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A theorem on derivations in semiprime rings

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Abstract

Let R be a semiprime ring with suitably-restricted torsion, U a nonzero left ideal of R and $D : R \to R$ a nonzero derivation. If for each $x \in U$, $[D(x), x]_n = \big[[\cdots[D(x), x]\cdots x], x\big] \in Z(R)$ with n fixed, then R must contain nonzero central ideals in case $D(U) \neq 0$.

In a recent paper, Lanski [3] gives an extension of a well-known theorem of Posner [4] by showing that a prime ring R is commutative if $[D(x), x]_n = 0$ for all x in a nonzero ideal of R. In [2], we studied the commutativity of semiprime rings with derivations and proved that a semiprime ring R must contain nonzero central ideals if $\big[[D(x), x], x\big] \in Z(R)$, which yields generalization of a theorem of Bell and Martindale [1]. The purpose of this note is to extend the result of Lanski [3, Theorem 1] from prime rings to semiprime rings with suitably-restricted additive torsion.

Throughout this paper, R denotes an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and $I_a(b)$ for $[b, a]$, and $[x, y]_n = \big[[\cdots[x, y], \cdots, y], y\big]$.

For easy reference we state two lemmas.

Lemma 1 ([2, Lemma 1]).

Let n be a positive integer, R be an $n!$ -torsion- free ring, and f be an additive map on R. For $i = 1, 2, \cdots, n$, let $F_i(X, Y)$ be a generalized polynomial which is homogeneous of degree i in the non-commuting indeterminates X and Y. Let $a \in R$ and (a) the additive subgroup generated by a. If

$$F_n\big(x, f(x)\big) + F_{n-1}\big(x, f(x)\big) + \cdots + F_1\big(x, f(x)\big) \in Z(R)$$

for all $x \in (a)$, then $F_i\big(a, f(a)\big) \in Z(R)$ for $i = 1, 2, \cdots, n$.

Lemma 2 ([5, Theorem]).

Let R be a ring and P a prime ideal such that $\mathrm{char}(R/P) = 0$ *or* $\mathrm{char}(R/P) \geq n$. *If $a_1, a_2, \cdots, a_{n+1}$ are elements of R such that $a_1 x a_2 x a_3 \cdots a_n x a_{n+1} \in P$ for all $x \in R$, then $a_i \in P$ for some $i = 1, 2, \cdots, n+1$.*

Theorem

Let n be a fixed positive integer, let R be a$(n+1)!$ -torsion- free semiprime ring, and let U be a nonzero left ideal of R. If R admits a nonzero derivation D such that $[D(x), x]_n \in Z(R)$ for all $x \in U$, then R contains nonzero central ideals or $D(U) = 0$.

Proof. For the proof, we need three steps.

Lemma A

$[D(x), x]_n = 0$ *for all* $x \in U$.

Proof. Linearizing the conditions $[D(x), x]_n \in Z(R)$ and using Lemma 1, we get

$$I_x^n\big(D(y)\big) + I_n^{n-1}\big([D(x), y]\big) + \cdots + I_x\big([[D(x), x]_{n-2}, y]\big) + \big[[D(x), x]_{n-1}, y\big] \in Z(R)\,.$$

Replacing y by x^2, noting that each term in the relation is $2x[D(x), x]_n$, we then have $2(n+1)x\big[D(x), x\big]_n \in Z(R)$, and $\big([D(x), x]_n\big)^2 = \big[[D(x), x]_{n-1}, x[D(x), x]_n\big] = 0$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we obtain $[D(x), x]_n = 0$. $\square$

Lemma B

$\big([D(x), x]_{n-1}\big)^2\, x = 0$ *for all* $x \in U$.

