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Range of the generalized Radon transform associated with partial differential operators

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Abstract

In this work we consider two partial differential operators, define a generalized Radon transform and its dual associated with these operators and characterize its range.

Introduction

K. Trimèche has proved in [10] that we can construct the classical Radon transform and its dual on $\mathbb{R}^2$ by using two partial differential operators on $]0,+\infty[\,\times\,]0,2\pi[$ and the integral representation of Mehler type of their eigenfunction regular at the point $(0,0)$. More precisely he considers the operators

$$\begin{cases} \Delta_1 = \dfrac{\partial}{\partial\theta} & , \ \theta \in \,]0,2\pi[\\ \Delta_2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2} & , \ r \in \,]0,+\infty[\end{cases}$$

The operator Δ_2 is the Laplacian on $\mathbb{R}^2$.

The eigenfunction regular at the point $(0,0)$ of these operators is the solution denoted $\varphi_{\mu,k}$ of the following system

$$\begin{cases} \Delta_1 u(r,\theta) = i \quad k \quad u(r,\theta) \quad , \ k \in \mathbb{Z} \\ \Delta_2 u(r,\theta) = -\mu^2 \quad u(r,\theta) \quad , \ \mu \in \mathbb{C} \\ -If \quad k \neq 0 \\ \qquad u(r,0) \underset{r\to 0}{\sim} \dfrac{(i\mu)^{|k|} r^{|k|}}{2^{|k|}\,|k|!} \ , \ \dfrac{\partial u}{\partial r}(r,0) \underset{r\to 0}{\sim} \dfrac{(i\mu)^{|k|} r^{|k|-1}}{2^{|k|}\,(|k|-1)!} \\ -If \quad k = 0 \\ \qquad u(0,0) = 1 \ , \quad \dfrac{\partial u}{\partial r}(0,0) = 0 \end{cases}$$

The function $\varphi_{\mu,k}$ is given by

$$\forall (r,\theta) \in [0,+\infty[\times [0,2\pi] , \qquad \varphi_{\mu,k}(r,\theta) = i^{|k|} e^{ik\theta} J_{|k|}(\mu r)$$

where $J_{|k|}$ is the Bessel function of the first kind and index $|k|$.

The function $\varphi_{\mu,k}$ possesses the following integral representation of Mehler type:

$$\varphi_{\mu,k}(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik\psi} e^{i\mu r \cos(\psi-\theta)} d\psi.$$

This integral representation can also be written in the form

$$\varphi_{\mu,k}(r,\theta) = \varphi_{\mu,k}\left(re^{i\theta}\right) = \varphi_{\mu,k}(y) = \frac{1}{2\pi} \int_{S^1} e^{i\mu<y,\omega>} \chi_k(\omega)\, d\omega$$

where $\chi_k\left(e^{i\theta}\right) = e^{ik\theta}$,d$w$ the measure on the unit circle S^1 and $< .,. >$ the euclidian scalar product on $\mathbb{R}^2$.

From this last integral representation we define the operator $\Re$, on the space $\mathcal{E}_*\left(\mathbb{R} \times S^1\right)$ (the space of C^∞-functions f on $\mathbb{R} \times S^1$ such that $f(-p,-\omega) = f(p,\omega)$) by

$$\forall y \in \mathbb{R}^2, \qquad \Re(f)(y) = \frac{1}{2\pi} \int_{S^1} f(< y,\omega >,\omega)\, d\omega$$

The operator $\Re$ is the classical dual Radon transform on $\mathbb{R}^2$.

For $(\mu,k) \in \mathbb{C} \times \mathbb{Z}$, we have

$$\forall y \in \mathbb{R}^2, \qquad \varphi_{\mu,k}(y) = \Re\left(e^{i\mu<.,.>}\chi_k\right)(y).$$

Let g be a function in $\mathcal{E}_*\left(\mathbb{R}\times S^1\right)$ and f a function in $\mathcal{D}\left(\mathbb{R}^2\right)$ (the space of C^∞-functions, with compact support). We have

$$\int\limits_{\mathbb{R}^2} f(y)\,\Re(g)(y)\,dy = \frac{1}{2\pi}\int\limits_{\mathbb{R}}\int\limits_{S^1} g(p,\omega)\,{}^t\Re(f)(p,\omega)\,dpd\omega\,.$$

where

$$(1)\qquad {}^t\Re(f)(p,\omega) = \int\limits_{<x,\omega>=p} f(x)\,dx$$

The operator ${}^t\Re$ is the classical Radon transform on $\mathbb{R}^2$.

We denote by $\mathbb{P}^2$ the space of all straight lines of $\mathbb{R}^2$. Each straight line $\xi\in\mathbb{R}^2$ is written

$$\xi = \left\{x\in\mathbb{R}^2/ < x,\omega >= p\right\}.$$

where ω is a unit vector of $\mathbb{R}^2$ and $p\in\mathbb{R}$. If we write ${}^t\Re(f)(\xi)$, instead of ${}^t\Re(f)(p,\omega)$, the relation (1) becomes

$${}^t\Re(f)(\xi) = \int\limits_{\xi} f(x)\,dx$$

where dx is the Lebesgue measure on the straight line ξ (See [3]).

D. Ludwig and S. Helgason have studied in [3], [4] the ranges of the transforms $\Re$ and ${}^t\Re$.

In this paper we consider the partial differential operators

$$\begin{cases} D_1 = \dfrac{\partial^2}{\partial\theta^2} + 4\alpha\cot g\theta\dfrac{\partial}{\partial\theta} \\ D_2 = \dfrac{\partial^2}{\partial y^2} + \left[2\left(2\alpha+1\right)\coth 2y\right]\dfrac{\partial}{\partial y} - \dfrac{1}{\mathrm{ch}^2 y}D_1 + \left(2\alpha+1\right)^2 \\ \alpha\in\mathbb{R}, \alpha\geq 0 \text{ and } (y,\theta)\in]0,+\infty[\times\left]0,\frac{\pi}{2}\right[. \end{cases}$$

The operator $D_2$is a part of the radial part of the Laplace-Beltrami operator on the homogeneous space $X = G/K$ where:

– For $\alpha = 1, G = Sp(1,1), K = Sp(1)\times Sp(2)$.

– For $\alpha = 2, G = Spin_0(1,8), K = Spin(7)$.

(See [1] and [6]).

