Collect. Math. **46**, 3 (1995), 303–314

On the convergence of semiiterative methods to the Drazin inverse solution of linear equations in Banach spaces

Nieves Castro González

Facultad de Informática, Universidad Politécnica de Madrid, Boadilla del Monte, 28660 Madrid, Spain

Received September 6, 1994. Revised May 3, 1995

Abstract

We consider general semiiterative methods (SIMs) to find approximate solutions of singular linear equations of the type $x = Tx + c$, where T is a bounded linear operator on a complex Banach space X such that its resolvent has a pole of order ν_1 at the point 1. Necessary and sufficient conditions for the convergence of SIMs to a solution of $x = Tx + c$, where c belongs to the subspace range $\mathcal{R}(I-T)^{\nu_1}$, are established. If $c \notin \mathcal{R}(I-T)^{\nu_1}$ sufficient conditions for the convergence to the Drazin inverse solution are described. For the class of normal operators in a Hilbert space, we analyze the convergence to the minimal norm solution and to the least squares minimal norm solution.

1. Introduction

The aim of this paper is to give some characterizations of the convergence of semiiterative methods to certain solutions of singular equations of the type

$$x = Tx + c\,, \tag{1}$$

where $T \in \mathcal{B}(X)$ is a bounded linear operator on a complex Banach space X such that $I - T$ is *singular* in the sense that the operator $(I-T)^{-1}$ does not exist in $\mathcal{B}(X)$.

The motivation for our work comes from Eiermann, Marek and Niethammer [1], who considered semiiterative methods for computing approximate solutions of singular linear systems of algebraic equations.

To be able to show an infinite dimensional analogue of results given in [1] we consider a special class of bounded operators which we next define.

Let $\mathbb{C}$ be the open complex plane, we denote the *spectrum* of the operator T by $\sigma(T)$ and its *resolvent* $R(\lambda, T) = (\lambda I - T)^{-1}$ at the point $\lambda \in \rho(T)$, $\rho(T) = \mathbb{C} \setminus \sigma(T)$. The *spectral radius* of T is denoted by $r(T)$.

We will assume throughout the paper that T belongs to the class

$$\mathbb{B} = \{T \in \mathcal{B}(X) : \text{ if } \lambda \in \pi\sigma(T), \text{ then it is a pole of } R(\lambda, T)\},$$

where $\pi\sigma(T)$ denotes the *spectral radius circle of* T, that is, the set of $\lambda \in \sigma(T)$ for which $\mid \lambda \mid = r(T)$.

Further, we will assume that $R(\lambda, T)$ has a pole of finite order ν_1 at the point 1.

A typical example of an operator satisfying these assumptions is a bounded operator T such that T^n is compact (for some $n \geq 1$) and $1 \in \sigma(T)$.

We remark that the above properties are always satisfied by a matrix T provided that 1 is an eigenvalue.

Given an initial guess $x_0 \in X$, we first consider for approximate solutions of (1) the iterative method

$$x_m = T x_{m-1} + c \quad (m \geq 1). \tag{2}$$

It is known that if the order $\nu_1 > 1$, or there exist other poles of the resolvent $\lambda \neq 1$ having modulus 1, or $r(T) > 1$, then the sequence $\{x_m\}_{m \geq 0}$ does not converge. These rather restrictive conditions lead us to consider semiiterative methods, based on the iterative method (2) and to investigate when they are suitable for solving the linear equations $x = Tx + c$. In this paper, we will focus our attention to find the approximate Drazin inverse solution.

An outline of the paper is as follows. Section 2 describes semiiterative methods (SIMs) and the concept of Drazin inverse operator. In section 3 a spectral decomposition of the iteration operator T is shown and SIMs are analyzed in terms of the components of this spectral decomposition and the two sequences of polynomials usually associated with a given semiiterative method. As a result, convergence theorems to Drazin inverse solutions are established.

2. Definitions and notation

We shall use the operational calculus in the algebra of linear bounded operators on a Banach space into itself. For $f \in \mathcal{F}(T)$, where $\mathcal{F}(T)$ is the set of functions which are analytic in some neighborhood of $\sigma(T)$, the function $f(T)$ is defined as ([4], Chap. 5)

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, T) d\lambda,$$

where Γ denotes a simple closed smooth curve with positive direction enclosing the spectrum of T in its interior.