Proof. From $[D(x), x]_n = 0$, we have $I_x^i\big([D(x), x]_i\big) = 0$ for $i + j \geq n$. Linearizing $[D(x), x]_n = 0$ and applying Lemma 1, we now have

$$I_x^n(D(y)) + I_x^{n-1}\big([D(x), y]\big) + \cdots + I_x\big([[D(x), x]_{n-2}, y]\big) + \big[[D(x), x]_{n-1}, y\big] = 0\,. \quad (1)$$

Since $I_x^n\big(D(y)\big) = xI_x^n\big(D(y)\big) + I_x^n\big(D(x), y\big)$,

$$I_x^{n-1}\big([D(x), xy]\big) = xI_x^{n-1}\big([D(x), y]\big) + I_x^{n-1}\big([D(x), x]y\big)\,;$$
$$I_x^k\big([[D(x), x]_{n-k-1}, xy]\big) = xI_x^k\big([D(x), x]_{n-k-1}, y\big) + I_x^k\big([D(x), x]_{n-k}, y\big)\,,$$

for $k = 1, 2, \cdots, n-2$.

Replacing y by xy in (1) yields

$$I_x^n\big(D(x)y\big) + I_x^{n-1}\big([D(x),x]y\big) + \cdots + I_x\big([D(x),x]_{n-1}y\big) = 0\,. \tag{2}$$

Taking $y = \big[D(x),x\big]_{n-2}x$ in (2), and noting $I_x^j\big([D(x),x]_{n-2}\big) = 0$ for $j \geq 2$ and $I_x^k(ab) = \sum_{j=0}^{k}\binom{k}{j}I_x^{k-j}(a)I_x^j(b)$, we then gain

$$n\big([D(x),x]_{n-1}\big)^2\ x + (n-1)\big([D(x),x]_{n-1}\big)^2\ x + \cdots + \big([D(x),x]_{n-1}\big)^2\ x = 0\,,$$

thus $\frac{n(n+1)}{2}\big([D(x),x]_{n-1}\big)^2\ x = 0$ and $\big([D(x),x]_{n-1}\big)^2\ x = 0\,.$ □

Lemma C

$[D(x),x]_{n-1} = 0$ *for all* $x \in U$.

Proof. Take a family $\Omega = \{p_\alpha | \alpha \in \Lambda\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$, and let $\Omega_1 = \{P_\alpha \in \Omega | D(U) \subseteq P_\alpha\}$.

For each $P \in \Omega_1$, and for each $P \in \Omega\backslash\Omega_1$ such that $0 < \mathrm{char}(R/P) \leq n+1$, we have $(n+1)!\ \big[D(x),x\big]_{n-1} \in P$ for all $x \in U$.

Suppose that there is a $P \in \Omega\backslash\Omega_1$ such that $\mathrm{char}(R/P) = 0$ or $\mathrm{char}(R/P) > n+1$, we shall show that $[D(x),x]_{n-1} \in P$. Firstly we show that $a^2 \in P$ implies $a \in P$ for $a \in U$. From $[D(x),x]_n = 0$, we arrive at

$$x^nD(x) + (-1)\binom{n}{1}x^{n-1}D(x)\,x + \cdots + (-1)^nD(x)\,x^n = 0\,. \tag{3}$$

For any $a \in U$ with $a^2 \in P$, replace x by ra in (3) and then right-multiply by a, we have $(ra)^n rD(a)a \in P$ for all $r \in R$, and apply Lemma 2 to conclude $D(a)a \in P$.

If we replace x by $ra + a$ in (3), and apply the condition that $D(ra+a)a \in P$, we get

$$\begin{aligned}&\big((ra)^n + a(ra)^{n-1}\big)D(ra+a)\\ &\qquad + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}\big((ra)^{n-k} + a(ra)^{n-k-1}\big)D(ra+a)(ra)^k\\ &\qquad + (-1)^nD(ra+a)(ra)^n \in P,\end{aligned}$$

and by [2, Lemma 2], we conclude that

$$\begin{aligned}&(ra)^nD(a) + a(ra)^{n-1}D(ra)\\ &\qquad + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}\big((ra)^{n-k}D(a)(ra)^k + a(ra)^{n-k-1}D(ra)(ra)^k\\ &\qquad + (-1)^nD(a)(ra)^n \in P\,,\end{aligned} \tag{4}$$

and

$$a(ra)^{n-1}D(a)+\sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a(ra)^{n-k-1}D(a)(ra)^k\in P\,. \tag{5}$$

Left-multiplying (5) by r, and in conjunction with (4), shows that

$$a(ra)^{n-1}D(ra)+\sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a(ra)^{n-k-1}D(ra)(ra)^k+(-1)^nD(a)(ra)^n\in P\,.$$

Left-multiplying this last condition by a, we have $(-1)^n aD(a)(ra)^n \in P$; and by Lemma 2, $aD(a)\in P$.