Using the precedent method which we have applied to construct the classical Radon transform and its dual on $\mathbb{R}^2$, we define the generalized Radon transform ${}^t\Re_\alpha$ and its dual $\Re_\alpha$ associated with the operators D_1, D_2, and we study their properties. Next we characterize the image by the transform ${}^t\Re_\alpha$ of the space $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$of square integrable functions on $[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$ with respect to the measure $W_{p,\alpha}(y,\theta)\,dyd\theta$, where

$$W_{p,\alpha}(y,\theta)=(\sin 2\theta)^{2\alpha}(\text{ch}y)^{4\alpha+3}\left[1-(\text{ch}y)^{-2}\right]^{p+\frac{1}{2}}\ ;\ p\in\mathbb{R}, p>-\frac{1}{2},\alpha\geq 0\,.$$

R. M. Perry has studied in [5] the same question for the Radon transform on the exterior of the unit disk. We can see that this transform which is denoted ${}^t\mathcal{X}_0$ in [8], is associated with the following partial differential operators

$$\begin{cases}\tilde{D}_1=\dfrac{\partial}{\partial\theta} & ,\ \theta\in\,]0,2\pi[\\ \tilde{D}_2=\dfrac{\partial^2}{\partial y^2}+[2\coth 2y]\dfrac{\partial}{\partial y}-\dfrac{1}{\text{ch}^2 y}\tilde{D}_1^2+1 & ,\ y\in]0,+\infty[\,.\end{cases}$$

For other generalized Radon transforms and their duals associated with partial differential operators we can see [8] [9] [10] .

The content of this paper is as follow

In the first section we give the solution $\varphi_{n,\mu}$ of the system

$$\begin{cases}D_1u(y,\theta)=-4n(n+2\alpha)u(y,\theta)\ ;\ n\in\mathbb{N}.\\ D_2u(y,\theta)=-\mu^2u(y,\theta)\ ;\ \mu\in\mathbb{C}.\\ u(0,0)=1,\dfrac{\partial}{\partial\theta}u(y,0)=0,\dfrac{\partial}{\partial y}u(0,\theta)=0,\text{ for all }(y,\theta)\in[0,+\infty[\times\left[0,\dfrac{\pi}{2}\right[;\end{cases}$$

and an integral representation of Mehler type of this solution.

The second section is devoted to the definition of the generalized dual Radon transform $\Re_\alpha$ associated with the operators D_1, D_2.

We define in the third section the generalized Radon transform ${}^t\Re_\alpha$ associated with the operators D_1, D_2 and we study its properties.

The last section is reserved for the characterization of the image of the space $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ by the generalized Radon transform ${}^t\Re_\alpha$.

1. Eigenfunction of the operators D_1, D_2

In this section we determine the eigenfunction of the operators D_1, D_2, regular at the point (0,0), and we give its integral representation of Mehler type using the generalized translation operator associated with the operator D_1 and the translation operator associated with the operator $\frac{d^2}{d\theta^2}$.

We consider the partial differential operators

$$\begin{cases} D_1 = \dfrac{\partial^2}{\partial\theta^2} + 4\alpha \cot g\theta \dfrac{\partial}{\partial\theta} \\ D_2 = \dfrac{\partial^2}{\partial y^2} + [2(2\alpha+1)\coth 2y]\dfrac{\partial}{\partial y} - \dfrac{1}{\mathrm{ch}^2 y} D_1 + (2\alpha+1)^2 \\ \alpha \in \mathbb{R}, \alpha \geq 0, (y,\theta) \in]0,+\infty[\times \left]0, \dfrac{\pi}{2}\right[\end{cases}$$

Theorem 1-1

The system of partial differential operators

$$(1-1)\begin{cases} D_1 u(y,\theta) = -4n(n+2\alpha)\, u(y,\theta) \ ; \ n \in \mathbb{N}. \\ D_2 u(y,\theta) = -\mu^2 u(y,\theta) \ ; \ \mu \in \mathbb{C}. \\ u(0,0) = 1, \dfrac{\partial}{\partial\theta} u(y,0) = 0, \dfrac{\partial}{\partial y} u(0,\theta) = 0, for\ all\ (y,\theta) \in [0,+\infty[\times \left[0, \dfrac{\pi}{2}\right[; \end{cases}$$

admits an unique solution $\varphi_{n,\mu}$ given by:

$$\varphi_{n,\mu}(y,\theta) = R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta))\,(\mathrm{ch}y)^n\, \varphi_\mu^{(\alpha,\alpha+n)}(y)$$

with $R_n^{(\alpha-1/2,\alpha-1/2)}$ is the Gegenbauer polynomial of degree n such that

$$R_n^{(\alpha-1/2,\alpha-1/2)}(1) = 1$$

and $\varphi_\mu^{(\alpha,\alpha+n)}$ is the Jacobi function defined by:

$$\psi(y) = \varphi_\mu^{(\alpha,\alpha+n)}(y) = {}_2F_1\left(\frac{1}{2}(2\alpha+n+1+i\mu), \frac{1}{2}(2\alpha+n+1-i\mu); \alpha+1; -\mathrm{sh}^2 y\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We put $\varphi_{n,\mu}(y,\theta) = R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta))(\text{ch}y)^n\,\psi(y)$, then the function $\varphi_{n,\mu}$ is a solution of the system (1-1) if and only if the function ψ is a solution of the differential equation:

$$\begin{cases} \dfrac{\partial^2}{\partial y^2}\psi(y) + [(2\alpha+1)\coth y + (2\alpha+2n+1)\text{th}y]\dfrac{\partial}{\partial y}\psi(y) \\ \qquad\qquad = -\left[\mu^2 + (2\alpha+2n+1)^2\right]\psi(y) \\ \psi(0) = 1, \dfrac{\partial}{\partial y}\psi(0) = 0. \end{cases}$$

or in [1] page 86, this differential equation admits an unique solution given by:

$$\psi(y) = \varphi_\mu^{(\alpha,\alpha+n)}(y) = {}_2F_1\left(\frac{1}{2}(2\alpha+n+1+i\mu), \frac{1}{2}(2\alpha+n+1-i\mu); \alpha+1; -\text{sh}^2 y\right)$$

where ${}_2F_1$ is the Gauss hypergeometric function. □

Remark 1-1. For $\alpha \geq 0$, the polynomial $R_n^{(\alpha-1/2,\alpha-1/2)}(\cos\omega)$, has the following expression using the Gauss hypergeometric function:
i) If $\alpha > 0$:

$$R_n^{(\alpha-1/2,\alpha-1/2)}(\cos\omega) = {}_2F_1\left(2\alpha+n, -n; \alpha+\frac{1}{2}; -\sin^2\frac{\omega}{2}\right), \ \omega \in \left[0, \frac{\pi}{2}\right[.$$

ii) If $\alpha = 0$:

$$R_n^{(-1/2,-1/2)}(\cos\omega) = {}_2F_1\left(n, -n; \frac{1}{2}; -\sin^2\frac{\omega}{2}\right) = T_n(\cos\omega), \ \omega \in \left[0, \frac{\pi}{2}\right[.$$

where T_n, is the Tchebycheff polynomial of the first kind and degree n.

