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$C-$nearest points and the drop property

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Abstract

For a closed convex set C with non-empty interior, we define the $C-$nearest distance from x to a closed set F. We show that, if there exists in the Banach space X a closed convex set with non-empty interior satisfying the drop property, then for all closed subset F of X, there exists a dense G_δ subset Γ of $X \setminus \{x; \rho(F,x) = 0\}$ such that every $x \in \Gamma$ has a $C-$nearest point in F. We also prove that every smooth (unbounded) convex set with the drop property has the smooth drop property.

1. Introduction

Let $(X, \|.\|)$ be a Banach space and B be the closed unit ball of X. Given $x \in X \setminus B$, the drop determined by x and B is the convex hull of the set $\{x\} \cup B$, denoted by $D(x,B)$. Daneš [3] proved that for every closed subset F at positive distance from B, there exists $x \in F$ such that $D(x,B) \cap F = \{x\}$, which is the drop theorem. It is not possible in general to replace the hypothesis "F at positive distance from B" by "F is disjoint from the unit ball". Rolewicz [16], introduced the drop property of the norm. He said that the norm $\|.\|$ has the drop property if for every closed set F disjoint from $B(\|.\|)$ there exists an $x \in F$ such that $D(x, B(\|.\|)) \cap F = \{x\}$. He proved that if the norm has the drop property then the space is reflexive.

A Banach space is said to have the Kadec-Klee property if, on the unit sphere, every weakly convergent sequence converges in norm. Montesinos [13], proved that the norm $\|.\|$ has the drop property if and only if the space is reflexive and the norm has the Kadec-Klee property.

Instead of drops formed from the closed unit ball, Kutzarova and Rolewicz [9] considered the drops formed from any closed (unbounded) convex set. They said a closed convex set C has the drop property if for every closed set F disjoint from C there exists an $a \in C$ such that $F \cap \operatorname{conv}(C \cup \{a\}) = \{a\}$. Here $D(x, C)$ denotes the set $\operatorname{conv}(C \cup \{x\})$ and it is called the drop determined by C and $x \in X \setminus C$. In [9], it was shown that a closed convex set having the drop property is compact or has a non-empty interior, and that every closed convex symmetric set having the drop property is bounded.

Kutzarova [8], showed that the space X is reflexive whenever X contains a non-compact bounded closed convex set with the drop property. Recently, Montesinos [14] and P. K. Lin [11] independently, extended this result showing that the same holds true for unbounded closed convex set.

Recently, we introduced in [12] the notion of smooth drop. A closed convex set D is called a smooth drop if 0 is in the interior of D and the Minkowski functional of D ($\rho(x) := \inf\left\{\lambda > 0; x\lambda^{-1} \in D\right\}$) is smooth. A smooth drop theorem was shown for spaces with smooth norms. Georgiev-Kutzarova-Maâden [7], said the norm $\|.\|$ has the smooth drop property if for every closed set F disjoint from $B(\|.\|)$ there exists a smooth drop D contains $B(\|.\|)$ such that $D \cap F$ is a singleton. They proved that if the space is reflexive and the norm is smooth with the Kadec-Klee property then the norm has the smooth drop property.

Let us fix a closed convex set C such that 0 is in the interior of C. For a closed set F of real Banach space X and $x \in X$ we define the $C-$nearest distance from x to F by $\rho(F, x) := \inf\{\rho(s - x); s \in F\}$ where ρ is the Minkowski functional of C. A point $z \in F$ is called a $C-$nearest point of F if there exists $x \in X \setminus \{x; \rho(F, x) = 0\}$ such that $\rho(F, x) = \rho(z - x)$. In this work we prove that if in addition C has the drop property then for all closed subset F of X, the set of points of $X \setminus \{x; \rho(F, x) = 0\}$ with $C-$nearest point in F contains a dense G_δ subset of $X \setminus \{x; \rho(F, x) = 0\}$, thus extending a result of Lau [10], [1]. We also prove that if C is smooth and has the drop property then it has the smooth drop property.

