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Some qualitative results for the linear theory of binary mixtures of thermoelastic solids

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Abstract

In this paper we study the linear thermodynamical problem of mixtures of thermoelastic solids. We use some results of the semigroup theory to obtain an existence theorem for the initial value problem with homogeneous Dirichlet boundary conditions. Continuous dependence of solutions upon the initial data and body forces is also established. We finish with a study of the asymptotic behavior of solutions of the homogeneous problem.

1. Introduction

The continuum theory of mixtures has been a subject of study in recent years. The works of Truesdell & Toupin [23], Kelly [17], Eringen & Ingram [9, 16], Green & Naghdi [11, 12], Müller [19] and Bowen & Wise [6] may be considered the starting point of the modern formulations of continuum thermomecanical theories of mixtures. Presentations of this theory can be found in the review articles of Bowen [5], Atkin & Craine [2], Bredford & Drumheller [4] and Rajagopal & Wineman [22].

This paper is concerned with the theory of binary mixtures of thermoelastic solids established by Ieşan [13], where a Lagrangian description is adopted. The equations are expressed in terms of quantities defined on the reference configuration Ω_0. In this theory the independent constitutive variables are the displacement gradients, relative displacement, temperature and temperature gradient.

We recall that uniqueness results in the linear theory of mixture of elastic solids without temperature effects have been presented by Atkin, Chadwick & Steel [3], Knops & Steel [18], and by Ieşan [14] in the case of nonsimple materials and by Quintanilla [21] in the nonlinear case. An existence theorem has been established by Ieşan & Quintanilla [15]. Dafermos [8] studied the asymptotic behavior of solutions of the equations of motion of a mixture of two linear homogeneous, isotropic materials. In [13] Ieşan also obtained an uniqueness theorem for the linear theory with thermal effects.

2. Basic equations

We consider a mixture of two interacting continua s_1 and s_2, that at time $t = 0$ occupies the region Ω_0 of Euclidean three-dimensional space. Let $\partial\Omega_0$ be the boundary of Ω_0. We refer the motion of each constituent to the reference configuration and a fixed system of rectangular Cartesian axes. In what follows, subscripts preceded by a comma denote partial differentiation with the corresponding Cartesian coordinate. We also use a superposed dot to denote partial differentiation respect to time. Greek indices are understood to range over the integers 1, 2. As usual, letters in boldface stand for tensors of order $p \geq 1$, and if $\mathbf{v}$ is of order p, we write $v_{i_1 \ldots i_p}$ for the components of $\mathbf{v}$ in the Cartesian coordinate frame.

The displacement of typical particles of s_1 and s_2 at time t are $\mathbf{u}$, $\mathbf{w}$, where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $\mathbf{w} = \mathbf{w}(\mathbf{y}, t)$, $\mathbf{x}, \mathbf{y} \in \Omega_0$. We assume that the particles under consideration occupy the same position at time $t = 0$, so that $\mathbf{x} = \mathbf{y}$. Let $\theta = \theta(\mathbf{x}, t)$ be the temperature of the material point $\mathbf{x}$ at time t. We denote by ρ_α the mass density of the constitutive s_α at time $t = 0$, $\mathbf{t}$ and $\mathbf{s}$ the partial stress tensors associated with the constituents s_1 and s_2, $\mathbf{p}$ the diffusive force vector, η the entropy density and $\mathbf{q}$ the heat flux vector.

The field equations of the theory consist of the equations of motion

$$\begin{aligned} t_{ji,j} - p_i + \rho_1 F_i^1 &= \rho_1 \ddot{u}_i, \\ s_{ji,j} + p_i + \rho_2 F_i^2 &= \rho_2 \ddot{w}_i, \end{aligned} \tag{1}$$

the equation of the energy

$$\rho T_0 \dot{\eta} = q_{i,i} + \rho r, \tag{2}$$

and the constitutive equations

$$\begin{aligned} t_{ji} =& A_{ji} + B_{ji} + A_{jr}u_{i,r} + B_{jr}w_{i,r} + (A_{jirs} + B_{rsji})e_{rs} \\ &+ (B_{jirs} + C_{jirs})g_{rs} + (D_{jir} + E_{jir} + C_j\delta_{ir} + b_j\delta_{ir})d_r - (\beta_{ji} + \gamma_{ji})\theta\,, \\ s_{ji} =& B_{ij} + B_{rj}u_{i,r} + B_{rsij}e_{rs} + C_{ijrs}g_{rs} + E_{ijr}d_r - \gamma_{ij}\theta - b_jd_i\,, \\ p_i =& C_i + c_ju_{i,j} + D_{rsi}e_{rs} + E_{rsi}g_{rs} + \alpha_{ij}d_j - \xi_i\theta + (b_jd_i)_{,j}\,, \end{aligned}$$

$$\rho\eta = D + \beta_{ij}e_{ij} + \gamma_{ij}g_{ij} + \xi_i d_i + a\theta\,, \tag{3}$$

$$q_i = k_{ij}\theta_{,j}, \tag{4}$$

where $\mathbf{F}^\alpha$ are the body force per unit mass acting on the constituent s_α, $\rho = \rho_1 + \rho_2$, T_0 is the constant absolute temperature of the body in the reference configuration and r is the external heat supply per unit mass per unit time. We also use the notation

