mapsto k * f$, one can show that $\|H_k^{(p)}\| \le N_p(k)\, c^{1/p} \alpha^{-1}$ by using an argument almost identical to the proof of the central transference result of [13]. However, we also wish to consider maximal operators of the form $f \mapsto \sup_j \|H_{k_j}^{(p)} f(\cdot)\|_X$ where $\{k_j\}_{j=1}^\infty \subset L^1(G)$. Analysis of these operators involves more delicate considerations.

Let $\{k_j\}$ be a finite or infinite sequence of functions in $L^1(G)$. We define $N_p^{(\mathrm{w})}(\{k_j\})$ to be the least $M \in [0,\infty]$ such that for all $y > 0$ and all $f \in L^p(G, X)$,

$$\lambda\left(\left\{u \in G : \sup_j \left\| \int_G k_j(u-v) f(v) d\lambda(v) \right\| > y\right\}\right) \le \left(\frac{M\|f\|_p}{y}\right)^p .$$

Note that for a finite sequence $N_p^{(\mathrm{w})}(\{k_j\}_{j=1}^N) \le \sum_{j=1}^N N_p^{(\mathrm{w})}(k_j) < \infty$.

Consider a finite sequence $\{k_j\}_{j=1}^N \subset L^1(G)$ such that for each $j \in \{1, \ldots, N\}$, k_j has compact support. Let K denote a compact subset of G containing $\bigcup_{j=1}^N \mathrm{supp}(k_j)$, and let V be any relatively compact open subset of G. The following technical lemma will permit us to consider the mapping $(u, \omega) \mapsto R_u^{(p)} f(\omega)$ as a jointly measurable mapping of $G \times \Omega$ into X which is necessary for the averaging arguments used in the sequel.

Lemma 3.2

Suppose R is a strongly continuous, distributionally controlled representation of G. Let $\{k_j\}_{j=1}^N$, K, and V be as in the preceding paragraph. Then for each

$f \in E^p$, there exists $\Omega_0 \subset \Omega$ a σ-finite measurable set and a strongly measurable function $F : G \times \Omega \to X$ such that

(3.2.1) $\|F(\cdot,\cdot)\| = 0$ *outside of* $(V - K) \times \Omega_0$;

(3.2.2) *for λ-almost all $u \in V - K$, $F(u,\omega) = (R_u^{(p)} f)(\omega)$ μ-a.e. on Ω;*

(3.2.3) *if $1 \le j \le N$, for all $s \in V$, $\int\limits_G k_j(u)F(s-u,\omega)\,d\lambda(u) = (R_s^{(p)} H_{k_j}^{(p)} f)(\omega)$ μ-a.e.*

Sketch of Proof. The proof of (3.2.1) and (3.2.2) parallels the case of $f \in L^p(\Omega)$ described in [8] and so will be omitted here. We show the proof of (3.2.3) in more detail since it varies from the proof used for the case of scalar-valued functions.

Fix $v_0 \in V$ and $j \in \{1,\dots,N\}$. By Hille's Theorem, [14, Theorem (II.2.6)], we have, $R_{v_0}^{(p)}\big(H_{k_j}^{(p)} f\big) = \int\limits_G k_j(u) R_{v_0-u}^{(p)} f\,d\lambda(u)$. Let $A \subset \Omega$ have finite measure and suppose B is any measurable subset of A. Since $\bigcup_{j=1}^N \operatorname{supp}(k_j) \subset K$, it follows that

$$\begin{aligned}
\int_B R_{v_0}^{(p)} H_{k_j}^{(p)} f(\omega)\,d\mu(\omega) &= \int_B \left(\int_K k_j(u)(R_{v_0-u}^{(p)} f)(\omega)\,d\lambda(u)\right) d\mu(\omega)\\
&= \int_K \int_B k_j(u)(R_{v_0-u}^{(p)} f)(\omega)\,d\mu(\omega)\,d\lambda(u)\\
&= \int_K \int_B k_j(u)F(v_0-u,\omega)\,d\mu(\omega)\,d\lambda(u) \qquad \text{(by (3.2.2))}\\
&= \int_B \left(\int_K k_j(u)F(v_0-u,\omega)\,d\lambda(u)\right) d\mu(\omega).
\end{aligned}$$

