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Four-term recurrence relations for hypergeometric functions of the second order. I

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Abstract

Four-term recurrence relations for hypergeometric functions of the second order are deduced from generating functions involving elementary functions. Generalisations are indicated and an example is given of a five-term recurrence for the confluent hypergeometric function.

1. Introduction

It has recently been pointed out by Yáñez, Dehesa and Zarzo (1994) that recurrence relations for hypergeometric functions of the second order are incompletely known except for certain three-term recurrence relations associated with orthogonality properties. This is all the more remarkable in view of the fact that these functions are of importance in many applications including mathematical physics.

The authors cited above have discussed certain four-term recurrence relations for these functions from the point of view of their associated differential equations. In this study, this matter is taken up beginning with certain generating functions which yield higher-order functions from which the required recurrences are deduced. If the higher-order functions are suitably selected, then they may be specialised to give the second-order functions concerned, and, in turn, the recurrences sought. The analysis used in this paper involves, in the main, the elementary manipulation of series.

The following functions occur in the subsequent analysis: The hypergeometric function of general order,

$$ {}_AF_B\big(a_1,\ldots,a_A;b_1,\ldots,b_B;x\big) = {}_AF_B\big((a);(b);x\big) = \sum \frac{\big((a),m\big)x^m}{\big((b),m\big)m!}\,, \tag{1.1}$$

the Kampé de Fériet function,

$$ F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b'); \\ \\ (c):(d);(d');\end{matrix}\quad x,y\right] = \sum \frac{\big((a),m+n\big)\big((b),m\big)\big((b'),n\big)x^m y^n}{\big((c),m+n\big)\big((d),m\big)\big((d'),n\big)m!\,n!}\,, \tag{1.2}$$

the Humbert function,

$$ \Phi_2(a,b;c;x,y) = \sum \frac{(a,m)(b,n)x^m y^n}{(c,m+n)m!\,n!} \tag{1.3}$$

and the first Appell function

$$ F_1\big(a,b,b';c;x,y\big) = \sum \frac{(a,m+n)(b,m)(b',n)x^m y^n}{(c,m+n)m!\,n!}\,. \tag{1.4}$$

See such works as Appell et Kampé de Fériet (1926), Erdélyi (1953), Slater (1966) and Exton (1976), for example, where hypergeometric functions are discussed in great depth.

All indices of summation run over all of the non-negative integers unless otherwise stated. Any values of parameters for which any expression does not make sense are tacitly excluded. As usual, the Pochhammer symbol (a,n) is given by

$$ (a,n) = a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a);\;\; (a,0) = 1\,. \tag{1.5}$$

The symbol $((a),n)$ represents the sequence $(a_1,n)\ldots(a_A,n)$.

2. Four-term recurrences for the Gauss functions ${}_2F_1$

We begin by considering the generating function

$$ V = (1-xt)^{-a}(1-yt)^{-b}(1-zt)^{-c} = \sum t^n F_n\,. \tag{2.1}$$

The product of three binomial functions is developed in powers of t, namely

$$(1-xt)^{-a}(1-yt)^{-b}(1-zt)^{-c} = \sum \frac{(a,p)(b,q)(c,r)}{p!\,q!\,r!}\, x^p y^q z^r t^{p+q+r}\,. \tag{2.2}$$

On replacing p by $n-q-r$, the series on the right becomes

$$\sum \frac{(a,n-q-r)(b,q)(c,r)}{p!\,q!\,r!}\, x^{n-q-r} y^q z^r t^n\,. \tag{2.3}$$

Within its domain of absolute convergence, this triple series can be re-arranged in the form

$$\sum t^n \left[\frac{(a,n)}{n!} x^n \sum \frac{(-n,q+r)(b,q)(c,r)}{(1-a-n,q+r)\;q!\,r!} \big(y/x\big)^q \big(z/x\big)^r\right]. \tag{2.4}$$

Hence, by comparison with (2.1) and using (1.4), it is clear that

$$\begin{aligned} F_n = F_n(x,y,z) &= F_n(a,b,c;x,y,z) \\ &= \frac{(a,n)}{n!} x^n\, F_1\big(-n,b,c;1-a-n;y/x,z/x\big)\,. \end{aligned} \tag{2.5}$$

This type of elementary manipulation of series has been used by many authors over a long time. See, for example, Watson (1944), Chapter 2, where an application to the theory of Bessel coefficients is cited.