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad g_{ij} = \frac{1}{2}(u_{j,i} + w_{i,j}), \quad d_i = u_i - w_i\,. \tag{5}$$

The coefficients in (4) have the following symmetries:

$$\begin{aligned} A_{ij} = A_{ji}, \quad \beta_{ij} = \beta_{ji}, \quad \alpha_{ij} = \alpha_{ji}, \quad D_{ijr} = D_{jir}, \\ A_{ijrs} = A_{jirs} = A_{rsij}, \quad B_{ijrs} = B_{jirs}, \quad C_{ijrs} = C_{rsij}. \end{aligned} \tag{6}$$

If we introduce the tensors:

$$\begin{aligned} p_i^0 = C_i, \qquad t_{ij}^0 = A_{ij} + B_{ij}, \qquad s_{ij}^0 = B_{ji}, \qquad b_{ij} = \beta_{ji} + \gamma_{ji}, \\ h_{ijr} = D_{jir} + E_{jir} + C_j\delta_{ir}, \qquad k_{ijr} = E_{ijr}, \qquad d_{ijrs} = C_{ijrs}, \\ a_{ijrs} = A_{js}\delta_{ir} + A_{jirs} + B_{rsji} + B_{jisr} + C_{jisr}, \\ b_{ijrs} = B_{js}\delta_{ir} + B_{jirs} + C_{jisr}, \end{aligned} \tag{7}$$

the constitutive equations become

$$\begin{aligned} t_{ji} &= t_{ji}^0 + a_{ijrs}u_{r,s} + b_{ijrs}w_{r,s} + h_{ijr}d_r - b_{ij}\theta + b_jd_i, \\ s_{ji} &= s_{ji}^0 + b_{rsij}u_{r,s} + d_{ijrs}w_{r,s} + k_{ijr}d_r - \gamma_{ij}\theta - b_jd_i, \\ p_i &= p_i^0 + h_{rsi}u_{r,s} + k_{rsi}w_{r,s} + \alpha_{ij}d_j - \xi_i\theta + (b_jd_i)_{,j}, \\ \rho\eta &= D + b_{ij}u_{i,j} + \gamma_{ij}w_{i,j} + \xi_id_i + a\theta, \end{aligned}$$

$$q_i = k_{ij}\theta_{,j}. \tag{8}$$

From (6) and (7) we get

$$a_{ijrs} = a_{rsij}, \qquad d_{ijrs} = d_{rsij}, \qquad \alpha_{ij} = \alpha_{ji}. \tag{9}$$

In view of (9), the equations of motion and the equation of the energy can be expressed in terms of the functions u_i, w_i, θ. We obtain the equations:

$$\begin{aligned}
\rho_1 \ddot{u}_i &= (a_{ijrs}u_{r,s} + b_{ijrs}w_{r,s} + h_{ijr}d_r - b_{ij}\theta)_{,j} - h_{rsi}u_{r,s} \\
&\quad - k_{rsi}w_{r,s} - \alpha_{ij}d_j + \xi_i\theta + H_i^1, \\
\rho_2 \ddot{w}_i &= (b_{ijrs}u_{r,s} + d_{ijrs}w_{r,s} + k_{ijr}d_r - \gamma_{ij}\theta)_{,j} + h_{rsi}u_{r,s} \\
&\quad + k_{rsi}w_{r,s} + \alpha_{ij}d_j - \xi_i\theta + H_i^2, \\
a\dot{\theta} &= \frac{1}{T_0}(k_{ij}\theta_{,j})_{,i} - b_{ij}\dot{u}_{i,j} - \gamma_{ij}\dot{w}_{i,j} - \xi_i\dot{d}_i + S,
\end{aligned}$$

where $H_i^1 = t_{ji,j}^0 - p_i^0 + F_i^1$, $H_i^2 = s_{ji,j}^0 + p_i^0 + F_i^2$, and $S = \rho r T_0^{-1}$.

It is convenient to introduce the following dimensionless quantities:

$$\bar{\mathbf{u}} = \frac{\mathbf{u}}{L}, \quad \bar{\mathbf{w}} = \frac{\mathbf{w}}{L}, \quad \bar{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \bar{t} = \frac{t}{\tau_0}, \quad \bar{\theta} = \frac{\theta}{\tilde{T}}, \quad \bar{T}_0 = \frac{t_0}{\tilde{T}}, \quad \bar{\rho}_\alpha = \frac{L^3}{m_0}\rho_\alpha,$$

where L, τ_0, $\tilde{T}$ and m_0 are constants with dimension of length, time, temperature and mass respectively. With these quantities we can write the equations in the form

