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Near smoothness of Banach spaces

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ABSTRACT

The aim of this paper is to discuss the concept of near smoothness in some Banach sequence spaces.

1. Introduction

In the geometric theory of Banach spaces the notion of smoothness plays very important and fundamental role (cf. [10, 14], for example).

This notion finds also numerous applications in other branches of nonlinear functional analysis and control theory, among others [4, 6, 12, 14].

In recent years the notion in question has been generalized in terms of compactness conditions by several authors [2, 5, 7, 15, 17].

Following the definition proposed in the paper [2] we will consider here the notion of near smoothness in some Banach sequence spaces such as $c_0(E_i)$ and $l^p(E_i)$. Particularly we show that these spaces are nearly smooth provided E_i 's have this

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property. Moreover, we indicate also some connections between the concepts of near smoothness and the duality mapping.

2. Notation, definitions and preliminary results

Throughout this paper we will always assume that $(E, \|\cdot\|)$ is an infinite dimensional real Banach space with the zero element θ . By E^* we denote the dual space of E . The symbols B_E and S_E stand for the unit ball and the unit sphere of E , respectively.

Further, fix $x \in S_E$ and $f \in S_{E^*}$. For a given number $\varepsilon \in [0, 1]$ consider the slices $F(f, \varepsilon)$ and $F^*(x, \varepsilon)$ defined in the following way

$$F(f, \varepsilon) = \{x \in B_E : f(x) \geq 1 - \varepsilon\},$$

$$F^*(x, \varepsilon) = \{g \in B_{E^*} : g(x) \geq 1 - \varepsilon\}.$$

Now, we can define the so-called modulus of near convexity [2] of the space E as the function $\beta_E : [0, 1] \longrightarrow [0, 1]$ given by

$$\beta_E(\varepsilon) = \sup \{\mu(F(f, \varepsilon)) : f \in S_{E^*}\},$$

where μ denotes the Hausdorff measure of noncompactness in the space E (cf. [4], for instance).

Similarly, the function $\Sigma_E : [0, 1] \longrightarrow [0, 1]$ defined by the formula

$$\Sigma_E(\varepsilon) = \sup \{\mu(F^*(x, \varepsilon)) : x \in S_E\}$$

will be called the modulus of near smoothness of the space E [2].

With help of the moduli introduced above we can define further concepts being useful in the geometric theory of Banach spaces (cf. [2]). For further goals we recall only those used in the sequel.

We say that the space E is nearly uniformly smooth (NUS, in short) if $\lim_{\varepsilon \rightarrow 0} \Sigma_E(\varepsilon) = 0$. In the case when $\lim_{\varepsilon \rightarrow 0} \mu(F^*(x, \varepsilon)) = 0$ for any $x \in S_E$ the space is said to be locally nearly uniformly smooth (LNUS). Similarly, the space E is referred to as nearly smooth (NS) whenever for any $x \in S_E$ the set $F_x^* = \{f \in S_{E^*} : f(x) = 1\}$ is compact.

Notice that $NUS \implies LNUS \implies NS$ but no converse implication is true (cf. [2, 17] and the examples given below).

On the other hand, taking into account that $F_x^* = F^*(x, 0)$ we can show that E is NS if and only if $\Sigma_E(0) = 0$.

We will say that the space E is nearly strictly convex (NSC) provided its unit sphere does not contain noncompact and convex sets. Keeping in mind this definition and the results given in [1, Lemma 2] we can easily see that E is NSC if and only if $\beta_E(0) = 0$.

On the other hand applying the following inequality [3]

$$\frac{1}{2}\beta_E(\varepsilon) \leq \Sigma_{E^*}(\varepsilon), \quad \varepsilon \in [0, 1]$$

we can deduce the following simple but handy result.

Lemma 1

A Banach space E is NSC if E^ is NS .*

In what follows we are going to point out certain relationship between the concept of near smoothness and the duality map.

Recall [4, 12] that the map $F : E \longrightarrow 2^{E^*}$ defined by

$$F(x) = \{f \in E^* : f(x) = \|x\|^2 = \|f\|^2\}$$

is called the duality map on the space E .

The duality map is frequently used in the theory of differential and integral equations in Banach spaces [4, 12] because it creates the possibility to formulate the so-called dissipative conditions.