Fubini's theorem justifies each change of the order of integration above since B and K each have finite measure. Because this equality holds for every measurable subset of A, we infer that $\int\limits_G k_j(u)F(v_0-u,\omega)\,d\lambda(u) = (R_{v_0}^{(p)} H_{k_j}^{(p)} f)(\omega)$ μ-a.e. on A (see [14, Corollary (II.2.5)]). Because each of these functions is only nonzero on a σ-finite subset of Ω, it follows that these functions are equal almost everywhere on Ω. □

Theorem 3.3

Let R be a strongly continuous μ-distributionally controlled representation of G. Let $\{k_j\}$ be a finite or infinite sequence of functions in $L^1(G)$. Then, for $p \in [1,\infty)$, we have that for all $y > 0$ and all $f \in E^p$,

$$\phi\left(\sup_j \|H_{k_j}^{(p)} f\| : y\right) \le \left(\frac{c}{\alpha}\right)^2 \left(\frac{N_p^{(\mathrm{w})}(\{k_j\})\,\|f\|}{\alpha^2 y}\right)^p \tag{3.3.1}$$

where c and α are the constants from (2.1).

Proof. We first consider the case of a finite sequence, $k_1, \dots, k_N \in L^1(G)$ each having compact support. Using scalar multiples of f if necessary, it suffices to prove (3.3.1) for $f \in E^p$ and $y = 1$. Let K denote a compact subset of G such that $\bigcup_{j=1}^N \operatorname{supp}(k_j) \subset K$. Then, given $\epsilon > 0$, take V to be a relatively compact open neighborhood of the identity in G such that $\frac{\lambda(V-K)}{\lambda(V)} < 1 + \epsilon$. Such a neighborhood exists by [16, Lemma (18.12)].

Averaging $\mu\left(\left\{ \max_{1\le j\le N} \|R_s^{(p)} H_{k_j}^{(p)} f(\omega)\| > 1 \right\}\right)$ over $s \in V$ as in the proof of the scalar-valued case, [5, Theorem (4.1)], one finds that there exists $s \in V$ such that

$$\mu\left(\left\{ \max_{1\le j\le N} \|R_s^{(p)} H_{k_j}^{(p)} f(\omega)\| > 1 \right\}\right) \le \left(N_p^{(\mathrm{w})}\left(\{k_j\}_{j=1}^N\right)\right)^p (1+\epsilon)\frac{c}{\alpha^p}\|f\|_p^p . \tag{3.3.2}$$

Let $\widetilde{R}^{(p)}$ denote the corresponding representation of G on $L^p(\Omega)$ described in (2.5). By (2.5.3), for every $u \in G$ the following relation holds μ-a.e. on Ω,

$$\max_{1\le j\le N} \|H_{k_j}^{(p)} f(\cdot)\| \le \alpha^{-1} \widetilde{R}_{-u}^{(p)} \left(\max_{1\le j\le N} \|H_{k_j}^{(p)} R_u^{(p)} f\| \right)(\cdot) . \tag{3.3.3}$$

Take $s \in V$ such that (3.3.2) holds for $\alpha^{-1} f$. Using (3.3.3) for this s gives

(3.3.4)
$$\mu\big(\big\{ \max_{1\le j\le N} \|H_{k_j}^{(p)} f(\omega)\| > 1\big\}\big) \le \mu\big(\big\{ \big| \widetilde{R}_{-s}^{(p)}\big(\max_{1\le j\le N} \|H_{k_j}^{(p)} R_s^{(p)} f(\omega)\|\big)\big| > \alpha\big\}\big) .$$