$$\begin{aligned}
\bar{\rho}_1 \ddot{\bar{u}}_i &= (\bar{a}_{ijrs}\bar{u}_{r,s} + \bar{b}_{ijrs}\bar{w}_{r,s} + \bar{h}_{ijr}\bar{d}_r - \bar{b}_{ij}\bar{\theta})_{,j} - \bar{h}_{rsi}\bar{u}_{r,s} \\
&\quad - \bar{k}_{rsi}\bar{w}_{r,s} - \bar{\alpha}_{ij}\bar{d}_j + \bar{\xi}_i\bar{\theta} + \bar{H}_i^1, \\
\bar{\rho}_2 \ddot{\bar{w}}_i &= (\bar{b}_{ijrs}\bar{u}_{r,s} + \bar{d}_{ijrs}\bar{w}_{r,s} + \bar{k}_{ijr}\bar{d}_r - \bar{\gamma}_{ij}\bar{\theta})_{,j} + \bar{h}_{rsi}\bar{u}_{r,s} \\
&\quad + \bar{k}_{rsi}\bar{w}_{r,s} + \bar{\alpha}_{ij}\bar{d}_j - \bar{\xi}_i\bar{\theta} + \bar{H}_i^2, \\
\bar{a}\dot{\bar{\theta}} &= \frac{1}{\bar{T}_0}(\bar{k}_{ij}\bar{\theta}_{,j})_{,i} - \bar{b}_{ij}\dot{\bar{u}}_{i,j} - \bar{\gamma}_{ij}\dot{\bar{w}}_{i,j} - \bar{\xi}_i\dot{\bar{d}}_i + \bar{S},
\end{aligned} \tag{10}$$

with the relations

$$\bar{a}_{ijrs} = \bar{a}_{rsij}, \qquad \bar{d}_{ijrs} = \bar{d}_{rsij}, \qquad \bar{\alpha}_{ij} = \bar{\alpha}_{ji}. \tag{11}$$

Now subscripts preceded by a comma denote partial differentiation respect $\bar{\mathbf{x}}$, superposed dot denotes partial differentiation respect to $\bar{t}$ and

$$\bar{a}_{ijrs} = a_{ijrs}\frac{L\tau_0^2}{m_0}, \qquad \bar{b}_{ijrs} = b_{ijrs}\frac{L\tau_0^2}{m_0}, \qquad \bar{d}_{ijrs} = d_{ijrs}\frac{L\tau_0^2}{m_0},$$

$$\bar{h}_{ijr} = h_{ijr}\frac{L^2\tau_0^2}{m_0}, \qquad \bar{k}_{ijr} = k_{ijr}\frac{L^2\tau_0^2}{m_0},, \qquad \bar{\alpha}_{ij} = \alpha_{ij}\frac{L^3\tau_0^2}{m_0}$$

$$\bar{b}_{ij} = b_{ij}\frac{L\tau_0^2\tilde{T}}{m_0}, \qquad \bar{\gamma}_{ij} = \gamma_{ij}\frac{L\tau_0^2\tilde{T}}{m_0},$$

$$\bar{k}_{ij} = k_{ij}\frac{\tau_0^3\tilde{T}}{m_0 L}, \qquad \bar{\xi}_i = \xi_i\frac{L^2\tau_0^2\tilde{T}}{m_0}, \qquad \bar{a} = a\frac{L\tau_0^2\tilde{T}^2}{m_0},$$

$$\bar{H}_i^\alpha = H_i^\alpha\frac{L^2\tau_0^2}{m_0}, \qquad \bar{S} = S\frac{\tau_0^2\tilde{T}}{m_0}.$$

In this way $\bar{\mathbf{u}}$, $\bar{\mathbf{w}}$, $\bar{\mathbf{d}}$, $\bar{\theta}$, $\bar{\mathbf{x}}$, $\bar{t}$ and the coefficients are dimensionless.

To these equations we need to add initial and boundary conditions.

Boundary conditions will be:

$$\bar{\mathbf{u}}(\mathbf{x},t) = 0, \quad \bar{\mathbf{w}}(\mathbf{x},t) = 0, \quad \bar{\theta}(\mathbf{x},t) = 0, \quad \text{on} \quad \partial\Omega_0 \times [0,t_1], \tag{12}$$

and initial conditions:

$$\begin{aligned} \bar{\mathbf{u}}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}), \quad \dot{\bar{\mathbf{u}}}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}),\\ \bar{\mathbf{w}}(\mathbf{x},0) &= \mathbf{w}_0(\mathbf{x}), \quad \dot{\bar{\mathbf{w}}}(\mathbf{x},0) = \mathbf{z}_0(\mathbf{x}),\\ \bar{\theta}(\mathbf{x},0) &= \theta_0(\mathbf{x}), \qquad\qquad \text{in} \quad \Omega_0, \end{aligned} \tag{13}$$

where $\mathbf{u}_0(\mathbf{x})$, $\mathbf{v}_0(\mathbf{x})$, $\mathbf{w}_0(\mathbf{x})$, $\mathbf{z}_0(\mathbf{x})$, $\theta_0(\mathbf{x})$ are prescribed fields.