For the properties of the duality map we refer to [4,12], for instance. Now, let us fix $x \in E$, $x \neq \theta$. Then we have

$$\begin{aligned} F(x) &= \{f \in E^* : f(x) = \|x\|^2 = \|f\|^2\} \\ &= \{\|x\|g : g \in S_{E^*}, g(x) = \|x\|\} \\ &= \|x\| \cdot \{g \in S_{E^*} : g(x) = \|x\|\} \\ &= \|x\| \cdot \{g \in S_{E^*} : g(x/\|x\|) = 1\} = \|x\| \cdot F_{(x/\|x\|)}^*. \end{aligned}$$

Moreover, it is easy to check that $F(\theta) = \{\theta\}$. Hence we deduce the following characterization.

Theorem 1

A space E is NS if and only if the duality map on E is compact valued.

In what follows we provide a few examples illustrating the concepts introduced above.

EXAMPLES 1 [2]: Let c_0 denote the classical space of real sequences converging to zero with maximum norm. Then it may be shown that $\Sigma_{c_0}(\varepsilon) = \varepsilon$ for $\varepsilon \in [0, 1]$. Thus c_0 is *NUS* space but not reflexive.

EXAMPLES 2: Consider the space c of real converging sequences with supremum norm. Take $x \in S_c$, $x = (1, 1, \dots)$. Then we have

$$\begin{aligned} F_x^* &= \{y \in S_{c^*} : y(x) = 1\} \\ &= \left\{ y = (y_i) \in S_{l^1} : y_1 \cdot \lim_{i \rightarrow \infty} x_i + \sum_{i=2}^{\infty} y_i x_i = 1 \right\} = \left\{ (y_i) \in S_{l^1} : \sum_{i=1}^{\infty} y_i = 1 \right\}. \end{aligned}$$

Particularly $F_x^* \supset \{e_i : i = 1, 2, \dots\}$, where $e_i = (\delta_{ij})$. Thus F_x^* is not compact which implies that c is not *NS*.

EXAMPLES 3: Take the space $l^p(l^{p_1}, l^{p_2}, \dots)$, where $p, p_i \in (1, \infty)$ ($i = 1, 2, \dots$) and $\lim_{i \rightarrow \infty} p_i = 1$. It was shown in [2] that this space is *LNUS* but not *NUS*.

EXAMPLES 4: Let $C = C[0, 1]$ be the classical space of real functions defined and continuous on the interval $[0, 1]$. Assume that C is endowed by the norm $\|\cdot\|$ defined as follows

$$\|x\| = \|x\|_C + \left(\int_0^1 |x(t)|^2 dt \right)^{1/2} + \sum_{i=1}^{\infty} \frac{1}{2^i} \sup \left\{ |x(t) - x(s)| : |t - s| \leq \frac{1}{i} \right\},$$

where $\|\cdot\|_C$ denotes the standard maximum norm. Since the norms $\|\cdot\|$ and $\|\cdot\|_C$ are equivalent, the space $(C, \|\cdot\|)$ is not reflexive. This fact in conjunction with results established in [16] and [1] allows us to infer that the space C^* is not *LNUS*. On the other hand the space $(C, \|\cdot\|)$ is *NSC* [8].

3. Near smoothness of the space $c_0(E_i)$

Assume that $(E_i, \|\cdot\|_{E_i})$ ($i = 1, 2, \dots$) is a sequence of infinite dimensional Banach spaces. Then we can consider the so-called product space $c_0(E_i) = c_0(E_1, E_2, \dots)$ which consists of all sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \dots$ and $\lim_{i \rightarrow \infty} \|x_i\|_{E_i} = 0$.

It is well known [10] that $c_0(E_i)$ forms a Banach space under the norm

$$\|x\|_{c_0} = \|(x_i)\|_{c_0} = \max \{ \|x_i\|_{E_i} : i = 1, 2, \dots \}.$$

The basic result of this section is contained in the following theorem.

Theorem 2

Let E_i be *NS* for every $i = 1, 2, \dots$. Then the space $c_0(E_i)$ is also *NS*.

Proof. Fix a point $x = (x_i) \in c_0 = c_0(E_i)$ such that $\|x\|_{c_0} = 1$. Denote

$$T = \{i \in \mathbb{N} : \|x_i\|_{E_i} = 1\}.$$

Obviously T is finite and nonempty. Without loss of generality we may assume that $T = \{1, 2, \dots, k\}$ for some natural number k .

Further, put $\gamma = \max\{\|x_i\|_{E_i} : i \geq k+1\}$. Obviously $\gamma < 1$.