Proposition 1-1

For $\alpha \geq 0, n \in \mathbb{N}$ and $\mu \in \mathbb{C}$, the function $(\text{ch}y)^n\varphi_\mu^{(\alpha,\alpha+n)}(y)$ possess the following integral representations of Mehler type:
i) If $\alpha > 0$:

$$(\text{ch}y)^n\varphi_\mu^{(\alpha,\alpha+n)}(y) = \frac{2^{-\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)(\text{sh}2y)^{2\alpha}}\int_0^y (\text{ch}2y - \text{ch}2s)^{\alpha-1/2}\cos(\mu s) \times R_n^{(\alpha-1/2,\alpha-1/2)}\left(\frac{\text{ch}s}{\text{ch}y}\right)ds$$

ii) If $\alpha = 0$:

$$(\mathrm{ch}y)^n \, \varphi_\mu^{(0,n)}(y) = \frac{2\sqrt{2}}{\pi} \int_0^y (\mathrm{ch}2y - \mathrm{ch}2s)^{-1/2} \cos(\mu s) \cos\left[n \,\mathrm{Arc}\cos\left(\frac{\mathrm{ch}s}{\mathrm{ch}y}\right)\right] ds.$$

Proof. i) If $\alpha > 0$:

From [6] page 8, for $n \in \mathbb{N}$, $\mu \in \mathbb{C}$ and $y > 0$, the function $(\mathrm{ch}y)^n \, \varphi_\mu^{(\alpha,\alpha+n)}(y)$ possess the integral representation:

$$(\mathrm{ch}y)^n \, \varphi_\mu^{(\alpha,\alpha+n)}(y) = \frac{2^{-\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)(\mathrm{sh}y)^{2\alpha}(\mathrm{ch}y)^{2\alpha}} \int_0^y (\mathrm{ch}2y - \mathrm{ch}2s)^{\alpha-1/2} \cos(\mu s)$$
$$\times {}_2F_1\left(2\alpha+n, -n; \alpha+\frac{1}{2}; \frac{\mathrm{ch}y - \mathrm{ch}s}{2\mathrm{ch}y}\right) ds\,.$$

Using the precedent remark we have:

$$R_n^{(\alpha-1/2,\alpha-1/2)}(\cos\omega) = {}_2F_1\left(2\alpha+n, -n; \alpha+\frac{1}{2}; -\sin^2\left(\frac{\omega}{2}\right)\right), \; \omega \in \left[0, \frac{\pi}{2}\right[\,.$$

Taking $\omega = \mathrm{Arc}\cos\frac{\mathrm{ch}s}{\mathrm{ch}y}$, we have:

$$\sin^2\left(\frac{\omega}{2}\right) = \frac{\mathrm{ch}y - \mathrm{ch}s}{2\mathrm{ch}y},$$

so that

$${}_2F_1\left(2\alpha+n, -n; \alpha+\frac{1}{2}; \mathrm{ch}y - \mathrm{ch}s2\mathrm{ch}y\right) = R_n^{(\alpha-1/2,\alpha-1/2)}\left(\frac{\mathrm{ch}s}{\mathrm{ch}y}\right).$$

ii) Similarly we get the result for $\alpha = 0$. □

Theorem 1-2

The function $\varphi_{n,\mu}, n \in \mathbb{N}, \mu \in \mathbb{C}$, possess the following integral representations of Mehler type:

i) If $\alpha > 0$:

$(1-2)$ $\varphi_{n,\mu}(y,\theta) =$

$$\begin{cases} \dfrac{2^{-\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)(\text{sh}2y)^{2\alpha}} \displaystyle\int_0^y (\text{ch}2y-\text{ch}2s)^{\alpha-1/2}\cos(\mu s)\, R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta)) \\ \qquad\qquad \times R_n^{(\alpha-1/2,\alpha-1/2)}(\cos\omega)\,ds \quad ;\ if\ y>0, \theta\in\left[0,\frac{\pi}{2}\right[\\ R_n^{(\alpha-1/2,\alpha-1/2)}\big(\cos(2\theta)\big) \quad ;\ if\ y=0, \theta\in\left[0,\frac{\pi}{2}\right[\end{cases}$$

ii) If $\alpha = 0$*:*

$$(1-3)\ \varphi_{n,\mu}(y,\theta) = \begin{cases} \dfrac{\sqrt{2}}{\pi}\displaystyle\int_0^y (\text{ch}2y-\text{ch}2s)^{-1/2}\cos(\mu s)\, T_n(\cos 2\theta)\, T_n(\cos\omega)\,ds \\ \qquad ;\ if\ y>0, \theta\in\left[0,\frac{\pi}{2}\right[\\ T_n(\cos(2\theta)) \quad ;\ if\ y=0, \theta\in\left[0,\frac{\pi}{2}\right[\end{cases}$$

with $\omega = \text{Arc}\cos\frac{\text{ch}s}{\text{ch}y}$.

Proof. We deduce the result from proposition 1-1 and theorem 1-1. □

Notations. We denote by
* $L^1(\sin^{2\alpha}(2\theta)d\theta)$ the space of measurable functions φ on $\left[0,\frac{\pi}{2}\right[$, satisfying

$$\int_0^{\pi/2} |\varphi(\theta)|\sin^{2\alpha}(2\theta)d\theta < +\infty.$$

* $L^1(\left[0,\frac{\pi}{2}\right[)$ the space of integrable functions φ on $\left[0,\frac{\pi}{2}\right[$ with respect to the measure dθ.
* $\tau_\theta^{(\alpha)}$, $\alpha > 0$, the generalized translation operator associated with the operator D_1 is defined on $L^1(\sin^{2\alpha}(2\theta)d\theta)$ by

$$(1-4) \qquad \tau_\theta^{(\alpha)}(\varphi)(\xi) = \int_0^{\pi/2} \varphi(\psi)K(\cos 2\theta, \cos 2\xi, \cos 2\psi)\sin^{2\alpha}(2\psi)d\psi$$

where the kernel K is given by:

$$K(\cos 2\theta,\cos 2\omega,\cos 2\psi)=\begin{cases}\dfrac{\Gamma\left(\alpha+\frac{1}{2}\right)\left[1-\cos^2 2\theta-\cos^2 2\omega-\cos^2 2\psi+2\cos 2\theta\cos 2\omega\cos 2\psi\right]^{\alpha-1}}{(\sin 2\theta\sin 2\omega\sin 2\psi)^{2\alpha-1}} & \\ \qquad ;\text{if } \ |\theta-\omega|<\omega<\theta+\omega \\ 0 & ;\text{otherwise}\end{cases}$$

(See [2], [7], page 116).