We shall assume that the constitutive fields are essentially bounded. We also need to make the following assumptions:

(i) The mass densities $\bar{\rho}_\alpha$ are strictly positive.
(ii) The function $\bar{a}$ is strictly positive.
(iii) The constitutives tensors $\bar{\mathbf{a}}$, $\bar{\mathbf{d}}$, $\bar{\alpha}$ satisfy the relations (11).
(iv) There exists a positive constant C_0 such that

$$\begin{aligned} &\int_{\Omega_0} \big(\bar{a}_{ijrs}\bar{u}_{i,j}\bar{u}_{r,s} + 2\bar{b}_{ijrs}\bar{u}_{i,j}\bar{w}_{r,s} + \bar{d}_{ijrs}\bar{w}_{i,j}\bar{w}_{r,s} + 2\bar{h}_{ijr}\bar{u}_{i,j}\bar{d}_r\\ &\qquad + 2\bar{k}_{ijr}\bar{w}_{i,j}\bar{d}_r + \bar{\alpha}_{ij}\bar{d}_i\bar{d}_j\big)dV\\ &\geq C_0\int_{\Omega_0}(\bar{u}_{i,j}\bar{u}_{i,j} + \bar{w}_{i,j}\bar{w}_{i,j} + \bar{d}_i\bar{d}_i)dV, \end{aligned} \tag{14}$$

for all $\bar{\mathbf{u}}, \bar{\mathbf{w}} \in [C_0^\infty(\Omega_0)]^3$.

(v) There exists a positive constant k such that

$$\int_{\Omega_0} \bar{k}_{ij}\bar{\theta}_{,i}\bar{\theta}_{,j}dV \geq k \int_{\Omega_0} \bar{\theta}_{,i}\bar{\theta}_{,i}dV, \tag{15}$$

for all $\bar{\theta} \in C_0^\infty(\Omega_0)$.

The mechanical interpretations of (i) and (ii) is obvious. Condition (15) is related to the well known property of a definite heat conductor. Assumption (14) is usual in the study of the well posed problems of mixtures of elastic solids [15], and it implies that the energy is positive.

3. The existence theorem

In this section we use results of the semigroups theory of linear operators to obtain an existence theorem to the equations of mixtures of thermoelastic solids. First we transform our boundary initial value problem to an abstract problem on a Hilbert space.

In order to simplify the notation, we suppress the accent $\bar{}$ in the equations (11)-(13) for the rest of the paper.

Let $\mathcal{Z} = \Big\{(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta);\ \mathbf{u}, \mathbf{w} \in \mathbf{W}_0^{1,2}(\Omega_0),\ \mathbf{v}, \mathbf{z} \in \mathbf{L}^2(\Omega_0),\ \theta \in L^2(\Omega_0)\Big\}$, where $\mathbf{L}^2(\Omega_0) = [L^2(\Omega_0)]^3$ and $\mathbf{W}_0^{1,2}(\Omega_0) = [W_0^{1,2}(\Omega_0)]^3$ where $W_0^{1,2}(\Omega_0)$ is the well know Sobolev space [1].

Let $\mathbf{v} = \dot{\mathbf{u}}$, $\mathbf{z} = \dot{\mathbf{w}}$ and consider the operators:

$$\begin{aligned}
M_i\mathbf{u} &= (a_{ijrs}u_{r,s} + h_{ijr}u_r)_{,j} - h_{rsi}u_{r,s} - \alpha_{ij}u_j, \\
N_i\mathbf{w} &= (b_{ijrs}w_{r,s} - h_{ijr}w_r)_{,j} - k_{rsi}w_{r,s} + \alpha_{ij}w_j, \\
C_i\theta &= -(b_{ij}\theta)_{,j} + \xi_i\theta, \\
P_i\mathbf{u} &= (b_{ijrs}u_{r,s} + k_{ijr}u_r)_{,j} + h_{rsi}u_{r,s} + \alpha_{ij}u_j, \\
Q_i\mathbf{w} &= (d_{ijrs}w_{r,s} - k_{ijr}w_r)_{,j} + k_{rsi}w_{r,s} - \alpha_{ij}w_j, \\
D_i\theta &= -(\gamma_{ij}\theta)_{,j} - \xi_i\theta,
\end{aligned}$$

$$F\mathbf{v} = -b_{ij}v_{i,j} - \xi_i v_i, \qquad G\mathbf{z} = -\gamma_{ij}z_{i,j} + \xi_i z_i, \qquad K\theta = (k_{ij}\theta_{,j})_{,i}.$$

Let $\mathcal{A}$ be the matrix operator with the domain $\mathcal{D} = \{(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta) \in \mathcal{Z} |$ $\mathcal{A}(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta)^t \in \mathcal{Z}\}$, defined by

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{Id} & 0 & 0 & 0 \\ \mathbf{M} & 0 & \mathbf{N} & 0 & \mathbf{C} \\ 0 & 0 & 0 & \mathbf{Id} & 0 \\ \mathbf{P} & 0 & \mathbf{Q} & 0 & \mathbf{D} \\ 0 & F & 0 & G & K \end{pmatrix},$$

where $\mathbf{M} = (M_i)$, $\mathbf{N} = (N_i)$, $\mathbf{C} = (C_i)$, $\mathbf{P} = (P_i)$, $\mathbf{Q} = (Q_i)$, $\mathbf{D} = (D_i)$, and $\mathbf{Id}$ is the identity operator.