We say that T has *property R_ν at the point μ* if the resolvent $R(\lambda, T)$ has a pole of finite order ν at the point $\lambda = \mu$. If this occurs, one can expand the resolvent into a Laurent series

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\nu} B_k(\mu)(\lambda - \mu)^{-k},$$

where $A_k(\mu)$ and $B_k(\mu)$ are defined as usual [4]. Moreover, if $\wp$ is a polynomial then one can obtain that

$$\frac{1}{2\pi i} \int_{\Gamma_\mu} \wp(\lambda) R(\lambda, T) d\lambda = \sum_{k=1}^{\nu} \frac{\wp^{(k-1)}(\mu)}{(k-1)!} B_k(\mu)\,, \tag{3}$$

where Γ_μ is a positively oriented circle with the center μ and such that no point of $\sigma(T)$ except μ lies on or inside Γ_μ.

Next, we need a few basic facts on generalized inverses (see, e.g., [3]).

Definition 2.1. Let $A \in \mathcal{B}(X)$ be an operator having the property R_{ν_0} at the point 0. Let $D \in \mathcal{B}(X)$ be the unique bounded operator satisfying the following conditions

$$(i)\;\; AD = DA; \quad (ii)\;\; DAD = D; \quad (iii)\;\; A^{\nu_0+1}D = A^{\nu_0}\,.$$

This operator is called the *Drazin inverse of* A and is denoted A^D. In particular, if $\nu_0 = 1$ then D is called *group inverse of* A and is denoted $A^\sharp$.

Moreover, the operator D is identified as $h_0(A)$, where

$$h_0(\lambda) = \begin{cases} \lambda^{-1}, & \text{if } \mid \lambda \mid \geq \delta > 0\,; \\ 0, & \text{otherwise}; \end{cases}$$

and where $[\sigma(A) \setminus \{0\}] \subset \{\lambda : \mid \lambda \mid > \delta > 0\}$.

By $\mathcal{R}(A)$ we denote the range of a linear operator A and by $\mathcal{N}(A)$ its nullspace.

Remark 2.2. Now, let H be a Hilbert space. If A is a linear operator in H, we denote by A^* the *adjoint* of A. For the special case of a bounded linear operator A with closed range, the *Moore-Penrose inverse of* A (see [3]), denoted by $A^\dagger$, can be characterized by the relations

$$(i)\;\; A^\dagger A A^\dagger = A^\dagger;\;\; (ii)\; A A^\dagger A = A;\;\; (iii)\; (AA^\dagger)^* = AA^\dagger;\;\; (iv)\; (A^\dagger A)^* = A^\dagger A\,.$$

Moreover, for each $b \in H$, the set of *least squares solutions of* $Ax = b$ (i.e., the set of all minimizers of $\parallel Ax - b \parallel$) is given by $A^\dagger b + \mathcal{N}(A)$, while the *best approximate solution of* $Ax = b$ (i.e., the least squares solution of minimal norm) is given by $A^\dagger b$.

Thus, if A is a normal operator having property R_{ν_0} at the point 0, which implies that $\nu_0 = 1$ and $\mathcal{R}(A)$ is a closed subspace, the above relations $(i) - (iv)$ for the operator $A^\dagger$ are equivalent to the conditions involving D in Definition 2.1 and, consequently, in this case $A^D = A^\dagger = A^\sharp$.

Now, we define semiiterative methods following Varga ([5], Chap. 5).

DEFINITION 2.3. Let $\mathcal{P} = (\pi_{m,i})_{m\geq 0, 0\leq i\leq m}$ be an infinite triangular matrix with elements from $\mathbb{C}$ such that

$$\sum_{i=0}^{m} \pi_{m,i} = 1 \qquad (m \geq 0).$$

A *semiiterative method* (SIM) with respect to the iterative method (2) is given by

$$y_m = \sum_{i=0}^{m} \pi_{m,i} x_i \quad (m \geq 0)\,. \tag{4}$$

Note that $y_0 = x_0$. Let us define the polynomial sequences

$$p_m(z) = \sum_{i=0}^{m} \pi_{m,i} z^i \quad (m \geq 0) \tag{5}$$

(note that $p_m(1) = 1$) and

$$q_{m-1}(z) = \frac{1 - p_m(z)}{1 - z} \quad (m \geq 1; q_{-1}(z) = 0)\ . \tag{6}$$

Then, for y_m defined by (4) we have the following representation in terms of these polynomials (see, e.g., [2])

$$y_m = p_m(T)y_0 + q_{m-1}(T)c \quad (m \geq 0)\,. \tag{7}$$

3. Convergence properties of semiiterative methods

First a lemma is proved, which plays an important role in our considerations. This shows spectral decompositions for the operator T and for polynomials on T which shall allow us to study the convergence properties of semiiterative methods in a similar way as it has been done in [1].