To fix our notation, denote by $B[x, r]$ (resp. $B(x, r)$) the closed (resp. open) ball with center x and radius r. Let C be a norm closed, convex and $0 \in \operatorname{int} C$. Denote by ρ the Minkowski functional of C. Denote by $F(C)$ the set of all linear continuous functionals $f \in X^*, f \neq 0$, which are bounded above on C, write $\rho_*(f) := \sup\{f(x); \rho(x) \leq 1\}$. For $f \in F(C)$ and $\delta > 0$, the slice $S(f, C, \delta)$ is defined by:

$$S(f, C, \delta) := \{x \in C; f(x) \geq M - \delta\}, \text{ where } M := \sup\{f(x); x \in C\}.$$

The Kuratowski measure of noncompactness $\alpha(A)$ of a set A in a Banach space X is the infimum of those $\varepsilon > 0$ for which there is a covering of A by a finite number of sets A_i such that $\mathrm{diam} A_i < \varepsilon$ (if A is not bounded, then we define $\alpha(A) = \infty$). We say that C has property (α) if

$$\alpha(S(f, C, \delta)) \longrightarrow 0 \text{ when } \delta \longrightarrow 0 \text{ for every } f \in F(C)\,.$$

It is known (see [9]) that if C has the property (α), then $S(f, C, \varepsilon)$ is bounded, whenever $\varepsilon > 0$ and $f \in F(C)$. In [9], Kutzarova and Rolewicz proved the following:

Theorem 1.1

Let X be a Banach space. Let C be a closed convex set with the drop property. Then C has property (α). If in addition X is reflexive and the interior of C is non-empty, the converse holds true.

Montesinos [14] and P.-K. Lin [11], showed this statement:

Theorem 1.2

Let X be a Banach space. Let $C \neq X$ be a closed convex set with the drop property. Then, if C is non-compact, X is reflexive.

In [12], a smooth drop theorem of Daneš type for spaces with smooth norms was shown. We include a simple proof of the smooth drop theorem (Theorem 1.5) derived from the Borwein and Preiss variational principle.

Let $\mathcal{E} := \{f; C^1 -$ smooth, convex, positive, $f(x) \to +\infty$ as $\|x\| \to +\infty$ and $\mu(f) < \infty\}$ where $\mu(f) := \sum_{n \geq 1} \frac{\|f\|_n}{2^n} + \|f'\|_\infty, \|f\|_n := \sup\{|f(x)|; \|x\| \leq n\}$.

In [6], the following variant of Borwein and Preiss variational principle [2], was proved using Deville's approach [5].

Theorem 1.3

Let $(X, \|.\|)$ be a Banach space with a smooth norm and consider a l.s.c. bounded below function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ with $f \neq +\infty$. Then the set $\{g \in \mathcal{E}; f + g$ attains its strong minimum on $X\}$ is residual in $\mathcal{E}$.

DEFINITION 1.4. Let X be a Banach space. A closed convex set D of X is said smooth drop if $0 \in \mathrm{int}D$ and the Minkowski functional of D is smooth.

Theorem 1.5

Let $(X, \|.\|)$ be a Banach space and assume that the dual norm is L.U.R. in X^*. Let C be a closed bounded convex set of X with 0 in its interior. Then for every closed set F at positive distance from C there exists a smooth drop D such that $D \cap F$ is a singleton and $C \subset D$.

Proof. Since C is bounded, without loss of generality we assume that $C \subset B$. Let $\varepsilon > 0$ such that $\mathrm{dist}^2(C, F) \geq 2\varepsilon$. Consider the function f defined on F by $f(x) := \mathrm{dist}^2(x, C)$. Now by the previous theorem there exists $g \in \mathcal{E}$ such that $f + g$ attains its strong minimum on F at x_0 and $\mu(g) \leq \varepsilon$. Then letting the set $D := \{x \in X; \mathrm{dist}^2(x, C) + g(x) \leq \mathrm{dist}^2(x_0, C) + g(x_0)\}$. So, it is clear that $D \cap F = \{x_0\}$. Moreover, if $x \in C$ then $(f+g)(x) = g(x) \leq 2\varepsilon \leq f(x_0) \leq (f+g)(x_0)$, so $C \subset D$. By hypothesis the dual norm is L.U.R. and C is a closed convex set then $\mathrm{dist}^2(x, C)$ is convex and smooth on X (for instance see p. 365 of [4]) and our proof is complete. □

2. Existence of $C-$nearest points

In this section we shall prove the following:

Theorem 2.1

Let C be a closed convex subset with the drop property. Let F be a non-empty closed subset of X. Then the set of points of $X \setminus \{x; \rho(F, x) = 0\}$ with $C-$nearest point in F contains the dense G_δ subset $\Omega(F)$ of $X \setminus \{x; \rho(F, x) = 0\}$.

When C is the unit ball of X we obtain Lau' theorem [10].