The following estimates are now easily obtained by applying (2.2), (3.3.4), and the criterion for choosing s:

$$\begin{aligned}
\mu\big(\big\{ \max_{1\le j\le N} H_{k_j}^{(p)} f(\omega)\| > 1\big\}\big) &\le c\alpha^{-2}\mu\big(\big\{ \max_{1\le j\le N} \|H_{k_j}^{(p)} R_s^{(p)} f(\omega)\| > \alpha\big\}\big) \\
&\le c\alpha^{-2}\mu\big(\big\{ \max_{1\le j\le N} \|H_{k_j}^{(p)} R_s^{(p)} ((1/\alpha) f)(\omega)\| > 1\big\}\big) \\
&\le \big(\frac{c}{\alpha}\big)^2 \left(\frac{N_p^{(\mathrm{w})}\left(\{k_j\}_{j=1}^N\right)}{\alpha^2}\right)^p \|f\|^p (1+\epsilon) .
\end{aligned}$$

The proof in this case is completed by letting $\epsilon \to 0$.

The arguments used to remove the conditions that each member of $\{k_j\}$ have compact support and that the sequence be finite are carried out just as in the scalar-valued case treated in [5]. □

We now consider the transference of strong-type bounds. Given a finite or infinite sequence $\{k_j\}$ of functions in $L^1(G)$, $N_p(\{k_j\})$ is defined as the least $M \in [0, \infty]$ such that for all $f \in L^p(G, \lambda, X)$,

$$\left\| \sup_j \left\| \int_G k_j(\,\cdot\, - v) f(v) d\lambda(v) \right\|_X \right\|_{L^p(\Omega)} \le M \|f\|_p \,.$$

If $N_p(\{k_j\}) < \infty$, the maximal operator for the sequence $\{k_j\}$ is said to be of strong-type (p,p). If the sequence is finite, $\{k_j\}_{j=1}^N$, then $N_p\big(\{k_j\}_{j=1}^N\big) \le \sum_{j=1}^N N_p(k_j) < \infty$.

Theorem 3.4

Let R be a strongly continuous μ-distributionally controlled representation of G. Let $\{k_j\}$ be a finite or infinite sequence of functions in $L^1(G)$. Then, for all $f \in E^p$, we have that,

$$\left\| \sup_j \|H_{k_j} f(\cdot)\| \right\|_{L^p(\Omega)} \le \left(\frac{c^2}{\alpha^{p+3}} \right)^{1/p} N_p(\{k_j\}) \, \|f\|_{E^p} \,. \tag{3.4.1}$$

Proof. As in the proof of the transference for weak-type bounds, it suffices to consider a finite sequence $\{k_j\}_{j=1}^N$ in $L^1(G)$ where each function has compact support. Also, let K be a compact subset of G with $\bigcup_{j=1}^N \operatorname{supp}(k_j) \subset K$. Given $\epsilon > 0$, take V to be a relatively compact open neighborhood of the identity in G such that $\frac{\lambda(V-K)}{\lambda(V)} < 1+\epsilon$.

From (2.2), the representation $\widetilde{R}^{(p)}$ acting on $L^p(\Omega)$ satisfies $\|\widetilde{R}_u^{(p)}\| \le (c\,\alpha^{-2})^{1/p}$ for all $u \in G$. This estimate and (2.4.3) imply that for all $s \in G$,

$$\left\| \max_{1 \le j \le N} \|H_{k_j} f(\cdot)\| \right\|_{L^p(\Omega)}^p \le \frac{c}{\alpha^3} \left\| \max_{1 \le j \le N} \|H_{k_j}^{(p)} R_s^{(p)} f(\cdot)\| \right\|_{L^p(\Omega)}^p \,. \tag{3.4.2}$$

Averaging (3.4.2) over $s \in V$ and estimating as in [3, Theorem (2.3)] implies the following,

$$\left\| \max_{1 \le j \le N} \|H_{k_j} f(\cdot)\| \right\|_{L^p(\Omega)}^p \le \frac{c^2}{\alpha^{3+p}} \left(N_p\big(\{k_j\}_{j=1}^N\big) \right)^p (1+\epsilon) \|f\|_{L^p(\Omega,X)}^p \,.$$

Letting $\epsilon \to 0$ completes the proof in this case. □

Remarks 3.5. In the case of scalar-valued function spaces, a representation need only be separation-preserving and uniformly bounded for the transference of strong-type bounds. So, there is reason to believe that the transference of strong type bounds holds with hypothesis on R weaker than ours. Our arguments rely heavily on (2.4.3), and it is for this reason that we obtain transference of strong-type bounds only for distributionally controlled representations.