Lemma 3.1

Let $T \in \mathcal{B}(X)$ be an operator having property R_{ν_1} at the point 1. Then,

$$T = T_P + T_N + T_R$$

and, moreover, if $\wp(z)$ is a polynomial, then

$$\wp(T) = \wp(1)T_P + \sum_{j=1}^{\nu_1 - 1} \frac{\wp^{(j)}(1)}{j!}\, T_N^j + \wp(T_R)(I - T_P)\,,$$

where T_P is a projection, $T_P^2 = T_P \neq \Theta$ (Θ is the null operator), whose range is $\mathcal{N}(I-T)^{\nu_1}$ and the range of $I - T_P$ is $\mathcal{R}(I-T)^{\nu_1}$; T_N is a nilpotent operator of order ν_1, $T_N^{\nu_1} = \Theta$; T_R is such that $(I - T_R)^{-1}$ exists (i.e., $1 \notin \sigma(T_R)$) and these operators also satisfy

$$T_P T_N = T_N T_P = T_N, \quad T_P T_R = T_R T_P = T_N T_R = T_R T_N = \Theta\,.$$

Proof. Under our hypothesis, if $B_1(1)$ is the first term in the main part of the Laurent series of the resolvent operator $R(\lambda, T)$ in a neighborhood of the point 1, then

$$\wp(T) = \wp(T)B_1(1) + \wp(T)(I - B_1(1))\,,$$

and further, applying the formula (3), we derive that

$$\wp(T)B_1(1) = \sum_{j=0}^{\nu_1-1} \frac{\wp^{(j)}(1)}{j!}(T-I)^j B_1(1)\,.$$

Finally, the setting

$$\begin{aligned} T_P &= B_1(1), \\ T_N &= (T-I)B_1(1), \\ T_R &= T(I - B_1(1))\,. \end{aligned}$$

completes the proof. □

Note that, because of the properties of the projection T_P, we can write

$$X = \mathcal{N}(I-T)^{\nu_1} \oplus \mathcal{R}(I-T)^{\nu_1}.$$

Lemma 3.2

Let T be as in Lemma 3.1. According to the terminology introduced above, the Drazin inverse of $I-T$ can then be expressed as

$$(I-T)^D = (I-T_R)^{-1}(I-T_P)\,.$$

Proof. We easily see that the operator on the right solves for D the equations in Definition 2.1. □

Throughout, by $\tilde{x}$ we denote the *Drazin inverse solution* of the equation (1) which is the unique vector that lies in the subspace $\mathcal{R}(I-T)^{\nu_1}$ and satisfies the equation $(I-T)x = (I-T_P)c$.

Now, invoking Lemma 3.1 with p_m instead of $\wp$ and substituting the expansion shown there in (7) and proceeding in the same way with q_{m-1} we obtain the following representation for the iterates y_m defined by (4)

$$\begin{aligned} y_m = &\left[T_P + \sum_{j=1}^{\nu_1-1} \frac{p_m^{(j)}(1)}{j!} T_N^j + p_m(T_R)(I-T_P)\right] y_0 \\ &+ \left[q_{m-1}(1)T_P + \sum_{j=1}^{\nu_1-1} \frac{q_{m-1}^{(j)}(1)}{j!} T_N^j + q_{m-1}(T_R)(I-T_P)\right] c\,. \end{aligned} \tag{8}$$