* $\tau_\theta^{(0)}$ the translation operator associated with the operator $\frac{d^2}{d\theta^2}$ is defined on $L^1(\left[0,\frac{\pi}{2}\right[)$ by:

$$(1-5)\qquad \tau_\theta^{(0)}(f)(\xi)=\frac{1}{2}\left[f(\theta+\xi)+f(\theta-\xi)\right]$$

Properties of $\tau_\theta^{(\alpha)}$, $\alpha\geq 0$:

i) For every functions φ and Φ in $\mathrm{L}^1(\sin^{2\alpha}(2\theta)d\theta)$, we have for $\alpha\geq 0$:

$$\int_0^{\pi/2}\varphi(\theta)\tau_\theta^{(\alpha)}(\Phi)(\xi)\sin^{2\alpha}(2\theta)d\theta=\int_0^{\pi/2}\Phi(\theta)\tau_\theta^{(\alpha)}(\varphi)(\xi)\sin^{2\alpha}(2\theta)d\theta\,.$$

ii) If $\alpha>0$:

$$\tau_\theta^{(\alpha)}\left(R_n^{(\alpha-1/2,\alpha-1/2)}\right)(\xi)=R_n^{(\alpha-1/2,\alpha-1/2)}\left(\cos 2\theta\right)R_n^{(\alpha-1/2,\alpha-1/2)}\left(\cos 2\xi\right)\,.$$

iii) If $\alpha=0$:

$$\tau_\theta^{(0)}\left(T_n\right)(\xi)=T_n\left(\cos 2\theta\right)T_n\left(\cos 2\xi\right).$$

(See [2] page 113).

Theorem 1-3

For every $n\in\mathbb{N}$, the relations(1-2),(1-3) can be written as follow
i) If $\alpha>0$:

$$\varphi_{n,\mu}(y,\theta)=\begin{cases}\dfrac{2^{\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}(\mathrm{sh}2y)^{-2\alpha}\displaystyle\int_0^y(\mathrm{ch}2y-\mathrm{ch}2s)^{\alpha-1/2}\cos(\mu s) & \\ \qquad\times\ \tau_\theta^{(\alpha)}\left(R_n^{(\alpha-1/2,\alpha-1/2)}\right)\left(\dfrac{\omega}{2}\right)ds & ;\,if\ \ y>0,\theta\in\left[0,\dfrac{\pi}{2}\right[\\ R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta)) & ;\,if\ \ y=0,\theta\in\left[0,\dfrac{\pi}{2}\right[\end{cases}$$

ii) If $\alpha = 0$:

$$\varphi_{n,\mu}(y,\theta) = \begin{cases} \dfrac{2\sqrt{2}}{\pi} \displaystyle\int_0^y (\mathrm{ch}2y - \mathrm{ch}2s)^{-1/2} \cos(\mu s) \\ \qquad\qquad \times\, \tau_\theta^{(0)}(T_n)\left(\dfrac{\omega}{2}\right) ds \quad ; \mathit{if}\ y > 0, \theta \in \left[0, \dfrac{\pi}{2}\right[\\ T_n(\cos(2\theta)) \qquad ; \mathit{if}\ y = 0, \theta \in \left[0, \frac{\pi}{2}\right[\end{cases}$$

with $\omega = \mathrm{Arc}\cos \frac{\mathrm{ch}s}{\mathrm{ch}y}$.

2. The generalized dual Radon transform associated with the operators D_1, D_2.

Using the integral representations of Mehler type of the function $\varphi_{n,\mu}$,we define in this section the generalized dual Radon transform associated with the operators D_1, D_2.

Notation. We denote by $C_*\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$. The space of functions $f(y,\theta)$, which are continuous on $\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and even with respect to y and θ.

DEFINITION 2-1. For $\alpha \geq 0$, we define the generalized dual Radon transform $\Re_\alpha$ associated with the operators D_1, D_2 on $C_*\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ by:

$$\Re_\alpha(f)(y,\theta) = \begin{cases} \dfrac{2^{\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} (\mathrm{sh}2y)^{-2\alpha} \displaystyle\int_0^y (\mathrm{ch}2y - \mathrm{ch}2s)^{\alpha-1/2} \\ \qquad\qquad \times\, \tau_\theta^{(\alpha)}(f(s,.))\left(\dfrac{\omega}{2}\right) ds \ ; \text{if } y > 0, \theta \in \left[0, \dfrac{\pi}{2}\right[\\ f(0,\theta) \qquad ; \text{if } y = 0, \theta \in \left[0, \frac{\pi}{2}\right[\end{cases}$$

with $\omega = \mathrm{Arc}\cos \frac{\mathrm{ch}s}{\mathrm{ch}y}$.

Remark 2-1. From theorem 1-2, we have for every $\alpha \geq 0$, $n \in \mathbb{N}, \mu \in \mathbb{C}$ and $(y,\theta) \in \mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$:

i) If $\alpha > 0$:

$$\varphi_{n,\mu}(y,\theta) = \Re_\alpha\left(\cos(\mu .)\, R_n^{(\alpha-1/2,\alpha-1/2)}(.)\right)(y,\theta).$$

ii) If $\alpha = 0$:

$$\varphi_{n,\mu}(y,\theta) = \Re_0 \left(\cos(\mu.)\, T_n(.)\right)(y,\theta).$$

Proposition 2-1

If $f(y,\theta) = R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta))\, h(y)$, *with* $n \in \mathbb{N}$ *and* h *an even continuous function on* $\mathbb{R}$, *then we have*
i) If $\alpha > 0$:

$$(2-1)\ \Re_\alpha(f)(y,\theta) = \frac{2^{\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} (\text{sh}2y)^{-2\alpha} R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta)) \times \int_0^y h(s)\,(\text{ch}2y - \text{ch}2s)^{\alpha-1/2} R_n^{(\alpha-1/2,\alpha-1/2)}\left(\frac{\text{ch}s}{\text{ch}y}\right) ds$$

ii) If $\alpha = 0$*:*

$$(2-2)\quad \Re_0(f)(y,\theta) = \frac{2\sqrt{2}}{\pi} T_n(\cos(2\theta)) \int_0^y h(s)\,(\text{ch}2y - \text{ch}2s)^{-1/2} T_n\left(\frac{\text{ch}s}{\text{ch}y}\right) ds$$

Proof. The result is a consequence of the definition 2-1 and the properties of the generalized translation operator $\tau_\theta^{(\alpha)}$, $\alpha \geq 0$. □

3. The generalized Radon transform associated with the operators D_1, D_2.

In this section we define the generalized Radon transform associated with the operators D_1, D_2 and we give its expression.