Corollary 2.2

Let $(X, \|.\|)$ be a reflexive Banach space and assume that the norm has the Kadec-Klee property. Let F be a non-empty closed subset of X. Then the set of points of $X \setminus F$ with nearest point in F contains a dense G_δ subset of $X \setminus F$.

Proof. Since the space is reflexive and the norm has the Kadec-Klee property, by Montesinos theorem [13] the unit ball has the drop property. The corollary is now a particular case of the theorem. □

As in the introduction, $C \neq X$ is a closed convex set with 0 in its interior and ρ is the "gauge" of C. Without loss of generality, throughout the paper, we assume that C contains the unit ball. For a closed non-empty set F of a Banach space X and $n \in \mathbb{N}$ we define

$$\begin{aligned}
L_n(F) &:= \{x \in X \setminus \{x; \rho(F,x) = 0\}; \exists \delta > 0, \exists\, x^* \in X^* \text{ with } \rho_*(x^*) = 1, \text{such that} \\
&\quad \inf\{< x^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta)C)\} > (1 - 2^{-n})\rho(F,x)\} \\
L(F) &:= \bigcap_n L_n(F) \\
\Omega(F) &:= \{x \in X \setminus \{x; \rho(F,x) = 0\};\ \text{there exists } x^* \in X^* \text{ with } \rho_*(x^*) = 1, \\
&\quad \text{such that for each } \varepsilon > 0,\ \text{there is } \delta > 0 \text{ so that} \\
&\quad \inf\{< x^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta)C)\} > (1 - \varepsilon)\rho(F,x)\}.
\end{aligned}$$

Definition 2.3. 1) A sequence (z_n) of elements in F is called a $C-$minimizing sequence in F for x if $\rho(F,x) := \inf\{\rho(z - x); z \in F\} = \lim_{n\to\infty} \rho(z_n - x)$.
2) For a lower semi continuous function f with $f(x)$ finite we define the subdifferential of f at x by $D^- f(x) := \left\{x^* \in X^*; \liminf_{y\to 0} \frac{f(x+y) - f(x) - <x^*,y>}{\|y\|} \geq 0\right\}$.

Proposition 2.4

Let F be a closed subset of $X, x \in X \setminus \{x; \rho(F,x) = 0\}$ and $x^* \in X^*$ such that $x^* \in D^-\rho(F,x)$. Then $\rho_*(-x^*) = 1$ and for each $C-$minimizing sequence (x_n) in F for x, $\lim_n < -x^*, x_n - x >= \lim_n \rho(x_n - x)$.

Proof. First remark that if $x^* \in D^-\rho(F,x)$ then $\rho_*(-x^*) \leq 1$. Indeed, clearly we have $\rho(F, x + ty) - \rho(F,x) \leq t\rho(-y)$. Then

$$\frac{\rho(F, x + ty) - \rho(F,x)}{t} \leq \rho(-y).$$

Let $x^* \in D^-\rho(F,x)$, then

$$\begin{aligned}
0 &\leq \frac{\rho(F, x + ty) - \rho(F,x)}{t} - < x^*, y > \\
&\leq \rho(-y) - < x^*, y >,
\end{aligned}$$

therefore

$$< -x^*, -y >\leq \rho(-y) \quad \text{for all} \quad y \in X.$$

This implies that $< -x^*, \frac{-y}{\rho(-y)} >\leq 1$ for all y with $\rho(-y) \neq 0$. Then

$$\rho_*(-x^*) := \sup \{< -x^*, z >; \rho(z) \leq 1\} \leq 1,$$

and the fact is proved.

Let $0 < t < 1$ and recall that $\rho(F, x) := \inf\{\rho(z - x); z \in F\}$. We have

$$\begin{aligned} \rho(F, x + t(x_n - x)) - \rho(F, x) &\leq \rho(x_n - x - t(x_n - x)) - \rho(F, x) \\ &= -t\rho(x_n - x) + [\rho(x_n - x) - \rho(F, x)] \,. \end{aligned}$$

Let $t_n := 2^{-n} + [\rho(x_n - x) - \rho(F, x)]^{1/2}$, it is clear that $t_n \longrightarrow 0$ and we have

$$\begin{aligned} \frac{-t_n\rho(x_n - x) + [\rho(x_n - x) - \rho(F, x)]}{t_n} &= -\rho(x_n - x) + \frac{(t_n - 2^{-n})^2}{t_n} \\ &\leq -\rho(x_n - x) + t_n \,. \end{aligned}$$