On the other hand, from (6) it follows that

$$p_m^{(j)}(1) = j!q_{m-1}^{(j-1)}(1) \quad (\forall j \geq 1)$$

and further, there holds a representation of y_m in terms of the polynomials $\{p_m\}_{m\geq 0}$, which will be useful later:

$$\begin{aligned} y_m = {} & T_P y_0 + \tilde{x} + p_m'(1) T_P c + p_m(T_R)[(I - T_P)y_0 - \tilde{x}] \\ & + \sum_{j=1}^{\nu_1 - 1} T_N^j \left(\frac{p_m^{(j)}(1)}{j!} y_0 + \frac{p_m^{(j+1)}(1)}{(j+1)!} c \right). \end{aligned} \tag{9}$$

In order to analyze the convergence of SIMs, we shall distinguish between two different cases. First we consider the case $c \in \mathcal{R}(I-T)^{\nu_1}$. Then the set of all solutions of equation (1) is given by $\tilde{x} + \mathcal{N}(I-T)$ and, moreover, $\tilde{x}$ is, in this case, a true solution. Necessary and sufficient conditions for the convergence of semiiterative methods to a solution are contained in the next theorem which is a generalization of Theorem 1 of [1].

Theorem 3.3

Let $T \in \mathbb{B}$ be an operator having the property R_{ν_1} at the point 1. We assume that $c \in \mathcal{R}(I-T)^{\nu_1}$. Then, the following three statements are equivalent:

(a) The sequence $\{y_m\}_{m\geq 0}$ of (4) converges, for any $y_0 \in \mathcal{N}(I-T) + \mathcal{R}(I-T)^{\nu_1}$, to a solution of $x = Tx + c$.

(b) $\lim_{m\to\infty} p_m(T)v = 0$ for every $v \in \mathcal{R}(I-T)^{\nu_1}$.

(c) $\lim_{m\to\infty} q_{m-1}(T)v = (I-T)^D v$ for each $v \in \mathcal{R}(I-T)^{\nu_1}$.

If one, and thus all of these conditions, are fulfilled, then

$$\lim_{m\to\infty} y_m = T_P y_0 + \tilde{x} = [I - (I-T)(I-T)^D]y_0 + (I-T)^D c.$$

Proof. (a) $\iff$ (b): Since $c \in \mathcal{R}(I-T)^{\nu_1}$, $T_P c = T_N c = 0$ follows from the properties of T_P and T_N shown in Lemma 3.1. Further, for any $y_0 \in \mathcal{N}(I-T) + \mathcal{R}(I-T)^{\nu_1}$, we have that $T_N y_0 = 0$. Thus, (9) reduces to

$$y_m = T_P y_0 + \tilde{x} + p_m(T_R)[(I - T_P)y_0 - \tilde{x}].$$

Now, for all $y_0 \in X$ there holds $(I - T_P)y_0 - \tilde{x} \in \mathcal{R}(I-T)^{\nu_1}$ and since the subspace $\mathcal{R}(I-T)^{\nu_1}$ is invariant under $p_m(T_R)$, (a) is equivalent to

$$\lim_{m\to\infty} p_m(T_R)v = 0$$

for all $v \in \mathcal{R}(I-T)^{\nu_1}$, which is analogous to that stated in (b).

(b) $\iff$ (c): From the definition (6) we get the relation $p_m(T)v = [I - (I-T)q_{m-1}(T)]v$. Now, using the expansions of $p_m(T)$ and $q_{m-1}(T)$ obtained via Lemma 3.1 and considering that, for any $v \in \mathcal{R}(I-T)^{\nu_1}$, there holds $T_P v = T_N v = 0$ and, by Lemma 3.2, $(I-T_R)^{-1}v = (I-T)^D v$, we conclude that this equivalence holds.

Finally, let us write $y_0 = y_{01} + y_{02}$, where $y_{01} \in \mathcal{N}(I-T) \subset \mathcal{N}(I-T)^{\nu_1}$ and $y_{02} \in \mathcal{R}(I-T)^{\nu_1}$. After applying the condition (b) to (9) we obtain

$$\lim_{m\to\infty} y_m = y_{01} + \tilde{x} = T_P y_0 + (I-T)^D c$$

and, because $T_P = I - (I-T)^D(I-T)$, the theorem is proved. □

The iterative method (2) can be seen as a special case of SIM; there we have $p_m(z) = z^m$ and $q_{m-1}(z) = 1 + z + \ldots + z^{m-1} = \frac{1-z^m}{1-z}$. In this case, the next corollary holds.