4. Applications

We look briefly at typical applications of the transference of maximal estimates proved in §3. Note that no additional hypotheses on the Banach space X were needed in proving (3.3) and (3.4). However, when considering applications involving the Hilbert transform and other singular integral operators, one must impose the condition $X \in$ UMD. We first introduce the appropriate definitions and notation. The principal results are Proposition (4.3) and Proposition (4.5).

Definition 4.1. A Banach space X has the *unconditionality property for martingale difference sequences* if for each $1 < p < \infty$, there exists a constant $C_p(X)$ such that for every X-valued martingale difference sequence $\{d_j\}$,

$$\Big\| \sum_{j=1}^{n} \epsilon_j d_j \Big\|_p \le C_p(X) \Big\| \sum_{j=1}^{n} d_j \Big\|_p \tag{4.1.1}$$

for all $n \in \mathbb{N}$ and for every $\{\epsilon_j\} \in \{-1,1\}^{\mathbb{N}}$. When (4.1.1) holds we write $X \in$ UMD.

It has been known for some time that the UMD condition is equivalent to boundedness of the Hilbert transform on $L^p(\mathbb{R}, X)$ for $1 < p < \infty$ (confer [9] and [11]). Also, there has been much work on the corresponding results for more general singular integral operators. With the transference theorems at our disposal, we now study maximal ergodic operators.

Definition 4.2. Suppose $k : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is locally integrable on its domain and satisfies the following where C_1 and C_2 are positive constants:

(4.2.1) $\lim_{\epsilon \to 0^+} \int_{\epsilon \le |x| \le 1/\epsilon} k(x)\, dx$ exists;

(4.2.2) $|k(x)| \le C_1/|x|$ for all $x \in \mathbb{R}^n \setminus \{0\}$;

(4.2.3) $|k(y) - k(y-x)| \le C_2 |x|/|y|^{n+1}$ for all $x, y \in \mathbb{R}^n$, $|y| \ge 2|x|$.

Let k_j be given by $k_j = k\, 1_{1/j \le |t| \le j}$. Let R denote a strongly continuous, distributionally controlled representation of $\mathbb{R}^n$ on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. For each $f \in L^p(\Omega, X)$, the function $M_k^{(p)}(f)(\cdot) = \sup_j \|H_{k_j}^{(p)} f(\cdot)\|_X$ is defined μ-a.e. on Ω as an extended real number. We call the mapping $f \mapsto M_k^{(p)}(f)$ the *transferred maximal operator on $L^p(\Omega, X)$*.

Proposition 4.3

Suppose k is a kernel on $\mathbb{R}^n$ satisfying (4.2.1)-(4.2.3) and $X \in$ UMD. Assume further that X has an unconditional basis $\{e_n\}$. Let R be a strongly continuous,

μ-distributionally controlled representation of $\mathbb{R}^n$ on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. Then, the following hold:

(4.3.1) *If $1 < p < \infty$, then $\|M_k^{(p)}(f)\|_p \le \left(\frac{c^2}{\alpha^{p+3}}\right)^{1/p} N_p(\{k_j\}) \|f\|_p$ for all $f \in E^p$;*

(4.3.2) *for all $f \in E^1$, and all $y > 0$, $\phi\big(M_k^{(1)}(f) : y\big) \le \left(\frac{c}{\alpha}\right)^2 \frac{N_1^{(\mathrm{w})}(\{k_j\})\, \|f\|}{\alpha^2 y}$.*

Proof. By Lemma (1) of [10], $N_p(\{k_j\}) < \infty$ for each $p \in (1, \infty)$. Thus, (4.3.1) follows immediately from (3.4). Since $N_1^{(\mathrm{w})}(\{k_j\}) < \infty$, (see [17]), (4.3.2) follows from (3.3). □

Definition 4.4. A function $\psi : \mathbb{Z} \to \mathbb{R}$ is called a *singular kernel on* $\mathbb{Z}$ if there exists constants $C_1, C_2 > 0$ such that the following hold:

(4.4.1) $\lim_{N\to\infty} \sum_{n=-N}^{N} \psi(n)$ exists;

(4.4.2) $\psi(0) = 0$ and $|\psi(n)| \le C_1/|n|$ for all $n \in \mathbb{Z} \setminus \{0\}$;

(4.4.3) $|\psi(n+1) - \psi(n)| \le C_2/n^2$ for all $n \in \mathbb{Z} \setminus \{0\}$.