By hypothesis $x^* \in D^-\rho(F, x)$ then

$$\liminf_{n \longrightarrow \infty} \frac{\rho(F, x + t_n(x_n - x)) - \rho(F, x)}{t_n} - < x^*, x_n - x >\geq 0.$$

Therefore

$$0 \leq \liminf_{n \longrightarrow \infty} -\rho(x_n - x) + t_n + < -x^*, x_n - x >$$

which implies that

$$\begin{aligned} \rho(F, x) := \lim_{n \longrightarrow \infty} \rho(x_n - x) &\leq \liminf_{n \longrightarrow \infty} < -x^*, x_n - x > \\ &\leq \rho_*(-x^*) \lim_{n \longrightarrow \infty} \rho(x_n - x) \\ &\leq \lim_{n \longrightarrow \infty} \rho(x_n - x) = \rho(F, x) \,. \end{aligned}$$

Therefore $\rho_*(-x^*) = 1$ and $\lim_{n \longrightarrow \infty} < -x^*, x_n - x >= \rho(F, x)$. The proof is complete. □

Proposition 2.5

Let $(X, \|.\|)$ be a Banach space. If $x \in X \setminus \{x; \rho(F, x) = 0\}$ and $D^-\rho(F, x) \neq \emptyset$ then $x \in \Omega(F)$.

Proof. Let $x^* \in D^-\rho(F,x)$. By Proposition 2.4, we have $\rho_*(-x^*) = 1$ and for each $C-$minimizing sequence (x_n) in F for x we have $< -x^*, x_n - x > \to \rho(F,x)$. So, for each $\varepsilon > 0$ there exists $\delta > 0$ so that $< -x^*, z - x >> \left(1 - \frac{\varepsilon}{2}\right)\rho(F,x)$ whenever $z \in F \cap (x + (\rho(F,x) + \delta)C)$. It follows that

$$\inf\{< -x^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta)C)\} > (1-\varepsilon)\rho(F,x)$$

and the proof is complete. □

Proposition 2.6

$L_n(F)$ is open for all n.

Proof. Let $x \in L_n(F)$. By the definition, there exist $x^* \in F(C)$, with $\rho_*(x^*) = 1$, and $\delta > 0$ such that

$$0 < \theta := \inf\{< x^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta)\, C)\} - \left(1 - 2^{-n}\right)\rho(F,x).$$

Let $\lambda > 0$ such that $\lambda < \min\left(\frac{\delta}{2}; \frac{\theta}{2}\right)$. Let y be fixed such that $\|x - y\| < \lambda$ and let $\delta' := \delta - 2\lambda$. Then it is easy to see that

$$F \cap (y + (\rho\,(F,y) + \delta')\, C) \subset F \cap (x + (\rho\,(F,x) + \delta)\, C)\,.$$

Let $A := F \cap (y + (\rho(F,y) + \delta')C)$ and let $z \in A$, then

$$\begin{aligned} < x^*, z - y > &= < x^*, z - x > + < x^*, x - y > \\ &\geq \theta + (1 - 2^{-n})\rho(F,x) + < x^*, x - y > \\ &\quad + (1 - 2^{-n})\rho(F,y) - (1 - 2^{-n})\rho(F,y) \\ &= (1 - 2^{-n})\rho(F,y) + \theta + (1 - 2^{-n})[\rho(F,x) - \rho(F,y)] + < x^*, x - y > \\ &\geq (1 - 2^{-n})\rho(F,y) + \theta - 2\lambda. \end{aligned}$$

Then $\inf\{< x^*, z - y >; z \in A\} > (1 - 2^{-n})\rho(F,y)$. Whence $B(x,\lambda) \setminus \{x; \rho(F,x) = 0\} \subset L_n(F)$, which proves that $L_n(F)$ is open and the proof is complete. □

Lemma 2.7

Let $(X, \|.\|)$ be a Banach space and assume that the closed convex set C has the property (α). Then $L(F) = \Omega(F)$.

Proof. It is easy, from the definitions of the two sets, to see that $\Omega(F) \subset L(F)$. Then we need to show the converse inclusion.