Corollary 3.4

Under the assumptions stated in Theorem 3.3, the iterative method (2) *converges, for any* $x_0 \in \mathcal{N}(I-T) + \mathcal{R}(I-T)^{\nu_1}$, *to a solution of* (1) *if and only if* $r(T) = 1$ *and* 1 *is the only point which lies on the spectral radius circle.*

Remark 3.5. We assume that $\pi\sigma(T) \setminus \{1\} = \{\lambda_1, \ldots, \lambda_p\}$ and T has property R_{ν_i} at the point λ_i, $1 \le i \le p$. Then, using the Laurent expansion of $R(\lambda, T)$ centered in λ_i and applying the formula (3), we have

$$p_m(T) = T_P + \sum_{j=1}^{\nu_1 - 1} \frac{p_m^{(j)}(1)}{j!} T_N^j + \sum_{i=1}^{p} \sum_{j=0}^{\nu_i - 1} \frac{p_m^{(j)}(\lambda_i)}{j!} B_{j+1}(\lambda_i) + \frac{1}{2\pi i} \int_\gamma p_m(\lambda) R(\lambda, T) d\lambda\,,$$

where γ is the smallest circle with its center at the origin and the radius such that the whole spectrum of T except $\pi\sigma(T) \cup \{1\}$ lies inside it.

From here, if Ω is an open set in $\mathbb{C}$ such that $1 \notin \Omega$ and such that $(\sigma(T) \setminus \pi\sigma(T)) \subset \Omega$, it follows that sufficient conditions for the fulfillment of the statement (b) given in Theorem 3.3 are

(b') $\{p_m(z)\}_{m\ge 0}$ converges to 0, uniformly on every compact subset of Ω and $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$, $1 \le i \le p$, $0 \le j \le \nu_i - 1$;

and, from the definition (6), we derive that for the fulfillment of (c) in Theorem 3.3 is sufficient that

(c') $\{q_{m-1}(z)\}_{m\ge 1}$ converges to $\frac{1}{(1-z)}$, uniformly on every compact subset of Ω and $\lim_{m\to\infty} q_{m-1}^{(j)}(\lambda_i) = \frac{1}{(1-\lambda_i)^{j+1}}$, $1 \le i \le p$, $0 \le j \le \nu_i - 1$.

Now, let $X = H$ be a Hilbert space with the norm induced by the scalar product defined in H. From the Remark 2.2 and Theorem 3.3 it follows the next corollary relative to the convergence of SIMs to the minimal norm solution.

Corollary 3.6

Let $T \in \mathbb{B}$ be a normal operator having property R_{ν_1} at the point 1. Assume that the conditions of Theorem 3.3 are satisfied. Then, if $c \in \mathcal{R}(I-T)$, the sequence $\{y_m\}_{m\geq 0}$ converges to the minimal norm solution of $x = Tx + c$ if and only if $y_0 \in \mathcal{R}(I-T)$.

EXAMPLES 3.7: We consider the SIM which corresponds to the Cesaro-summability method

$$y_m = \frac{1}{m+1}\sum_{i=0}^{m} x_i \quad (m \geq 0).$$

Here, the polynomials p_m have the form

$$p_m(z) = \frac{1}{m+1}\sum_{i=0}^{m} z^i \quad (m \geq 0).$$

These polynomials satisfy $\lim_{m\to\infty} p_m(z) = 0$ if, and only if, $z \in \{z \in \mathbb{C} : \mid z \mid \leq 1\}\backslash\{1\}$ and, for $j \geq 1$, $\lim_{m\to\infty} p_m^{(j)}(z) = 0$ if and only if $z \in \{z \in \mathbb{C} : \mid z \mid < 1\}$. Now, from Theorem 3.3 and Remark 3.5 it follows that, if $c \in \mathcal{R}(I-T)^{\nu_1}$, the sequence $\{y_m\}_{m\geq 0}$ converge to a solution of equation $x = Tx + c$ whenever the operator T is such that $r(T) = 1$ and the poles on the spectral radius circle of T are simple poles $(\nu_i = 1,\ i = 1, \ldots, p)$.

The case $c \notin \mathcal{R}(I-T)^{\nu_1}$ is treated below. In this case, we generalize Theorem 2 of [1].