We note that $(\mathbf{W}_0^{1,2} \cap \mathbf{W}^{2,2}) \times \mathbf{W}_0^{1,2} \times (\mathbf{W}_0^{1,2} \cap \mathbf{W}^{2,2}) \times \mathbf{W}_0^{1,2} \times (W_0^{1,2} \cap W^{2,2})$ is a dense subset of $\mathcal{Z}$ which is contained in $\mathcal{D}$.

The boundary initial value problem (10)-(13) can be transformed into the following abstract equation in the Hilbert space $\mathcal{Z}$

$$\frac{d\omega}{dt} = \mathcal{A}\,\omega(t) + \mathcal{F}(t), \qquad \omega(0) = \omega_0, \tag{16}$$

where $\mathcal{F} = (0, \mathbf{H}^1, 0, \mathbf{H}^2, S)$, $\omega_0 = (\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0, \mathbf{z}_0, \theta_0)$.

We introduce the following inner product in $\mathcal{Z}$

$$\begin{aligned}
&\langle (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta), (\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*, \mathbf{z}^*, \theta^*) \rangle \\
&\qquad = \int_{\Omega_0} \Big\{ \rho_1 \mathbf{v}\mathbf{v}^* + \rho_2 \mathbf{z}\mathbf{z}^* + a\theta\theta^* + A[(\mathbf{u}, \mathbf{v}), (\mathbf{u}^*, \mathbf{v}^*)] \Big\} dV,
\end{aligned}$$

where

$$\begin{aligned}
A[(\mathbf{u}, \mathbf{v}), (\mathbf{u}^*, \mathbf{v}^*)] &= a_{ijrs} u_{i,j} u^*_{r,s} + b_{ijrs}(u_{i,j} w^*_{r,s} + u^*_{i,j} w_{r,s}) + d_{ijrs} w_{i,j} w^*_{r,s} \\
&\quad + h_{ijrs}(u_{i,j} d^*_r + u^*_{i,j} d_r) + k_{ijrs}(w_{i,j} d^*_r + w^*_{i,j} d_r) + \alpha_{ij} d_i d^*_j.
\end{aligned}$$

In view of (14) if follows that the norm induced by A is equivalent to the usual norm in $\mathbf{W}_0^{1,2}(\Omega_0) \times \mathbf{W}_0^{1,2}(\Omega_0)$. Thus $\langle\ ,\ \rangle$ defines a norm equivalent to the usual norm in $\mathcal{Z}$.

Lemma 3.1

The operator $\mathcal{A}$ satisfies the property

$$\langle \mathcal{A}\omega, \omega \rangle \leq 0$$

for any $\omega \in \mathcal{D}$.

Proof. Let $\omega = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta) \in \mathcal{D}$. Using the divergence theorem and the boundary conditions we find that

$$\begin{aligned}
\langle \mathcal{A}\omega, \omega \rangle &= \int_{\Omega_0} \Big\{ v_i(t_{ji,j} - p_i) + z_i(s_{ji,j} + p_i) + A[(\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})] \\
&\qquad - \theta\big(b_{ij} v_{i,j} + \gamma_{ij} z_{i,j} + \xi_i(v_i - z_i)\big) + \frac{1}{T_0}\theta q_{i,i} \Big\} dV \\
&= \int_{\Omega_0} \Big\{ A[(\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{z})] - v_{i,j}(t_{ji} + b_{ij}\theta) - z_{i,j}(s_{ji,j} + \gamma_{ij}\theta) \\
&\qquad - (v_i - z_i)(p_i + \xi_i\theta) - \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j} \Big\} dV \\
&= -\frac{1}{T_0} \int_{\Omega_0} k_{ij}\theta_{,i}\theta_{,j} dV.
\end{aligned}$$

Lemma 3.1 follows from condition (15). □

Lemma 3.2

The operator $\mathcal{A}$ satisfies the range condition

$$\mathrm{Rang}(\mathcal{I}d - \mathcal{A}) = \mathcal{Z}.$$

Proof. Let $\omega^* = (\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*, \mathbf{z}^*, \theta^*) \in \mathcal{Z}$. We must show that the equation

$$\omega - \mathcal{A}\omega = \omega^*,$$

has a solution $\omega = (\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}, \theta) \in \mathcal{D}$. From the definition of the operator $\mathcal{A}$ we obtain the system

$$\mathbf{u} - \mathbf{v} = \mathbf{u}^*,$$

$$\mathbf{w} - \mathbf{z} = \mathbf{w}^*,$$

$$\mathbf{v} - (\mathbf{M}\mathbf{u} + \mathbf{N}\mathbf{w} + \mathbf{C}\theta) = \mathbf{v}^*,$$