A number of functional relations involving F_n may be obtained from (2.1) by taking partial derivatives in a judicious manner. In the present context, we take partial derivatives with respect to t, when it follows that

$$\begin{aligned} \frac{\partial V}{\partial t} &= ax(1-xt)^{-a-1}(1-yt)^{-b}(1-zt)^{-c} + by(1-xt)^{-a}(1-yt)^{-b-1}(1-zt)^{-c} \\ &\quad + cz(1-xt)^{-a}(1-yt)^{-b}(1-zt)^{-c-1} \\ &= \sum nt^{n-1}F_n\,. \end{aligned} \tag{2.6}$$

Re-arranging, we have

$$\begin{aligned} &\big[ax(1-yt)(1-zt) + by(1-xt)(1-zt) + cz(1-xt)(1-yt)\big] \sum t^n F_n \\ &\qquad - (1-xt)(1-yt)(1-zt) \sum nt^{n-1}F_n = 0\,. \end{aligned} \tag{2.7}$$

Equate coefficients of t^n, and obtain the four-term recurrence relation

$$\begin{aligned} (n+1)F_{n+1} &= \big[ax+by+cz+(x+y+z)n\big]\,F_n \\ &\quad - \big[(a+b-1)xy+(a+c-1)xz+(b+c-1)yz+(xy+xz+yz)n\big]F_{n-1} \\ &\quad + (a+b+c-2+n)\,xyzF_{n-2}\,. \end{aligned} \tag{2.8}$$

The relation (2.8) may be made applicable to the Gauss function ${}_2F_1$ by observing that

$$F^{A:1;1}_{C:0;0}\left[\begin{matrix}(a):b;b'; \\ \\ (c):-;-;\end{matrix}\quad x,x\right] = {}_{A+1}F_C\big((a),b+b';(c);x\big)\,, \tag{2.9}$$

as given by Srivastava and Karlsson (1985), page 28, for example. It follows now that, replacing $b+c$ by b and putting $z=y$,

$$F_n = \frac{(a,n)x^n}{n!}\ {}_2F_1\big(-n,-1,b;1-a-n;y/x\big)\,. \tag{2.10}$$

With this specialisation, (2.8) may be written as

$$\begin{aligned}&(a+n)(a+n-1)(a+n-2)x^3\,{}_2F_1(-n-1,b;-a-n;y/x)\\ &=\big[ax+by+(x+2y)n\big](a+n-1)(a+n-2)x^2\,{}_2F_1\big(-n,b;1-a-n;y/x\big)\\ &-\big[(2a+b-2)xy+(b-1)y^2+y(2x+y)n\big](a+n-2)nx\,{}_2F_1\big(1-n,b;2-a-n;y/x\big)\\ &+(a+b-2+n)xy^2n(n-1)\,{}_2F_1\big(2-n,b;3-a-n;y/x\big)\,.\end{aligned} \tag{2.11}$$

The bilateral generating function

$$V_1 = \big(1-x/t\big)^{-a}\big(1-y/t\big)^{-b}\big(1-zt\big)^{-c} = \sum_{n=-\infty}^{\infty} t^n\,G_n \tag{2.12}$$

generates a non-polynomial form of the Appell function F_1. Proceeding as above, it is found that

$$G_n = \frac{(c,n)}{n!}\ z^nF_1\big(c+n,a,b;1+n;xz,yz\big)\,, \tag{2.13}$$

and by taking the partial derivative of (2.12) with respect to t

$$\begin{aligned}&\big[-axt^{-2}(1-y/t)(1-zt)-byt^{-2}\big(1-x/t\big)(1-zt)+cz\big(1-x/t\big)\big(1-y/t\big)\big]\\ &\qquad\times\sum_{n=-\infty}^{\infty} t^n\,G_n\\ &=\big(1-x/t\big)\big(1-y/t\big)(1-zt)\sum_{n=-\infty}^{\infty} nt^{n-1}G_n\,.\end{aligned} \tag{2.14}$$