Notation. We denote by $C_{*,c}\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ the subspace of $C_*\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ consists of compact support functions.

Proposition 3-1

Let $g \in C_*\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$ *and* $f \in C_{*,c}\left(\mathbb{R} \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right)$, *then for every* $\alpha \geq 0$, *we have:*

$$\int_0^{+\infty}\int_0^{\pi/2} f(y,\theta)\Re_\alpha(g)(y,\theta)(\sin 2\theta)^{2\alpha}(\text{sh}2y)^{2\alpha+1}\,d\theta dy =$$
$$\int_0^{+\infty}\int_0^{\pi/2} g(s,\psi)\Bigg[\frac{2^{\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}(\text{sh}2y)^{-2\alpha}$$
$$\times\int_s^{+\infty}(\text{ch}2y-\text{ch}2s)^{\alpha-1/2}\tau_\psi^{(\alpha)}(f(y,.))\Big(\frac{\omega}{2}\Big)\,\text{sh}2y dy\Bigg](\sin 2\psi)^{2\alpha}\,d\psi ds$$

with $\omega = \text{Arc}\cos\left(\frac{\text{ch}s}{\text{ch}y}\right)$.

Proof. We put

$$K_\alpha(s,y) = \frac{2^{\alpha+3/2}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}(\text{ch}2y-\text{ch}2s)^{\alpha-1/2}(\text{sh}2y)^{-2\alpha}$$
$$A_\alpha(y) = (\text{sh}2y)^{2\alpha+1}\ .$$

From the definition 2-1, we have

$$I = \int_0^{+\infty}\int_0^{\pi/2} f(y,\theta)\Re_\alpha(g)(y,\theta)(\sin 2\theta)^{2\alpha}A_\alpha(y)\,d\theta dy$$
$$= \int_0^{+\infty}\int_0^{\pi/2} f(y,\theta)\left[\int_0^y K_\alpha(s,y)\,\tau_\theta^{(\alpha)}(g(s,.))\left(\frac{\omega}{2}\right)ds\right](\sin 2\theta)^{2\alpha}A_\alpha(y)\,d\theta dy\,.$$

Using Fubini's theorem, we get

$$I = \int_0^{+\infty}\int_0^{y}\left[\int_0^{\pi/2} f(y,\theta)\tau_\theta^{(\alpha)}(g(s,.))\left(\frac{\omega}{2}\right)(\sin 2\theta)^{2\alpha}\,d\theta\right]K_\alpha(s,y)\,A_\alpha(y)\,dsdy\,.$$

From the property 1 of $\tau_\theta^{(\alpha)}$, it follows

$$I=\int_0^{+\infty}\int_0^{y}\left[\int_0^{\pi/2} g(s,\theta)\tau_\theta^{(\alpha)}\left(f\left(y,.\right)\right)\left(\frac{\omega}{2}\right)\left(\sin 2\theta\right)^{2\alpha}d\theta\right]K_\alpha\left(s,y\right)A_\alpha\left(y\right)dsdy\,.$$

By the theorem of changing variables, we deduce

$$I=\int_0^{+\infty}\int_0^{+\infty}\int_0^{\pi/2} g(s,\theta)\tau_\theta^{(\alpha)}\left(f\left(y,.\right)\right)\left(\frac{\omega}{2}\right)\left(\sin 2\theta\right)^{2\alpha}K_\alpha\left(s,y\right)A_\alpha\left(y\right)d\theta dsdy\,.$$

The Fubini's theorem implies

$$I=\int_0^{+\infty}\int_0^{\pi/2} g(s,\theta)\left[\int_s^{+\infty}\tau_\theta^{(\alpha)}\left(f\left(y,.\right)\right)\left(\frac{\omega}{2}\right)K_\alpha\left(s,y\right)A_\alpha\left(y\right)dy\right]\left(\sin 2\theta\right)^{2\alpha}d\theta ds\,.$$

We get the result by replacing K_α and A_α by their expressions. □

Definition 3-1. For $\alpha\geq 0$, we define the generalized Radon transform ${}^t\Re_\alpha$ associated with the operators D_1, D_2 on $C_{*,c}\left(\mathbb{R}\times\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\right)$ by

$$(3-1)\quad {}^t\Re_\alpha\left(f\right)\left(s,\gamma\right)=\frac{2^{\alpha+1/2}\Gamma\left(\alpha+1\right)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(\text{sh}2s\right)^{-2\alpha}\int_s^{+\infty}\left(\text{ch}2y-\text{ch}2s\right)^{\alpha-1/2}\times\tau_\gamma^{(\alpha)}\left(f\left(y,.\right)\right)\left(\frac{\omega}{2}\right)\text{sh}2ydy$$

with $\omega=\text{Arc}\cos\left(\frac{\text{ch}s}{\text{ch}y}\right)$.

Proposition 3-2

If $f(y,\theta)=R_n^{(\alpha-1/2,\alpha-1/2)}\left(\cos(2\theta)\right)h(y)$ *, with* $n\in\mathbb{N}$ *and* h *an even continuous function on* $\mathbb{R}$ *with compact support, then*

i) For $\alpha>0$*:*

$$(3-2)\quad {}^t\Re_\alpha\left(f\right)\left(s,\gamma\right)=\frac{2^{\alpha+3/2}\Gamma\left(\alpha+1\right)}{\sqrt{\pi}\Gamma(\alpha+1/2)}R_n^{(\alpha-1/2,\alpha-1/2)}\left(\cos(2\gamma)\right)\times\int_s^{+\infty}h(y)\left(\text{ch}2y-\text{ch}2s\right)^{\alpha-1/2}R_n^{(\alpha-1/2,\alpha-1/2)}\left(\frac{\text{ch}s}{\text{ch}y}\right)\text{sh}2ydy$$

ii) For $\alpha = 0$:

$$(3-3) \quad {}^t\Re_0 (f)(s,\gamma) = \frac{2\sqrt{2}}{\pi} T_n(\cos(2\gamma)) \times \int_s^{+\infty} h(y)(\mathrm{ch}2y - \mathrm{ch}2s)^{-1/2} T_n\left(\frac{\mathrm{ch}s}{\mathrm{ch}y}\right) \mathrm{sh}2y dy$$

Proof.