Theorem 3.8

Let $T \in \mathbb{B}$ be an operator having the property R_{ν_1} at the point 1. Let us fix an integer s, $1 \leq s \leq \nu_1$. Assume that one of the following conditions, which are equivalent, is fulfilled:

(A) $\lim_{m\to\infty} p_m(T)v = [I-(I-T)(I-T)^D]v$ *for any* $v \in \mathcal{N}(I-T)^{s+1} + \mathcal{R}(I-T)^{\nu_1}$.

(B) $\lim_{m\to\infty} q_{m-1}(T)v = (I-T)^D v$ *for any* $v \in \mathcal{N}(I-T)^s + \mathcal{R}(I-T)^{\nu_1}$.

Then, given $c \in \mathcal{R}(I-T)^{\nu_1 - s}$, the sequence $\{y_m\}_{m\geq 0}$ converges, for any $y_0 \in \mathcal{N}(I-T)^{s+1} + \mathcal{R}(I-T)^{\nu_1}$, to

$$\hat{x}(y_0) = [I-(I-T)(I-T)^D]y_0 + (I-T)^D c. \tag{10}$$

Proof. First we are going to show that (A)$\Longleftrightarrow$(B): Assume that condition (A) holds. Let $v \in \mathcal{N}(I-T)^s + \mathcal{R}(I-T)^{\nu_1}$. Let $v = v_1 + v_2$, with $v_1 \in \mathcal{N}(I-T)^s$ and $v_2 \in \mathcal{R}(I-T)^{\nu_1}$. The properties of the operators T_P, T_N and T_R, established in Lemma 3.1, imply that

$$T_P v_1 = v_1,\;\; T_N^j v_1 = 0 \text{ for all } j \geq s,\;\; T_R v_1 = T_P v_2 = T_N v_2 = 0 \;\text{ and } T_R v_2 = v_2.$$

Therefore, using Lemma 3.1 to expand $q_{m-1}(T)$ together the definition of q_{m-1} (6) and after substituting the above relations we obtain,

$$\begin{aligned} q_{m-1}(T)v &= q_{m-1}(1)v_1 + \sum_{j=1}^{s-1} \frac{q_{m-1}^{(j)}(1)}{j!} T_N^j v_1 + q_{m-1}(T_R)v_2 \\ &= \sum_{j=1}^{s} \frac{p_m^{(j)}(1)}{j!} T_N^{j-1} v_1 + (I-T_R)^{-1}(I-p_m(T_R))v_2. \end{aligned}$$

Now, by on hypothesis it follows that $p_m(T)v_2 = p_m(T_R)v_2 \to 0$ as $m \to \infty$. Moreover, $p_m(T)w = T_P w + \sum_{j=1}^{s-1} \frac{p_m^{(j)}(1)}{j!} T_N^j w \to w$ as $m \to \infty$ for any $w \in \mathcal{N}(I-T)^{s+1}$ and, hence, $\lim_{m\to\infty} p_m^{(j)}(1) = 0,\quad 1 \leq j \leq s$. Thus, we conclude

$$\lim_{m\to\infty} q_{m-1}(T)v = (I-T_R)^{-1}v_2 = (I-T)^D v\,.$$

For the converse, from the definition (6) we get

$$p_m(T)v = [I - (I-T)q_{m-1}(T)]v\,.$$

Hence, it follows that (B) implies (A).

Now suppose $c \in \mathcal{R}(I-T)^{\nu_1 - s}$. Then there exists $b \in X$ such that $c = (I-T)^{\nu_1 - s} b$. Let $b = b_{\mathcal{N}} + b_{\mathcal{R}}$, with $b_{\mathcal{N}} \in \mathcal{N}(I-T)^{\nu_1}$ and $b_{\mathcal{R}} \in \mathcal{R}(I-T)^{\nu_1}$. We can write $c = (I-T)^{\nu_1-s} b_{\mathcal{N}} + (I-T)^{\nu_1-s} b_{\mathcal{R}}$. Thus, $(I-T)^s (I-T)^{\nu_1-s} b_{\mathcal{N}} = (I-T)^{\nu_1} b_{\mathcal{N}} = 0$ and $(I-T)^{\nu_1-s} b_{\mathcal{R}} \in \mathcal{R}(I-T)^{2\nu_1-s} = \mathcal{R}(I-T)^{\nu_1}$. This implies that

$$c \in \mathcal{R}(I-T)^{\nu_1} + \mathcal{N}(I-T)^s\,.$$

Therefore, taking the limit $m \to \infty$ in (7) and applying the conditions (A) and (B), we obtain the desired result. □

We note that $(I-T)\hat{x}(y_0) = T_P y_0 + (I-T_P)c$ and, hence, we conclude the following corollary.