Let $x \in L(F) := \bigcap_n L_n(F)$, then $x \in L_n(F)$ for all n. Hence there exist $x_n^* \in F(C), \rho_*(x_n^*) = 1$ and $\delta_n > 0$ such that

$$\inf\{< x_n^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta_n)C)\} > (1 - 2^{-n})\rho(F,x)\,.$$

Let x^* be a ω^*-cluster point of (x_n^*). Let $K_n := \overline{F \cap (x + (\rho(F,x) + \delta_n)C)}^{\omega}$. We claim that K_n is bounded. Indeed, let $z \in K_n$, then $z \in x + (\rho(F,x) + \delta_n)C$ and $< x_n^*, z-x >> (1-2^{-n})\rho(F,x)+\delta_n-\delta_n = (\rho(F,x)+\delta_n)-(2^{-n}\rho(F,x)+\delta_n)$. This implies that $z \in S\left(x_n^*, x + (\rho(F,x) + \delta_n)C, \delta_n + 2^{-n}\rho(F,x)\right)$. Since C has property (α), the slice is bounded (see [9], [14]) and the claim is proved. By Theorem 1.2, the space is reflexive, therefore the set K_n is weakly compact. Without loss of generality we assume that the sequence (δ_n) is nonincreasing and goes to zero. Thus $K := \bigcap_{n \geq 1} K_n$ is non-empty. Let $z \in K$, then $< x_n^*, z - x >> (1 - 2^{-n})\rho(F,x)$ and $\rho(z-x) \leq \rho(F,x)+\delta_n$ for all $n \geq 1$. So, $< x^*, z-x >\geq \rho(F,x)$ and $\rho(z-x) \leq \rho(F,x)$. We know that $\rho_*(x^*) \leq 1$, then we deduce that $\rho(F,x) \leq < x^*, z-x > \leq \rho_*(x^*)\rho(z-x) \leq \rho(F,x)$. So, $\rho_*(x^*) = 1$ and $\rho(F,x) =< x^*, z - x >= \rho(z-x)$ whenever $z \in K$.

Let $\varepsilon > 0$ and set $A(\varepsilon) := \left\{z; < x^*, z - x >> \left(1 - \frac{\varepsilon}{2}\right)\rho(F,x)\right\}$ which is weakly open and it is clear that $K \subset A(\varepsilon)$. By the weak compactness of the set K_n and that the sequence (K_n) is nonincreasing, there exists some n_0 such that $K_{n_0} \subset A(\varepsilon)$. This implies that

$$\inf\{< x^*, z - x >; z \in F \cap (x + (\rho(F,x) + \delta_{n_0})C)\} > (1 - \varepsilon)\,\rho(F,x)$$

and x^* is as required. The proof is complete. □

In the previous lemma we are used just that the space X is reflexive and C satisfies that $S(x^*, C, \delta)$ is bounded whenever $x^* \in F(C)$ and $\delta > 0$. In particular if C has property (α) this true (see [9], [14]).

Proposition 2.8

Let $(X, \|.\|)$ be an Asplund space. Then $\Omega(F)$ is dense in $X \setminus \{x; \rho(F,x) = 0\}$.

Proof. Since X is an Asplund space and it is clear that the function $\rho(F,x)$ is Lipschitz, then by Preiss theorem [15], $\rho(F,x)$ is Fréchet smooth on a dense subset of $X \setminus \{x; \rho(F,x) = 0\}$ and Proposition 2.5 completes the proof. □

Combining all the previous things, we see that we have proved the following:

Theorem 2.9

Let C be a closed convex subset with the drop property and $0 \in intC$. Let F be a non-empty closed subset of X. Then the set $\Omega(F) = L(F)$ is a dense G_δ subset of $X \setminus \{x; \rho(F,x) = 0\}$.

Proof of theorem 2.1. Let $x \in \Omega(F)$. Set $x^* \in X^*$ such that $\rho_*(x^*) = 1$, given by the definition of $\Omega(F)$. So, for all $n \geq 1$ there is $\delta_n > 0$ such that

$$\inf \{< x^*, z - x >; z \in F \cap (x + (\rho(F, x) + \delta_n)C)\} > \left(1 - 2^{-n}\right) \rho(F, x).$$

For each $n \geq 1$, choose $z_n \in F \cap (x + (\rho(F, x) + \delta_n)C)$. Since C has property (α), then the space is reflexive and the slices are bounded. Therefore (z_n) has a weakly converging subsequence. We assume that without generality the sequence (δ_n) goes to zero and $z_n \longrightarrow z$ in the weak topology, then $< x^*, z_n - x > \longrightarrow < x^*, z - x >$.