We deduce this result from definition 3-1 and the properties 2, 3 of the generalized translation operator $\tau_\gamma^{(\alpha)}$. □

Corollary 3-1

For $k, n \in \mathbb{N}$, we have

i) If $\alpha > 0$:

$${}^t\Re_\alpha \left\{(\mathrm{ch}y)^{-2\alpha-k-2} R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta))\right\}(s,\gamma) = \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} (\mathrm{ch}s)^{-k-1} \times R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\gamma)) \int_0^1 R_n^{(\alpha-1/2,\alpha-1/2)}(t)\left(1-t^2\right)^{\alpha-1/2} t^k dt$$

ii) If $\alpha = 0$:

$${}^t\Re_0\left\{(\mathrm{ch}y)^{-k-2} T_n(\cos(2\theta))\right\}(s,\gamma) = \frac{4}{\pi}(\mathrm{ch}s)^{-k-1} T_n(\cos(2\gamma)) \times \int_0^1 T_n(t)\left(1-t^2\right)^{-1/2} t^k dt$$

Notation. For $k, n \in \mathbb{N}$, we put:

(3 − 4)

$$C_\alpha(n,k) = \begin{cases} \dfrac{2^{\alpha+1}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \displaystyle\int_0^1 R_n^{(\alpha-1/2,\alpha-1/2)}(t)\left(1-t^2\right)^{\alpha-1/2} t^k dt & ;\text{if } \alpha > 0 \\ \dfrac{4}{\pi} \displaystyle\int_0^1 T_n(t)\left(1-t^2\right)^{-1/2} t^k dt & ;\text{if } \alpha = 0 \end{cases}$$

Proposition 3-3

i) If $\alpha \geq 0$, we have: $C_\alpha(n,k) = 0$, if $n+k$ even and $k < n$.

ii) If $k \geq n$, we have:

$$(3-5) \qquad C_\alpha(n,k) = \frac{2^{2\alpha-n}\Gamma(\alpha+1)\Gamma(k+1)\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\alpha+1+\frac{k+n}{2}\right)\Gamma(k-n+1)} \quad ;\ if \quad \alpha > 0$$

$$(3-6) \qquad C_0(n,k) = \frac{2^{1-k}\Gamma(k+1)}{\sqrt{\pi}\Gamma\left(\frac{k+n}{2}+1\right)\Gamma\left(\frac{k-n}{2}+1\right)} \quad ;\ if \quad \alpha = 0$$

4. Characterization of the range of the generalized Radon transform ${}^t\Re_\alpha$.

In this section we characterize the range of the generalized Radon transform ${}^t\Re_\alpha$ associated with the operators D_1, D_2. The method used has been applied by R. M. Perry in [5] to characterize the range of the Radon transform on the exterior of the unit disk.

Notations. We denote by:

i) $Q_n^*(a,b;x)$, for $Re(a) > -1, Re(b) > -1$, the polynomial of degree n satisfying:

$$\begin{cases} \int_0^1 x^a(1-x)^b Q_n^*(a,b;x)\, x^k dx = 0 & ;\text{if} \quad 0 \leq k < n \\ \int_0^1 x^a(1-x)^b Q_n^*(a,b;x)\, x^n dx > 0 & ;\text{if} \quad k = n \\ \int_0^1 x^a(1-x)^b \left[Q_n^*(a,b;x)\right]^2 dx = 1. & \end{cases}$$

(See [5]).

ii) $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$the space of square integrable functions on$[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$ with respect to the measure $W_{p,\alpha}(y,\theta)\,dyd\theta$, where:

$$W_{p,\alpha}(y,\theta) = (\sin 2\theta)^{2\alpha}(\text{ch}y)^{4\alpha+3}\left[1-(\text{ch}y)^{-2}\right]^{p+1/2} \ ;\ p \in \mathbb{R},\ p > -\frac{1}{2}, \alpha \geq 0.$$

Lemma 4-1

The polynomial $Q_n^*(a,b;x)$ *has the following expansion*

$$(4-1)\qquad Q_n^*(a,b;x)=\sum_{k=0}^{n} q_{n,k}^*(a,b)\,x^k$$

where

$$(4-2)\qquad q_{n,k}^*(a,b)=\frac{(-1)^{n-k}\,\Gamma(a+b+n+k+1)}{\Gamma(n-k+1)\,\Gamma(k+1)\,\Gamma(a+k+1)}\times\left[\frac{(a+b+2n+1)\,\Gamma(n+1)\,\Gamma(a+n+1)}{\Gamma(b+n+1)\,\Gamma(a+b+n+1)}\right]^{1/2}$$

(*See* [5]).

Theorem 4-1

We consider the functions

i) For $\alpha>0$*:*

$$f_{m,n}^{p,\alpha}(y,\theta)=(\mathrm{ch}y)^{-2\alpha-\Delta m-2}\,R_m^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta))\,Q_n^*\left(\Delta m-\frac{1}{2},p;(\mathrm{ch}y)^{-2}\right)$$

ii) If $\alpha=0$*:*

$$f_{m,n}^{p}(y,\theta)=(\mathrm{ch}y)^{-\Delta m-2}\,T_m(\cos(2\theta))\,Q_n^*\left(\Delta m-\frac{1}{2},p;(\mathrm{ch}y)^{-2}\right)$$

where $p\in\mathbb{R}, n,m\in\mathbb{N}$ *and* $\Delta m=\begin{cases}0 & ,if\quad m \text{ is even}\\ 1 & ,if\quad m \text{ is odd}\end{cases}.$

Then for fixed α *and* p*, the system* $\left\{f_{m,n}^{p,\alpha}, m,n\in\mathbb{N}\right\}$, *is an orthogonal complete system in* $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$.

Proof. We get the result from the orthogonality and the completion of the systems

$$\left\{R_m^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta)), m\in\mathbb{N}\right\}\ \text{and}\ \left\{Q_n^*(a,b;x), n\in\mathbb{N}\right\}. \ \square$$

Remark 4-1. For fixed m,n and p, we have

$$\left\|f_{m,n}^{p,\alpha}\right\|_{L_1^2}^2=\begin{cases}\dfrac{2^{-2\alpha}\pi\Gamma(m+1)}{(m+\alpha)\,\Gamma(m+2\alpha)} & ;\text{if}\quad \alpha>0\\[2mm] \dfrac{\pi}{4} & ;\text{if}\quad \alpha=0.\end{cases}$$

In the following we shall evaluate the generalized Radon transform ${}^t\Re_\alpha\left(f^{p,\alpha}_{m,n}\right)$ in terms of the functions $h^\alpha_{m,k}$ given for all $(y,\theta)\in[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$ by

$$h^\alpha_{m,k}(y,\theta)=\begin{cases}(\mathrm{ch}y)^{-2\alpha-k-2}R_n^{(\alpha-1/2,\alpha-1/2)}(\cos(2\theta)) & ;\text{if}\quad \alpha>0\\ (\mathrm{ch}y)^{-k-2}T_n(\cos(2\theta)) & ;\text{if}\quad \alpha=0\,.\end{cases}$$

Term-by-term application of the corollary 3-1 and using the linearity of the generalized Radon transform, we obtain the following result.