Since $aD(a)$ and $D(a)a$ are in P, we obtain

$$(ra+ara)^nD(ra+ara)-\big((ra)^n+a(ra)^n\big)D(ra)\in P\,;$$
$$(ra+ara)^{n-k}D(ra+ara)(ra+ara)^k-\big((ra)^{n-k}+a(ra)^{n-k}\big)D(ra)(ra)^k\in P$$

for $k=1,2,\cdots,n-1$;

$$D(ra+ara)(ra+ara)^n-\big(D(ra)+D(ara)\big)(ra)^n\in P\,.$$

Substituting $ra+ara$ for x in (3), we have

$$\big((ra)^n+a(ra)^n\big)D(ra)+\sum_{k=1}^{n-1}(-1)^k\binom{n}{k}\big((ra)^{n-k}+a(ra)^{n-k}\big)D(ra)(ra)^k$$
$$+(-1)^n\big(D(ra)+D(ara)\big)(ra)^n\in P\,,$$

and using the condition $[D(ra),ra]_n=0$, we obtain $D(a)(ra)^{n+1}\in P$, so that Lemma 2 yields either $a\in P$ or $D(a)\in P$. For all $x\in R$, $axa\in U$ and $(xax)^2\in P$, that is, axa satisfies our original hypotheses on a, therefore for each $x\in R$, either $axa\in P$ or $D(axa)\in P$. Since the sets $\{x\in R|axa\in P\}$ and $\{x\in R|D(axa)\in P\}$ are additive subgroups of R, we conclude that either $aRa\subseteq P$ or $D(aDa)\subseteq P$. The former implies that $a\in P$ and in this event we are done. We assume henceforth that

$$a\notin P,\ D(a)\in P\quad\text{and}\quad D(aRa)\subseteq P\,.$$

It follows immediately that $aD(ya)\in P$ for all $y\in R$. Substituting ya for x in (3), $(-1)^nD(ya)(ya)^n\in P$ and $D(ya)(ya)^n\in P$. Now, right-multiplying the equation $D(axya)=D(ax)(ya)+axD(ya)$ by $(ya)^n$, we see that

$$D(axya)(ya)^n=D(ax)(ya)^{n+1}+axD(ya)(ya)^n\,.$$

Since $D(axya) \in P$ and $D(ya)(ya)^n \in P$, we have $D(ax)(ya)^{n+1} \in P$ for all $x, y \in R$. By Lemma 2, we obtain $D(ax) \in P$ and $aD(x) \in P$ for all $x \in R$. Therefore $aD(xy) = aD(x)y + axD(y) \in P$, $axD(y) \in P$ and $D(y) \in P$ for all $y \in R$, contradicting the hypothesis that $P \notin \Omega_1$. Hence $a \in P$. Henceforth $(P+U)/P$ contains no left zero divisors of R in view of [2, Lemma 3], but by Lemma A, we get $[D(x), x]_{n-1}\, x = x[D(x), x]_{n-1}$, and Lemma B shows that $\big(x[D(x), x]_{n-1}\big)^2 = \big([D(x), x]_{n-1}x\big)^2 = 0$, so that $x[D(x), x]_{n-1} \in P$ and $[D(x), x]_{n-1} \in P$.

Now, we establish that $(n+1)![D(x), x]_{n-1} \in P$ for all $P \in \Omega$ and for all $x \in U$, thus $(n+1)![D(x), x]_{n-1} \in \cap P_\alpha = \{0\}$; hence $[D(x), x]_{n-1} = 0$. □

We finish the proof of our theorem by induction on n. Assume inductively that $[D(x), x]_{n-1} = 0$ implies $[D(x), x] = 0$, by Lemma C, then $[D(x), x]_n = 0$ implies $[D(x), x] = 0$ as well. The existence of a nonzero central ideal follows from [1] in case $D(U) \neq 0$. □

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