$$\mathbf{z} - (\mathbf{P}\mathbf{u} + \mathbf{Q}\mathbf{w} + \mathbf{D}\theta) = \mathbf{z}^*,$$

$$\theta - (F\mathbf{v} + G\mathbf{z} + K\theta) = \theta^*. \tag{17}$$

From the two first equations we obtain the system

$$\mathbf{u} - (\mathbf{M}\mathbf{u} + \mathbf{N}\mathbf{w} + \mathbf{C}\theta) = \mathbf{u}^* + \mathbf{v}^*,$$

$$\mathbf{z} - (\mathbf{P}\mathbf{u} + \mathbf{Q}\mathbf{w} + \mathbf{D}\theta) = \mathbf{w}^* + \mathbf{z}^*,$$

$$\theta - (F\mathbf{v} + G\mathbf{z} + K\theta) = \theta^* - F\mathbf{u}^* - G\mathbf{w}^*. \tag{18}$$

To study this system for the unknowns $\mathbf{u}$, $\mathbf{v}$ and θ, we introduce the bilinear form on $\mathbf{W}_0^{1,2}(\Omega_0) \times \mathbf{W}_0^{1,2}(\Omega_0) \times W_0^{1,2}(\Omega_0)$

$$\begin{aligned}
\mathcal{B}\big[(\mathbf{u}, \mathbf{w}, \theta), (\hat{\mathbf{u}}, \hat{\mathbf{w}}, \hat{\theta})\big]
&= \langle (\mathbf{u} - (\mathbf{M}\mathbf{u} + \mathbf{N}\mathbf{w} + \mathbf{C}\theta), \mathbf{z} - (\mathbf{P}\mathbf{u} + \mathbf{Q}\mathbf{w} + \mathbf{D}\theta), \theta - (F\mathbf{v} \\
&\quad + G\mathbf{z} + K\theta)), (\rho_1 \hat{\mathbf{u}}, \rho_2 \hat{\mathbf{w}}, a\hat{\theta}\rangle_{\mathbf{L}^2 \times \mathbf{L}^2 \times L^2}.
\end{aligned}$$

It is easy to see that $\mathcal{B}$ is bounded. We note that

$$\begin{aligned}\mathcal{B}\big[(\mathbf{u},\mathbf{w},\theta),(\mathbf{u},\mathbf{w},\theta)\big] &= \int_{\Omega_0} \Big\{\rho_1 u_i u_i + \rho_2 w_i w_i + a\theta^2 + u_{i,j}(t_{ji}+b_{ij}\theta)\\ &\qquad - w_{i,j}(s_{ji}+\gamma_{ij}\theta) + (u_i - w_i)(p_i + \xi_i\theta)\\ &\qquad + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}\Big\}\, dV\\ &= \int_{\Omega_0} \Big\{\rho_1 u_i u_i + \rho_2 w_i w_i + a\theta^2 + A[(\mathbf{u},\mathbf{w}),(\mathbf{u},\mathbf{w})]\\ &\qquad + \frac{1}{T_0} k_{ij}\theta_{,i}\theta_{,j}\Big\}\, dV,\end{aligned}$$

so that $\mathcal{B}$ is coercive.

Clearly

$$(\mathbf{v}^* + \mathbf{u}^*, \mathbf{z}^* + \mathbf{w}^*, \theta^* - F\mathbf{u}^* - G\mathbf{w}^*) \in \mathbf{W}_0^{-1,2}(\Omega_0) \times \mathbf{W}_0^{-1,2}(\Omega_0) \times W^{-1,2}(\Omega_0).$$

The Lax-Milgram theorem proves the existence of the solution $(\mathbf{u},\mathbf{w},\theta) \in \mathbf{W}_0^{1,2}(\Omega_0) \times \mathbf{W}_0^{1,2}(\Omega_0) \times W_0^{1,2}(\Omega_0)$ of system (17).

The lemma is proved by taking $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$ and $\mathbf{z} = \mathbf{w} - \mathbf{w}^*$. □

Theorem 3.1

The operator $\mathcal{A}$ generates a contractive semigroup in $\mathcal{Z}$.

Proof. The proof follows from the previous lemmas and the Lumer-Phillips corollary to Hille-Yosida theorem. □

Theorem 3.2

Assume that $H_i^\alpha, S \in C^1(\mathbb{R}^+, L^2(\Omega_0)) \cap C^0(\mathbb{R}^+, W_0^{1,2}(\Omega_0))$ and $\omega_0 \in \mathcal{D}$. Then, there exists a unique solution $\omega \in C^1(\mathbb{R}^+, \mathcal{Z})$ with values in $\mathcal{D}$ of the boundary initial value problem (16).