If the powers of t are collected, and coefficients of t^n equated, we have

$$\begin{aligned}czG_n = &\big[cy(x+z)-z(ax+by)\big]G_{n+1}+\big[ax+by+(a+b-c)xyz\big]G_{n+2}\\ &-(a+b)xyG_{n+3}+\big[1+z(x+y)\big](n+1)G_{n+1}\\ &-(x+y+xyz)(n+2)G_{n+2}\\ &+xy(n+3)G_{n+3}\,.\end{aligned} \tag{2.15}$$

Put $y = x$ and replace $a + b$ by a, when G_n reduces to

$$\frac{(c,n)}{n!}\, z^n \;{}_2F_1(c+n, a; 1+n; xz)\,. \tag{2.16}$$

After some algebra, (2.15) yields a second four-term recurrence relation for a Gauss function ${}_2F_1$. This is

$$\begin{aligned}
&c(n+1)(n+2)(n+3)\;{}_2F_1(c+n, a; 1+n; xz)\\
&\quad + \big[cx(x+z) + 2(1-a)xz + 1 + (1+2xz)n\big]\\
&\quad \times (c+n)(n+2)(n+3)\;{}_2F_1(c+n+1, a; 2+n; xz)\\
&\quad + \big[(a-4)x + (a-c-2)x^2z - 2x(2+xz)n\big](c+n)(c+n+1)(n+3)z\\
&\quad \times\;{}_2F_1(c+n+2, a; 3+n; xz)\\
&\quad + (3-a+n)x^2(c+n)(c+n+1)(c+n+2)z^2\\
&\quad \times\;{}_2F_1(c+n+3, a; 4+n; xz) = 0\,.
\end{aligned} \tag{2.17}$$

3. Recurrence relations for confluent hypergeometric functions

Methods along the same lines as those applied in the previous section may be used to obtain four-term recurrence relations connecting certain contiguous confluent hypergeometric functions.

Beginning with the generating function

$$V_2 = (1-xt)^{-a}(1-yt)^{-b}\exp(zt) = \sum t^n K_n\,, \tag{3.1}$$

it may readily established that

$$\begin{aligned}
K_n &= \sum \frac{(a,p)(b,p)(-n,p+q)}{p!\,q!\,n!}\,(-x/z)^p(-y/z)^q z^n\\
&= \frac{z^n}{n!}\,F^{1:1;1}_{0:0;0}\begin{bmatrix} -n : a; b; & \\ & -x/z, -y/z \\ - : -; -; & \end{bmatrix}.
\end{aligned} \tag{3.2}$$

This last polynomial may be regarded as a two variable generalisation of the Charlier polynomial. Compare Erdélyi (1953) Vol. II page 226.

Again, taking partial derivatives of (3.1) with respect to t, we have

$$\big[ax/(1-xt) + by/(1-yt) + z\big]\,V_2 = \sum nt^{n-1}K_n\,. \tag{3.3}$$

Proceeding in a fashion precisely parallel to that of Section 2, we have eventually, after some algebra,

$$(n+1)K_{n+1} = (ax+z+2xn)K_n - \big[(a-1)x^2 - 2zx + x^2 n\big]K_{n-1} + zx^2\, K_{n-2}\,, \quad (3.4)$$

after letting $y = x$ and replacing $a + b$ by a.

Interpreting (3.4) in terms of the function ${}_2F_0$, it follows that

$$\begin{aligned} z^2\, {}_2F_0\big(-n-1, a; -; -x/z\big) = {} & (ax + z + 2xn)z\ {}_2F_0\big(-n, a; -; -x/z\big) \\ & - \big[(a-1)x^2 - 2xz + x^2\big]n\ {}_2F_0\big(-n+1, a; -; -x/z\big) \\ & + x^2 n(n-1)\ {}_2F_0\big(-n+2, a; -; -x/z\big)\,. \end{aligned} \quad (3.5)$$

The terminating series ${}_2F_0(-n, a; -; y)$ can be reversed to give

$$(a, n)(-y)^n\ {}_1F_1\big(-n; 1-a-n; -1/y\big)\,, \quad (3.6)$$

which is more easily recognisable as being of confluent hypergeometric form.