Conditions (4.4.1)-(4.4.3) are the discrete analogs of (4.2.1)-(4.2.3). In this case the truncated kernels are given by $\psi_j = \psi\, 1_{[-j,j]\cap\mathbb{Z}}$ for each $j \in \mathbb{N}$. Let R be a distributionally controlled representation of $\mathbb{Z}$ on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. The *discrete transferred maximal operator* is defined by $M_\psi^{(p)} f(\cdot) = \sup_j \| \sum_{n=-j}^{j} \psi(n) R_{-n}^{(p)} f(\cdot)\|_X$ μ-a.e. on Ω for each $f \in L^p(\Omega, X)$. We now state a discrete version of Proposition (4.3).

Proposition 4.5

Suppose ψ is a singular kernel on $\mathbb{Z}$ and let $X \in$ UMD. Assume further that X has an unconditional basis $\{e_n\}$. Let R be a μ-distributionally controlled representation of $\mathbb{R}^n$ acting on $L^1(\Omega, X) \cap L^\infty(\Omega, X)$. Then, the following hold:

(4.5.1) *If $1 < p < \infty$, then $\|M_\psi^{(p)}(f)\|_p \le \left(\frac{c^2}{\alpha^{p+3}}\right)^{1/p} N_p(\{\psi_j\}) \|f\|_p$ for all $f \in E^p$;*

(4.5.2) *for all $f \in E^1$, and all $y > 0$, $\phi\big(M_\psi^{(1)}(f) : y\big) \le \left(\frac{c}{\alpha}\right)^2 \frac{N_1^{(\mathrm{w})}(\{\psi_j\})\, \|f\|}{\alpha^2 y}$.*

Proof. First recall that any representation of $\mathbb{Z}$ is strongly continuous since $\mathbb{Z}$ is discrete. It is well known that to show $N_p(\{\psi_j\}) < \infty$ for $1 < p < \infty$, and $N_1^{(\mathrm{w})}(\{\psi_j\}) < \infty$, one transfers the estimates from corresponding kernels on $\mathbb{R}$ satisfying (4.2.1)-(4.2.3). The interested reader can refer to [1, Theorem (2.4)] for one proof of this fact. Now, (4.5.1) follows from (3.4) while (4.5.2) is a consequence of (3.3). □

Remarks 4.6. (i) Proposition (4.5) generalizes the maximal ergodic theorems appearing in [1]. In [1], the maximal estimates (4.3.1) and (4.3.2) are only proved for the special case when $(\Omega, \mathcal{F}, \mu)$ is a probability space and $\mathbb{Z}$ acts on $L^p(\Omega, X)$ by $R_n^{(p)} f = f \circ U^n$ where $U : \Omega \to \Omega$ is a bijective measure preserving mapping. Such actions would always produce distributionally controlled representations while the representation described in (2.6.i) herein provides an example of an action of $\mathbb{Z}$ which is more general.

(ii) To apply maximal estimates such as those in (4.3) and (4.6) for proving almost everywhere convergence results on $L^p(\Omega, X)$, one would also need to prove the a.e. convergence for f in a dense subset of $L^p(\Omega, X)$. When $1 < p < \infty$ and X is reflexive, the existence of a suitable subspace is proved as [2, Theorem (2.1)]. For $p = 1$, [4, Theorem (3.19)] proves the corresponding result for $X = \mathbb{R}$ and $\mu(\Omega) < \infty$, but the same argument applies whenever X is reflexive. Since every UMD space is reflexive, (4.3) and (4.5) imply corresponding almost everywhere convergence results.

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