Now, take $f = (f_i) \in F_x^*$. This means that $\|f\|_{l^1(E_i^*)} = \sum_{i=1}^{\infty} \|f_i\|_{E_i^*} = 1$ and $f(x) = 1$. Hence we get

$$\begin{aligned} 1 &= f(x) = f_1(x_1) + f_2(x_2) + \dots \\ &\leq \|f_1\|_{E_1^*} \|x_1\|_{E_1} + \|f_2\|_{E_2^*} \|x_2\|_{E_2} + \dots \\ &\leq \|f_1\|_{E_1^*} + \dots + \|f_k\|_{E_k^*} + \gamma(\|f_{k+1}\|_{E_{k+1}^*} + \dots). \end{aligned} \quad (1)$$

Denote $t = \|f_{k+1}\|_{E_{k+1}^*} + \|f_{k+2}\|_{E_{k+2}^*} + \dots$. Then, from the inequalities (1) we obtain that

$$1 \leq 1 - t + \gamma t = 1 + (\gamma - 1)t.$$

Consequently $(\gamma - 1)t \geq 0$. Since $\gamma < 1$ we get that $t = 0$. Thus, from (1) we have

$$1 = f_1(x_1) + f_2(x_2) + \dots + f_k(x_k) \leq \|f_1\|_{E_1^*} + \|f_2\|_{E_2^*} + \dots + \|f_k\|_{E_k^*} \leq 1.$$

Hence

$$f_1(x_1) + \dots + f_k(x_k) = \|f_1\|_{E_1^*} + \dots + \|f_k\|_{E_k^*} = 1.$$

In particular, the above equality implies

$$f_i(x_i) = \|f_i\|_{E_i^*} \quad (2)$$

for $i = 1, 2, \dots, k$.

In what follows fix $i \in \{1, 2, \dots, k\}$ and consider the set

$$F_i^* = \{f_i \in B_{E_i^*} : f_i(x_i) = \|f_i\|_{E_i^*}\}.$$

Let $F_{x_i}^*$ be defined as previously. Then, in view of the assumption we infer that $F_{x_i}^*$ is a compact subset of $S_{E_i}^*$.

Next observe that

$$F_i^* \subset \bigcup_{0 \leq \lambda \leq 1} \lambda F_{x_i}^*. \quad (3)$$

Indeed, take $f_i \in F_i^*$, $f_i \neq \theta$. By (2) we have that $f_i(x_i) = \|f_i\|_{E_i^*}$ so $g_i = f_i/\|f_i\|_{E_i^*}$ is a member of $S_{E_i^*}$. Obviously $g_i(x_i) = 1$ which implies that $g_i \in F_{x_i}^*$. But $f_i = \|f_i\|_{E_i^*} g_i \in \bigcup_{0 \leq \lambda \leq 1} \lambda F_{x_i}^*$ since $\|f_i\|_{E_i^*} \leq 1$. Now, notice that

$$\bigcup_{0 \leq \lambda \leq 1} \lambda F_{x_i}^* = \text{Conv}(\{\theta\} \cup F_{x_i}^*),$$

where the symbol $\text{Conv } X$ denotes the convex closed hull of X . Hence, in virtue of (3) and the Mazur theorem we conclude that F_i^* is compact ($i = 1, 2, \dots, k$).

Finally, by the criterion of compactness in the space $l^1(X_i)$ due to Leonard [13] we infer that the set F_x^* is compact.

This completes the proof. \square

From the above theorem we obtain, for example, that the spaces $c_0(c_0, c_0, \dots)$ and $c_0(l^{p_1}, l^{p_2}, \dots)$ are NS provided $p_i > 1$ for $i = 1, 2, \dots$.

4. Near smoothness of the space $l^p(E_i)$

As in the previous section we can consider the Banach sequence space $l^p(E_i) = l^p(E_1, E_2, \dots)$ ($1 \leq p < \infty$) consisting of all sequences $x = (x_i)$, $x_i \in E_i$ ($i = 1, 2, \dots$) such that

$$\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p < \infty$$

and furnished by the norm

$$\|x\| = \|(x_i)\| = \left(\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p \right)^{1/p}.$$

It turns out that near smoothness in the space $l^p(E_i)$ behaves similarly as in the space $c_0(E_i)$.

More precisely, we have the following result.

Theorem 3

If E_i is NS for any $i = 1, 2, \dots$ then $l^p(E_i)$ is also NS for $1 < p$.

Proof. Take $x \in S_{l^p(E_i)}$ i.e. $x = (x_i)$, where

$$\|x\| = \left(\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p \right)^{1/p} = \sum_{i=1}^{\infty} \|x_i\|_{E_i}^p = 1.$$

Consider the set

$$\begin{aligned} F_x^* &= \{f = (f_i) \in S_{(l^p(E_i))^*} : f(x) = 1\} \\ &= \{f = (f_i) \in S_{l^q(E_i^*)} : f_1(x_1) + f_2(x_2) + \dots = 1\}, \end{aligned}$$

where $1/p + 1/q = 1$.