Since the semigroup defined by the operator $\mathcal{A}$ is contractive, we have the estimate

$$\|\omega(t)\|_{\mathcal{Z}} \le \Big(\|\omega_0\|_{\mathcal{Z}} + \int_0^t \big(\|H^1(\tau)\|_{\mathbf{L}^2(\Omega_0)} + \|H^2(\tau)\|_{\mathbf{L}^2(\Omega_0)} + \|S(\tau)\|_{L^2(\Omega_0)}\big)d\tau\Big),$$

which proves the continuous dependence of the solutions upon initial data and body forces. Thus, under the hypotheses (i)-(v) the problem is well posed.

4. Asymptotic behavior of solutions

In this section we study the asymptotic behavior of solutions, whose existence has been proved previously, in the homogeneous case ($\mathbf{H}^\alpha = 0$, $S = 0$). We assume the hypotheses on the thermoelastic coefficients stated in the former Section.

We recall that for any semigroup of contractions, such that its generator $\mathcal{A}$ has the only fixed point 0 and whose orbits are precompact, the orbits trend to the ω-limit set (see [8]). The structure of the ω-limit set is determined by the eigenvectors of eigenvalues $i\lambda$ ($\lambda \in \mathbb{R}$) in the closed subspace

$$\mathcal{L} =<< \left\{(\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z},\theta) | \langle \mathcal{A}(\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z},\theta), (\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z},\theta)\rangle = 0\right\} >>,$$

where $<< C >>$ denotes the closed vector space spanned by the set C.

Lemma 4.1

The operator

$$(\mathcal{I}d - \mathcal{A})^{-1}$$

is compact.

Proof. Let $(\hat{\mathbf{u}}^n, \hat{\mathbf{v}}^n, \hat{\mathbf{w}}^n, \hat{\mathbf{z}}^n, \hat{\theta}^n)$ be a bounded sequence in $\mathcal{Z}$ and let be $(\mathbf{u}^n, \mathbf{v}^n, \mathbf{w}^n, \mathbf{z}^n, \theta^n)$ the sequence of solutions of the system (17). We have

$$\begin{aligned} \mathcal{B}[(\mathbf{u}^n,\mathbf{w}^n,\theta^n),(\mathbf{u}^n,\mathbf{w}^n,\theta^n)] &= \langle (\mathbf{g}^n,\mathbf{h}^n,\eta^n),(\mathbf{u}^n,\mathbf{w}^n,\theta^n)\rangle_{\mathbf{L}^2\times\mathbf{L}^2\times L^2} \\ &\leq R(\mathcal{B}[(\mathbf{u}^n,\mathbf{w}^n,\theta^n),(\mathbf{u}^n,\mathbf{w}^n,\theta^n)])^{1/2}, \end{aligned} \tag{19}$$

where $\mathbf{g}^n = \hat{\mathbf{u}}^n + \hat{\mathbf{v}}^n$, $\mathbf{h}^n = \hat{\mathbf{w}}^n + \hat{\mathbf{z}}^n$ and $\eta^n = \hat{\theta} - F\hat{\mathbf{u}}^n - G\hat{\mathbf{v}}^n$ which is bounded sequence in $\mathbf{L}^2 \times \mathbf{L}^2 \times L^2$. Inequality (19) implies that $(\mathbf{u}^n, \mathbf{w}^n, \theta^n)$ is a bounded sequence in $\mathbf{W}_0^{1,2}(\Omega_0) \times \mathbf{W}_0^{1,2}(\Omega_0) \times W_0^{1,2}(\Omega_0)$. The theorem of compacity of Rellich-Kondrachoff (see [1], [7]) implies that there exists a converging subsequence in $\mathbf{L}^2 \times \mathbf{L}^2 \times L^2$. In a similar way $\mathbf{v}^{n_j} = \mathbf{u}^{n_j} - \hat{\mathbf{u}}^{n_j}$ and $\mathbf{z}^{n_j} = \mathbf{w}^{n_j} - \hat{\mathbf{w}}^{n_j}$ has a converging subsequence in $\mathbf{L}^2 \times \mathbf{L}^2$. Thus, we conclude the existence of a subsequence

$$(\mathbf{u}^{n_{j_k}}, \mathbf{v}^{n_{j_k}}, \mathbf{w}^{n_{j_k}}, \mathbf{z}^{n_{j_k}}, \theta^{n_{j_k}})$$

which converges in $\mathcal{Z}$. □

The previous lemma implies that the orbits starting in $\mathcal{D}(\mathcal{A})$ are precompact (see [20]). From inequalities (14), (15) it is easy to check out that $\mathcal{A}^{-1}\{0\} = \{0\}$. Now, we can state a theorem on the asymptotic behavior of solutions.