From the generating function

$$V_3 = \big(1 - x/t\big)^{-a}(1 - y/t)^{-b}\, \exp(zt) = \sum_{n=-\infty}^{\infty} t^n\, M_n\,, \quad (3.7)$$

it is found that

$$M_n = \frac{z^n}{n!}\, \Phi_2\big(a, b; 1+n; xz, yz\big)\,. \quad (3.8)$$

Proceeding as above, taking the special case $y = x$, and replacing $a+b$ by a, so that

$$M_n = \frac{z^n}{n!}\ {}_1F_1(a; 1+n; xz)\,, \quad (3.9)$$

we see that

$$zM_{n-2} + (1-2xz-n)\, M_{n-1} + \big(x^2 z - ax + 2xz\big)\, M_n + (a-1-n)\, x^2 M_{n+1} = 0\,. \quad (3.10)$$

This last expression may be written in the following confluent hypergeometric form:

$$\begin{aligned} & (n-1)n(n+1)\ {}_1F_1(a; n-1; xz) + (1 - 2xz - n)n(n+1)\ {}_1F_1(a; n; xz) \\ & \quad + (xz - a + 2n)(n+1)xz\ {}_1F_1(a; 1+n; xz) \\ & \quad + (a-1-n)x^2x^2\ {}_1F_1(a; 2+n; xz) = 0. \end{aligned} \quad (3.11)$$

4. Generalisations and conclusion

The expressions deduced in the previous sections may readily be generalised to give recurrence relations of any number of terms greater than two by employing generating functions with any given number of binomial factors. It must be pointed out, however, that the complexity of the algebra involved increases rapidly as the number of terms rises.

As an example, we discuss a generalisation of (3.7), namely,

$$V_4 = \bigl(1 - x_1/t\bigr)^{-a_1} \ldots \bigl(1 - x_m/t\bigr)^{-a_m} \ \exp(zt) = \sum_{n=-\infty}^{\infty} t^n \ {}_mM_n$$
$$= \sum \frac{\bigl(a_1, p_1\bigr) \ldots \bigl(a_m, p_m\bigr) x_1^{p_1} \ldots x_m^{p_m} \ z^r}{p_1! \ldots p_m!\, r!}\, t^{r - p_1 - \ldots - p_m} \,. \tag{4.1}$$

Put $r = n + p_1 + \ldots + p_m$, when we see that the previous series becomes

$$\sum \frac{\bigl(a_1, p_1\bigr) \ldots \bigl(a_m, p_m\bigr) x_1^{p_1} \ldots x_m^{p_m} z^{n + p_1 + \ldots + p_m}}{p_1! \ldots p_m! \bigl(n + p_1 + \ldots + p_m\bigr)!}\,, \tag{4.2}$$

and it is clear that

$${}_mM_n = \frac{z^n}{n!} \Phi_2^{(m)}\bigl(a_1, \ldots, a_m; 1 + n; x_1 z, \ldots, x_m z\bigr)\,. \tag{4.3}$$

The multiple confluent hypergeometric is given by

$$\Phi_2^{(n)}\bigl(a_1, \ldots, a_n; b; x_1, \ldots, x_n\bigr) = \sum \frac{\bigl(a_1, r_1\bigr) \ldots \bigl(a_n, r_n\bigr) x_1^{r_1} \ldots x_n^{r_n}}{\bigl(b, r_1 + \ldots + r_n\bigr) r_1! \ldots r_n!}\,. \tag{4.4}$$

see Exton (1976) page 42, for example.

By *a* multiple application of (2.9), it will be seen that

$$\Phi_2^{(n)}\bigl(a_1, \ldots, a_n; b; x, \ldots, x\bigr) = {}_1F_1\bigl(a_1 + \ldots + a_n; b; x\bigr)\,. \tag{4.5}$$

Proceeding as in the previous sections, a recurrence relation for the confluent function involving $m+2$ terms may now be obtained after a great deal of algebra. Similar generalisations for the other results given in Sections 2 and 3 may be developed. It must be stressed that other recurrences for the hypergeometric functions exist. It is hoped to discuss such results in subsequent work.

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