Thus, taking arbitrarily $f \in F_x^*$ we have

$$\|f\| = \|(f_i)\| = \sum_{i=1}^{\infty} \|f_i\|_{E_i^*}^q = 1.$$

Hence, applying the Hölder inequality we get

$$\begin{aligned} 1 &= f_1(x_1) + f_2(x_2) + \dots \leq \|f_1\|_{E_1^*} \|x_1\|_{E_1} + \|f_2\|_{E_2^*} \|x_2\|_{E_2} + \dots \\ &\leq \left(\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p \right)^{1/p} \left(\sum_{i=1}^{\infty} \|f_i\|_{E_i^*}^q \right)^{1/q} = 1. \end{aligned}$$

This yields

$$\begin{aligned} 1 &= f_1(x_1) + f_2(x_2) + \dots = \|f_1\|_{E_1^*} \|x_1\|_{E_1} + \|f_2\|_{E_2^*} \|x_2\|_{E_2} + \dots \\ &= \left(\sum_{i=1}^{\infty} \|x_i\|_{E_i}^p \right)^{1/p} \left(\sum_{i=1}^{\infty} \|f_i\|_{E_i^*}^q \right)^{1/q} = 1. \end{aligned} \tag{4}$$

Consequently we deduce that

$$f_i(x_i) = \|f_i\|_{E_i^*} \|x_i\|_{E_i} \tag{5}$$

for every $i = 1, 2, \dots$

On the other hand keeping in mind (4) and using the well-known property concerning the equality sign in the Hölder inequality [9] we infer that

$$\|x_i\|_{E_i}^p = \|f_i\|_{E_i^*}^q \tag{6}$$

for all $i = 1, 2, \dots$

Now, let us fix arbitrarily $\varepsilon > 0$. Then we can find $n_0 \in \mathbb{N}$ such that

$$\sum_{i=n_0}^{\infty} \|x_i\|_{E_i}^p \leq \varepsilon.$$

Hence, if we choose an arbitrary element $f = (f_i) \in F_x^*$ and use of (6) we get

$$\sum_{i=n_0}^{\infty} \|f_i\|_{E_i^*}^q \leq \varepsilon. \quad (7)$$

Next, fix $i \in \mathbb{N}$ and take $y_i = x_i / \|x_i\|_{E_i}$ provided $x_i \neq \theta$. Then the set

$$F_{y_i}^* = \{g \in S_{E_i^*} : g(y_i) = 1\}$$

is compact in view of the assumptions, since $y_i \in S_{E_i}$. Keeping in mind (5) and repeating the same reasoning as in the proof of Theorem 2 we can show that the set

$$F_i^* = \{f_i \in B_{E_i^*} : f_i(x_i) = \|f_i\|_{E_i^*} \|x_i\|_{E_i}\}$$

is contained in the set $\text{Conv}(\{\theta\} \cup F_{y_i}^*)$. This shows that F_i^* is compact for all $i \in \mathbb{N}$ such that $x_i \neq 0$. In the case when $x_i = 0$ the compactness of F_i^* follows easily from (6).

Now, let us take the projection $p_i : l^q(E_1^*, E_2^*, \dots) \longrightarrow E_i^*$, $p_i(f_1, f_2, \dots) = f_i$.

Obviously $p_i(F_x^*) = F_i^*$. This implies that $p_i(F_x^*)$ is compact for every $i = 1, 2, \dots$

Finally, combining the above assertion with (7) and taking into account the Leonard criterion of compactness [13] we obtain that the set F_x^* is compact.

Thus the proof is complete. \square

As an immediate corollary we get that the spaces $l^p(c_0, c_0, \dots)$ and $l^p(l^{p_1}, l^{p_2}, \dots)$ are NS whenever $p > 1$ and $p_i > 1$ for $i = 1, 2, \dots$

At the end let us pay our attention to some open problems which can be raised in the light of the results obtained.

These problems can be formulated as follows:

1. Assume that E_i is NUS for $i = 1, 2, \dots$. Is the space $c_0(E_i)$ NUS ?
2. Is the space $c_0(E_i)$ $LNUS$ provided the spaces E_i ($i = 1, 2, \dots$) are such?
3. Suppose E_i is NUS for $i = 1, 2, \dots$. Consider the function $r(\varepsilon) = \sup\{\Sigma_{E_i}(\varepsilon) : i = 1, 2, \dots\}$, $\varepsilon \in [0, 1]$. Under the assumption that all the spaces E_i are reflexive and $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$ it was proved in [3] (cf. also [11]) that the space $l^p(E_i)$ is NUS . Is this result true without the assumption on reflexivity?

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