Proposition 4-1

For $\alpha\geq 0$ and $(y,\theta)\in[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$, we have

$$(4-3)\qquad f^{p,\alpha}_{m,n}(y,\theta)=\sum_{k=0}^{n}q^*_{n,k}\left(\Delta m-\frac{1}{2},p\right)h^\alpha_{m,(\Delta m+2k)}(y,\theta)$$

with $q^*_{n,k}\left(\Delta m-\frac{1}{2},p\right)$, given by lemma 4-1.

Proof. The result is a consequence of the expression of functions $f^{p,\alpha}_{m,n}$ and the lemma 4-1. □

Corollary 4-1

For all $m,n\in\mathbb{N},(s,\gamma)\in[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$, we have
i) For $\alpha>0$:

$${}^t\Re_\alpha\left(f^{p,\alpha}_{m,n}\right)(s,\gamma)=\sum_{k=0}^{n}q^*_{n,k}\left(\Delta m-\frac{1}{2},p\right)C_\alpha(m,\Delta m+2k)(\mathrm{ch}s)^{-2k-\Delta m-1}\times R_m^{(\alpha-1/2,\alpha-1/2)}(\cos(2\gamma))$$

ii) For $\alpha=0$:

$${}^t\Re_0\left(f^{p}_{m,n}\right)(s,\gamma)=\sum_{k=0}^{n}q^*_{n,k}\left(\Delta m-\frac{1}{2},p\right)C_0(m,\Delta m+2k)(\mathrm{ch}s)^{-2k-\Delta m-1}\times T_m(\cos(2\gamma))$$

with $C_\alpha(m,\Delta m+2k)$, given by the relations (3-5),(3-6) and (3-7).

Remark 4-2. Let $a \in \mathbb{R}$, if $[a]$ means the entire part of a, then from the remark 3-1, i) and the corollary 4-1, we have for $\alpha \geq 0$:

$$(4-4) \qquad {}^t\Re_\alpha\left(f^{p,\alpha}_{m,n}\right) \equiv 0 \qquad ;\text{if} \quad n < \left[\frac{m}{2}\right].$$

Theorem 4-2

For all $(s,\gamma) \in [0,+\infty[\times \left[0,\frac{\pi}{2}\right[$, *we have*
i) For $\alpha > 0$*:*
$(4-5)$

$${}^t\Re_\alpha\left(f^{p,\alpha}_{m,n}\right)(s,\gamma) = \begin{cases} 0 & ;if\ n < \left[\frac{m}{2}\right] \\ d^{p,\alpha}_{m,n} R^{(\alpha-1/2,\alpha-1/2)}_m\left(\cos(2\gamma)\right)(\text{ch}s)^{-m-1} & \\ \qquad \times Q^*_{n-\left[\frac{m}{2}\right]}\left(m+\alpha, p-\alpha-\frac{1}{2}; (\text{ch}s)^{-2}\right) & ;if\ n \geq \left[\frac{m}{2}\right] \end{cases}$$

with

$$(4-6) \quad d^{p,\alpha}_{m,n} = \frac{2^\alpha \Gamma(\alpha+1)}{\sqrt{\pi}}$$
$$\times \left[\frac{\Gamma(n+1)\Gamma\left(n+\Delta m+\frac{1}{2}\right)\Gamma\left(p+n-\alpha+\frac{\Delta m-m+1}{2}\right)\Gamma\left(p+n+\frac{\Delta m+m+1}{2}\right)}{\Gamma(p+n+1)\Gamma\left(p+n+\Delta m+\frac{1}{2}\right)\Gamma\left(n+\frac{\Delta m-m+2}{2}\right)\Gamma\left(n+\alpha+\frac{\Delta m+m+2}{2}\right)}\right]^{1/2}$$

ii) For $\alpha = 0$*:*

$$(4-7) \quad {}^t\Re_0\left(f^{p}_{m,n}\right)(s,\gamma) = \begin{cases} 0 & ;if\ n < \left[\frac{m}{2}\right] \\ d^{p}_{m,n} T_m\left(\cos(2\gamma)\right)(\text{ch}s)^{-m-1} & \\ \qquad \times Q^*_{n-\left[\frac{m}{2}\right]}\left(m, p-\frac{1}{2}; (\text{ch}s)^{-2}\right) & ;if\ n \geq \left[\frac{m}{2}\right] \end{cases}$$

with

$$(4-8) \quad d^{p}_{m,n} = \frac{2}{\sqrt{\pi}}$$
$$\times \left[\frac{\Gamma(n+1)\Gamma\left(n+\Delta m+\frac{1}{2}\right)\Gamma\left(p+n+\frac{\Delta m-m+1}{2}\right)\Gamma\left(p+n+\frac{\Delta m+m+1}{2}\right)}{\Gamma(p+n+1)\Gamma\left(p+n+\Delta m+\frac{1}{2}\right)\Gamma\left(n+\frac{\Delta m-m+2}{2}\right)\Gamma\left(n+\frac{\Delta m+m+2}{2}\right)}\right]^{1/2}$$

Proof. We get the result from the corollary 4-1 and the relations (4-1),...,(4-4). □

Notation. We denote by $L_2^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ the space of square integrable functions on $[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$ with respect to the measure $W'_{p,\alpha}(s,\gamma)\,dsd\gamma$, where:

$$W'_{p,\alpha}(s,\gamma)=(\sin 2\gamma)^{2\alpha}(\text{ch}s)^{-2\alpha}\left[1-(\text{ch}s)^{-2}\right]^{p-\alpha}\quad;\ \alpha\geq 0,\ p\in\mathbb{R},p>\alpha-\frac{1}{2}.$$