We have $\rho(F, x)\,(1 - 2^{-n}) \leq \rho(z_n - x) \leq \rho(F, x) + \delta_n$. So $\lim_n \rho(z_n - x) = \rho(F, x)$ and (z_n) is a $C-$minimizing sequence in F for x.

On the other hand, we have $\rho(F, x)\left(1 - 2^{-n}\right) \leq < x^*, z_n - x > \leq (1 + \delta_n)\,\rho(F, x)$, which proves that

$$< x^*, z - x > = \lim_n < x^*, z_n - x > = \rho(F, x).$$

Whence $\rho(z - x) \geq < x^*, z - x > = \rho(F, x) = \lim_n \rho(z_n - x)$. By the $\omega-$l.s.c. of ρ we have that $\lim_n \rho(z_n - x) \geq \rho(z - x)$. Therefore $\rho(z - x) = \lim_n \rho(z_n - x)$.

We have $z_n - x \longrightarrow z - x$ weakly and $\rho(z_n - x) \longrightarrow \rho(z - x)$. Since C has the property (α) then it is clear that every support point is a point of continuity, then $z_n - x \longrightarrow z - x$ in the norm topology. Since F is closed, z belongs to F. By the $\omega-$l.s.c. of ρ, we have

$$\rho(F, x) \leq \rho(z - x) \leq \liminf_{n \to \infty} \rho(z_n - x) \leq \rho(F, x).$$

Then the theorem is proved. □

3. The smooth drop property

In this section, we shall prove that every smooth (unbounded) convex set with the drop property has the smooth drop property (Theorem 3.3). The tool used is the geometrical result of Theorem 3.2.

Let us first give some definitions.

DEFINITION 3.1. Let X be a Banach space.
1) A closed convex set D of X with non-empty interior, is said Fréchet (resp. Gâteaux) smooth if the Minkowski functional of $D_0 := D - x_0$ for some $x_0 \in \text{int}D$, is Fréchet (resp. Gâteaux) smooth.
2) A closed convex set C of X said has the Fréchet (resp. Gâteaux) quasi-smooth drop property if for all closed set F of X such that $F \cap C = \emptyset$, there exists a Fréchet (resp. Gâteaux) smooth convex set D such that $\text{conv}(D \cup C) \cap S$ is a singleton.
3) A sequence (x_n) in X is called a $C-$stream if $x_{n+1} \in \text{conv}\left(\{x_n\} \cup C\right) \setminus C$, for all n.

Theorem 3.2

Let $(X, \|.\|)$ be a Banach space. Let C be a closed Fréchet (resp. Gâteaux) smooth convex subset with the drop property and assume that $0 \in \mathrm{int}C$. Let F be a closed non-empty subset of X at a positive distance from C. Let $\mu > 1$ and $z \in F$. Then there exists a Fréchet (resp. Gâteaux) smooth convex subset C_0 of X such that conv$(C_0 \cup C) \cap F = \{z_0\}$ and $C_0 \subset D(\mu z, C)$.

Theorem 3.3

Let $(X, \|.\|)$ be a Banach space and assume that the norm has the Kadec-Klee property and its dual norm is L.U.R. (resp. R.). Let C be a closed Fréchet (resp. Gâteaux) smooth convex with 0 in its interior and has the drop property, then C has the Fréchet (resp. Gâteaux) quasi-smooth drop property.

Proof. Let F be a closed subset of X such that $F \cap C = \emptyset$.

Case 1. There exists $z \in X$ such that

$$F_1 := F \cap \mathrm{int}D(z, C) \neq \emptyset \text{ and } \quad \mathrm{dist}(F_1, C) > 0$$

then we just apply Theorem 3.2 and the proof is complete in this case.

Case 2. Case 1 is not satisfied.

We can define inductively a sequence (x_n) in F with $x_1 \in F$ arbitrary such that

$$(1 + \mu_{n+1})x_{n+1} \in D\left((1 + \mu_{n+1})\, x_n, C\right) \cap (1 + \mu_{n+1})\, C,$$

for some $\mu_n > 0, \mu_n \longrightarrow 0$.

Fact : Let C be a closed convex subset of X with the drop property. Then every $C-$stream has a convergent subsequence.

Assume that there exists a $C-$stream (x_n) with no norm convergent subsequence. Then the set $A := \{x_n; n \geq 1\}$ is closed. Now $x_{n+1} \in D(x_n, C)$ for all n and we see that is no $n \geq 1$ such that $D(x_n, C) \cap A = \{x_n\}$, so C does not have the drop property and the fact is proved.