Corollary 3.9

The vector $\hat{x}(y_0)$ of (10) is a solution of the equation $x = Tx + (I-T_P)c$, for any $y_0 \in \mathcal{N}(I-T) + \mathcal{R}(I-T)^{\nu_1}$. Moreover, $\hat{x}(y_0)$ is the Drazin inverse solution of $x = Tx + c$ if and only if $y_0 \in \mathcal{R}(I-T)^{\nu_1}$.

Remark 3.10. Under the assumption stated in Remark 3.5, property (b') together with

(P1) $\lim_{m\to\infty} p_m^{(j)}(1) = 0, \quad 1 \le j \le s$

are sufficient conditions for the fulfillment of the statement (A) in Theorem 3.8.

Similarly, property (c') together with

(Q1) $\lim_{m\to\infty} q_{m-1}^{(j)}(1) = 0, \quad 1 \le j \le s-1$

are sufficient conditions for the fulfillment of the statement (B) in Theorem 3.8.

Moreover, from the proof of Theorem 3.8, it turns out that the conditions (P1), and thus conditions (Q1), are also necessary.

From Theorem 3.8 and Remark 2.2 we obtain the next result relative to find approximate least squares solutions considering semiiterative methods in the context of Hilbert spaces.

Corollary 3.11

Let $T \in \mathcal{B}(H)$ be a normal operator having property R_{ν_1} at the point 1. Assume that the conditions of Theorem 3.8 are satisfied. Then the sequence $\{y_m\}_{m\ge 0}$ converges, for any $y_0 \in H$, to a least squares solution of $(I-T)x = c$. Moreover, it converges to the minimal norm least squares solution if and only if $y_0 \in \mathcal{R}(I-T)$.

EXAMPLES 3.12: In this example we assume that $\nu_1 = 1$. In the case $c \notin \mathcal{R}(I-T)$ the SIM considered in Example 3.7 does not converge. Note that $p'_m(1) = \frac{m}{2}$. Thus, condition (P1) in Remark 3.10 is not fulfilled. We construct a sequence of polynomials of degree $m+1$ $\{\tilde{p}_{m+1}\}_{m\ge 0}$, $\tilde{p}_{m+1}(z) = \wp_1(z)p_m(z)$, satisfying, besides the requirement $\tilde{p}_{m+1}(1) = 1$, $\tilde{p}'_{m+1}(1) = 0$. We get

$$\tilde{p}_{m+1}(z) = \left(-\frac{m}{2} + \frac{m+2}{2}\right)\frac{1}{m+1}\sum_{i=0}^{m} z^j \quad (m \ge 0).$$

Because, for $j \ge 0$, $\lim_{m\to\infty} \tilde{p}_m^{(j)}(z) = 0$ if and only if $|\, z \,| < 1$, the terms $\tilde{y}_m$ corresponding to the SIM induced by the polynomials $\{\tilde{p}_m\}_{m\ge 1}$ converge, for any $y_0 \in X$, to the Drazin inverse solution of $x = Tx + c$ whenever $(\sigma(T) \setminus \{1\}) \subset \{z \in \mathbb{C} : |\, z \,| < 1\}$.

Acknowledgment. The author would like to thank Professor Ivo Marek for suggesting the subject of this paper and for helpful discussions during her stay in the Caroline University of Prage.

References

1. M. Eiermann, I. Marek and W. Niethammer, On the solution of singular linear systems of algebraic equations by semi-iterative methods, *Numer. Math.* **53** (1988), 265–283.
2. M. Eiermann and W. Niethammer, On the construction of semiiterative methods, SIAM J. *Numer. Anal.* **20** (1983), 1153–1160.
3. M. Z. Nashed, Inner, outer, and generalized inverses in Banach and Hilbert spaces, *Numer. Funct. Anal. Optim.* **9** (1987), 261–325.
4. A. E. Taylor, *Introduction to functional analysis*, John Wiley, New York, 1963.
5. R. S. Varga, *Matrix iterative analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.