Remark 4-3. If we take $d_{m,n}^{p,\alpha}=0$,for $n<\left[\frac{m}{2}\right]$, then we have

$$\left\|{}^t\Re_\alpha\left(f_{m,n}^{p,\alpha}\right)\right\|_{L_2^2}^2=\begin{cases}\dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\alpha+\frac{1}{2}\right)}{(m+\alpha)\,\Gamma(\alpha)}\left(d_{m,n}^{p,\alpha}\right)^2 & ;\text{if}\quad\alpha>0\\ \dfrac{\pi}{4}\left(d_{m,n}^{p}\right)^2 & ;\text{if}\quad\alpha=0\end{cases}$$

Lemma 4-2

i) For fixed $\alpha\geq 0,\ p\in\mathbb{R},\ p>\alpha-\frac{1}{2}$, for large m and n with $n\geq\left[\frac{m}{2}\right]$, there exist two positive constants $C_1(p)$ and $C_2(p)$ such that:

$$(4-9)\qquad C_1(p)\leq d_{m,n}^{p,\alpha}\left[(n+1)^{\alpha+1/2}\left(\frac{n+1}{n-\frac{m}{2}+1}\right)\right]^{(p-\alpha)/2-1/4}\leq C_2(p).$$

ii) The generalized Radon transform ${}^t\Re_\alpha$ associated with the operators D_1,D_2, is a compact operator from $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ into $L_2^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$.

Proof. i) The result is a consequence of the following property of the Γ function:

For large $x>0$, there exist $a_1,a_2>0$, such that:

$$a_1\leq\frac{\Gamma(x)}{x^{x-1/2}e^{-x}}\leq a_2.$$

ii) We have the result from i) and the fact that if $p>\alpha-\frac{1}{2}$, the function $d_{m,n}^{p,\alpha}$ is bounded as a function of m and n. □

Remark 4-4. If $f\in L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$, then from the theorem 4-1, for all $(y,\theta)\in[0,+\infty[\times\left[0,\frac{\pi}{2}\right[$, we have for $\alpha\geq 0$:

$$(4-10)\qquad f(y,\theta)=\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\gamma_{m,n}^{p,\alpha}\ f_{m,n}^{p,\alpha}(y,\theta)$$

with

$$(4-11)\qquad \gamma_{m,n}^{p,\alpha}=\left\|f_{m,n}^{p,\alpha}\right\|_{L_1^2}^{-2}<f,\ f_{m,n}^{p,\alpha}>_{L_1^2}.$$

Furthermore the function ${}^t\Re_\alpha(f)$ belongs to $L_2^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ and we have:

$$(4-12) \qquad {}^t\Re_\alpha(f)(s,\gamma)=\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\gamma_{m,n}^{p,\alpha}\ {}^t\Re_\alpha\left(f_{m,n}^{p,\alpha}\right)(s,\gamma).$$

Lemma 4-3

For $n\geq\left[\frac{m}{2}\right]$, the coefficients $\gamma_{m,n}^{p,\alpha}$ are given by
i) For $\alpha>0$:

$$\gamma_{m,n}^{p,\alpha}=\frac{2(m+1)\Gamma(\alpha)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(d_{m,n}^{p,\alpha}\right)^{-2}<{}^t\Re_\alpha(f),{}^t\Re_\alpha\left(f_{m,n}^{p,\alpha}\right)>_{L_2^2}$$

ii) For $\alpha=0$:

$$\gamma_{m,n}^{p}=\frac{4}{\pi}\left(d_{m,n}^{p}\right)^{-2}<{}^t\Re_0(f),{}^t\Re_0\left(f_{m,n}^{p}\right)>_{L_2^2}$$

Proof. We have the result from the relation (4-12) and the remark 4-3. □

Remark 4-5. From lemma 4-3, we see that for $n<\left[\frac{m}{2}\right]$, we can't deduce $\gamma_{m,n}^{p,\alpha}$, $\alpha\geq 0$, from ${}^t\Re_\alpha(f)$, so there exists in $L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ a subspace S_p of functions such that their transform by ${}^t\Re_\alpha$ vanish.

Proposition 4-2

For fixed $\alpha\geq 0$ and $p>\alpha-\frac{1}{2}$, the system of functions

$$\left\{\left[\frac{2(m+\alpha)\Gamma(\alpha)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\right]^{1/2}\left(d_{m,n}^{p,\alpha}\right)^{-1}\ {}^t\Re_\alpha\left(f_{m,n}^{p,\alpha}\right), m\in\mathbb{N}, n\geq\left[\frac{m}{2}\right]\right\},$$

is an orthonormal complete system in $L_2^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$.

Proof. The result is a consequence of theorems 4-2, 4-3 and the completion of the systems $\left\{R_m^{(\alpha-1/2,\alpha-1/2)}(\cos(2\gamma)), m\in\mathbb{N}\right\}$ and $\left\{Q_n^*(a,b;x), n\in\mathbb{N}\right\}$. □

Theorem 4-3

Let $g\in L_2^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$, then we have that $g={}^t\Re_\alpha(f)$, with $f\in L_1^2\left([0,+\infty[\times\left[0,\frac{\pi}{2}\right[\right)$ if and only if the coefficients $\gamma_{m,n}^{p,\alpha}$ given by the lemma 4-3, satisfy the condition

$$(4-13) \qquad \sum_{m=0}^{+\infty}\sum_{n\geq\left[\frac{m}{2}\right]}\left|\gamma_{m,n}^{p,\alpha}\right|^2<+\infty.$$

Remark 4-6.

From the relation (4-9), the relation (4-13) is equivalent to

$$(4-14) \qquad \sum_{m=0}^{+\infty} \sum_{n \geq \left[\frac{m}{2}\right]} \left|\xi_{m,n}^{p,\alpha}\right|^2 (n+1)^{\alpha+1} \left(\frac{n+1}{n-\frac{m}{2}+1}\right)^{p-\alpha-1/2} < +\infty$$

where $\xi_{m,n}^{p,\alpha}$ are the coefficients of g in the basis

$$\left\{ \left[\frac{2(m+\alpha)\Gamma(\alpha)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\right]^{1/2} \left(d_{m,n}^{p,\alpha}\right)^{-1} {}^t\Re_\alpha\left(f_{m,n}^{p,\alpha}\right), m \in \mathbb{N}, n \geq \left[\frac{m}{2}\right] \right\}.$$

Corollary 4-2

Let $g \in L_2^2\left([0,+\infty[\times \left[0,\frac{\pi}{2}\right[\right)$, *if* $g = {}^t\Re_\alpha(f)$, *with* $f \in L_1^2\left([0,+\infty[\times \left[0,\frac{\pi}{2}\right[\right)$ *then* $g = {}^t\Re_\alpha(f+h)$, *for all* $h \in S_p$.

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