For completing the proof, let $y_n := (1 + \mu_{n+1})\, x_n$. Then the sequence (y_n) is a $C-$stream. Since C has the drop property, (y_n) has a norm convergent subsequence denoted also by (y_n). So $y_n \longrightarrow y$ in F. But we have $\mathrm{dist}(y_n, C) \longrightarrow 0$ and C is closed, therefore we deduce that $y \in C$, which gives a contradiction with the fact that $C \cap S = \emptyset$. Therefore the case 2 is impossible and our theorem is proved. □

In order to prove Theorem 3.2, we shall need the following lemmas:

Lemma 3.4

Let $(X, \|.\|)$ be a Banach space. Let C be a closed convex subset of X containing the unit ball. Let $x_1 \in X \setminus C, x_2 \in \{tx_1; t \in (0,1)\} \setminus C$ and $\eta \in (1, \rho(x_2))$, where ρ is the Minkowski functional of C. Let $M := 2\sup\{\max(\rho(x_1 - c), \rho(c - x_1)); c \in D(x_1, C)\}$. Then for $0 < \varepsilon < \frac{\eta-1}{M}\frac{\rho(x_1-x_2)}{\rho(x_1)}$, we have $[D(x_2, C) + \varepsilon B] \setminus \eta C \subset \mathrm{int}D(x_1, C)$.

Proof. Let $0 < \varepsilon_2 < 1$ such that $x_2 = (1-\varepsilon_2)x_1$. Let $x \in D(x_2, C) \setminus \eta C$, then $x = \lambda x_2 + (1-\lambda)c$ for some $\lambda \in [0,1]$ and $c \in C$, with $\rho(x) > \eta$. Without loss of generality we assume that $\lambda \in (0,1)$. So, $x - c = \lambda(x_2 - c)$ which implies

$$\begin{aligned} \eta - 1 < |\rho(x) - \rho(c)| &\leq \rho(x-c) = \lambda\rho(x_2 - c) \leq \lambda[\rho(x_2 - x_1) + \rho(x_1 - c)] \\ &\leq 2\lambda \sup\{\max(\rho(x_1 - c), \rho(c - x_1)); c \in D(x_1, C)\} = \lambda M. \end{aligned}$$

Therefore $\frac{\eta-1}{M} < \lambda$. By the definition of ε_2 we see that $\varepsilon_2 = \frac{\rho(x_1 - x_2)}{\rho(x_1)}$.

Let $0 < \varepsilon < \frac{\eta-1}{M}\varepsilon_2$. Let $y \in x + \varepsilon B$ and $y_2 := \frac{y}{\lambda} - \frac{1-\lambda}{\lambda}c$. Whence $\lambda\|y_2 - x_2\| = \|y - x\| \leq \varepsilon$. This means that

$$\|y_2 - x_2\| \leq \frac{\varepsilon}{\lambda} < \frac{M\varepsilon}{\eta - 1}$$

and by the definition of ε, we can see that $\|y_2 - x_2\| < \varepsilon_2$. Then we have proved that $y_2 \in x_2 + \varepsilon_2 \mathrm{int}B = (1-\varepsilon_2)x_1 + \varepsilon_2\mathrm{int}B \subset \mathrm{int}D(x_1, C)$. Therefore $y = \lambda y_2 + (1-\lambda)c \in \mathrm{int}D(x_1, C)$, for all $y \in x + \varepsilon B$. Then $[D(x_2, C) + \varepsilon B] \setminus \eta C \subset \mathrm{int}D(x_1, C)$ and the proof is complete. □

Lemma 3.5

Let $(X, \|.\|)$ be a Banach space and assume that the dual norm is L.U.R. (resp. R.) in X^*. Let C be a closed convex set, then $C_1 := C + \varepsilon B$ is a Fréchet (resp. Gâteaux) smooth set for all $\varepsilon > 0$.

Proof. It is easy to see that $\overline{C_1} = \left\{x;\ \mathrm{dist}^2(x, C) \leq \varepsilon^2\right\}$. Since the dual norm is L.U.R. (resp. R.) and C is a closed convex set then the function $f(x) := \mathrm{dist}^2(x, C)$ is Fréchet (resp. Gâteaux) smooth on X (see p. 365 of [4]). □

Proof of Theorem 3.2. Since X is reflexive, X has an equivalent norm having the Kadec-Klee property and its dual norm is L.U.R. (resp. R.) in X^*. The drop property of C and the property (α) are invariant under isomorphisms. We therefore can assume without loss of generality that the original norm $\|.\|$ has the Kadec-Klee property and its dual is L.U.R. (resp. R.).

Case 1 : For some x_1 we have

$$D_1 := F \cap D(\mu z, C) \cap \rho(x_1)\mathrm{int}C \subset \mathrm{int}D(\mu z, C).$$

Let $d := \inf\{\rho(x); x \in D_1\}$, it is clear that $d < \rho(x_1)$. Since C has the drop property then Theorem 2.1, yields the existence of a dense G_δ subset Γ of $X \setminus \{\rho(F, x) = 0\}$

such that every $x \in \Gamma$ has a $C-$nearest point in $\overline{D}_1$. Let $0 < \delta < \min\left(1, \frac{d-1}{2}, \frac{\rho(x_1)-d}{2}\right)$ and $\lambda > 0$ such that $B[0,\lambda] \subset \delta C$. So we can choose a point a in $\delta C \cap B[0,\lambda]$, such that a has a $C-$nearest point z_0 in D_1, i.e.

$$\rho(z_0 - a) = \inf\{\rho(z-a); z \in D_1\}.$$

Let $x \in a + \rho(z_0 - a)C$. So we have

$$\begin{aligned}\rho(D_1, a) &:= \inf\{\rho(x-a); x \in D_1\} = \rho(z_0 - a)\\ &\leq \rho(-a) + \inf\{\rho(x); x \in D_1\} \leq \delta + d.\end{aligned}$$

Therefore

$$\rho(x) \leq \rho(x-a) + \rho(a) \leq \rho(z_0 - a) + \rho(a) \leq \delta + d + \delta \leq \rho(x_1).$$

Thus

$$a + \rho(z_0 - a)C \subset \rho(x_1)C.$$

Let $x \in C$, then $\rho(z_0 - a) \geq |\rho(z_0) - \rho(-a)| \geq d - \delta \geq 1 + \delta \geq 1 + \rho(-a) \geq \rho(x) + \rho(-a) \geq \rho(x-a)$. This means that $x \in a + \rho(z_0 - a)C$ and consequently we have proved that

$$C \subset a + \rho(z_0 - a)C \subset \rho(x_1)C.$$

By our construction we have $(a + \rho(z_0 - a)C) \cap F \supset \{z_0\}$ and $F \cap \mathrm{int}(a + \rho(z_0 - a)C) = \emptyset$. Let

$$C_0 := \{x; \rho^2(x-a) + \lambda\|x - z_0\|^2 \leq \rho^2(z_0 - a)\}$$

then C_0 is Fréchet (resp. Gâteaux) smooth contained in $a + \rho(z_0 - a)C$ and it is clear that $C_0 \cap \partial[a + \rho(z_0 - a)C] = \{z_0\}$, and for λ small we can check that $C_0 \subset D(\mu z, C)$. Thus $\mathrm{conv}(C_0 \cup C) \cap F = \{z_0\}$.

Case 2 : Case 1 is not satisfied.

Choose $x_2 \in F \cap \mathrm{int}D(\mu z, C)$ and $\alpha > 0$ such that $x_1 := (1+\alpha)x_2 \in D(\mu z, C)$. Then by Lemma 3.4 there exists $\varepsilon > 0$ such that $[D(x_2, C) + \varepsilon B] \cap F \subset \mathrm{int}D(x_1, C)$.

Let $F_2 := D\left(x_2, C + \frac{\varepsilon}{2}B\right) \cap F$. Since C has the drop property then it is easy to see that $C + \frac{\varepsilon}{2}B$ also has the drop property (for more see [9] and [11]) and by Lemma 3.5 is a Fréchet (resp. Gâteaux) smooth set. So we can choose $a_1 \in F_2$ such that $D\left(a_1, C + \frac{\varepsilon}{2}B\right) \cap F_2 = \{a_1\}$. Let $0 < \alpha < \frac{\varepsilon}{4}$ and $a_2 := ta_1$ for some $t \in (0,1)$ such that $\|a_1 - a_2\| = \frac{\alpha}{2}$ and let $\eta > 0$ such that $B[a_2, \eta] \subset \mathrm{int}D\left(x_2, C + \frac{\varepsilon}{2}B\right)$. Now it is easy to see that $C_1 := \mathrm{conv}\left(C + \frac{\varepsilon}{2}B \cup B\left[a_2, \eta\right]\right)$ satisfies the condition of the first case. Our theorem is proved. □

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