Theorem 4.1

Let $(\mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0, \mathbf{z}_0, \theta_0) \in \mathcal{D}(\mathcal{A})$ *and* $(\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t), \mathbf{z}(t), \theta(t))$ *a solution of the boundary-initial-value problem (16) for* $\mathcal{F} = 0$. *Then*

$$\theta(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \quad \text{in} \quad L^2(\Omega_0).$$

Moreover

$$\mathbf{u}(t),\ \mathbf{w}(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \quad \text{in} \quad \mathbf{W}_0^{1,2}(\Omega_0),$$

$$\mathbf{v}(t),\ \mathbf{z}(t) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty \quad \text{in} \quad \mathbf{L}^2(\Omega_0),$$

if the system

$$\begin{aligned} \mathbf{M}\mathbf{u} + \mathbf{N}\mathbf{w} + n^2\mathbf{u} &= 0, \\ \mathbf{P}\mathbf{u} + \mathbf{Q}\mathbf{w} + n^2\mathbf{w} &= 0, \\ F\mathbf{u} + G\mathbf{w} &= 0, \qquad \text{on} \quad \Omega_0, \\ \mathbf{u} = 0,\ \mathbf{w} &= 0, \qquad \text{on} \quad \partial\Omega_0, \end{aligned} \tag{20}$$

has the unique solution $\mathbf{u} = 0$, $\mathbf{w} = 0$ *on* Ω_0.

Proof. To prove the theorem we have study the structure of the ω-limit set. Thus, we must to study the equation

$$\hat{\mathcal{A}}\omega = in\omega \qquad \text{for some} \quad n \in \mathbb{R}, \tag{21}$$

$\omega \in \mathcal{D}(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}} = \mathcal{A}_{|\mathcal{L}}$ is the generator of a group on $\mathcal{L}$. If $\omega \in \mathcal{L}$ then $\langle \mathcal{A}\omega, \omega \rangle = 0$. Using inequality (15) we find $\theta = 0$ and the asymptotic behavior of the temperature is proved. The fact that equation (21) has nontrivial solution is equivalent to the fact that the system (20) has $\mathbf{u} = 0$ and $\mathbf{v} = 0$ as the unique solution.

It is natural to expect that generically the system (20) only admits the trivial solution. Nevertheless for certain materials and geometries it is possible to obtain non trivial solutions.

To show this, let us to consider an homogeneous isotropic thermoelastic mixture with a center of symmetry. We have (see [13])

$$t_{ij} = (\lambda + \nu)e_{rr}\delta_{ij} + 2(\mu + \xi)e_{ij} + (\alpha + \nu)g_{rr}\delta_{ij} + (2\kappa + 2\gamma + 2\xi)g_{ij} - (\beta + m)\theta\delta_{ij},$$

$$s_{ij} = 2\xi e_{ij} + \alpha g_{rr}\delta_{ij} + 2(\kappa + \gamma)g_{ij} - m\theta\delta_{ij},$$

$$p_i = \xi d_i,$$

$$\rho\eta = \beta e_{rr} + m g_{rr} + a\theta,$$

$$q_i = k\theta_{,i}.$$

Thus, we find the following relations

$$a_{ijrs} = (\lambda + \alpha + 2\nu)\delta_{ij}\delta_{rs} + (2\kappa + \xi)\delta_{ir}\delta_{js} + (2\mu + 2\gamma + 3\xi)\delta_{is}\delta_{jr},$$

$$b_{ijrs} = (\alpha + \nu)\delta_{ij}\delta_{rs} + (2\kappa + \xi)\delta_{is}\delta_{jr} + (2\mu + \xi)\delta_{ir}\delta_{js},$$

$$d_{ijrs} = \alpha\delta_{ij}\delta_{rs} + 2\kappa\delta_{ir}\delta_{js} + 2\gamma\delta_{is}\delta_{jr},$$

$$b_{ij} = (\beta + m)\delta_{ij}, \qquad \gamma_{ij} = m\delta_{ij}, \qquad \alpha_{ij} = m\delta_{ij},$$

$$h_{ijr} = k_{ijr} = 0, \qquad \xi_i = 0,$$

where λ, α, ν, κ, μ, γ, β, and m are real constants.

If we look for solutions of the form $\mathbf{u} = \epsilon\mathbf{w}$ ($\epsilon \in \mathbb{R}$), then whenever $\beta + 2\epsilon m \neq 0$, system (20) becomes

$$(2\kappa + \xi(1+\epsilon) + 2\gamma\epsilon)u_{i,jj} + (\rho_1 n^2 + \epsilon m - m)u_i = 0,$$

$$(2\epsilon\kappa + \xi + 2\gamma)u_{i,jj} + (\rho_2 n^2 - \epsilon m + m)u_i = 0,$$

$$(\beta + 2\epsilon m)u_{j,j} = 0.$$

The two first equations are the same whenever

$$(2\kappa + \xi(1+\epsilon) + 2\gamma\epsilon)(\rho_2 n^2 - \epsilon m + m) = (\rho_1 n^2 + \epsilon m - m)(2\epsilon\kappa + \xi + 2\gamma).$$

This equation gives two possible solutions

$$\epsilon_\alpha = \epsilon_\alpha(\kappa, \xi, \gamma, n, m, \rho_1, \rho_2).$$

Thus we have to solve the problem

$$\begin{aligned} u_{i,jj} + \varpi_\alpha u_i &= 0, \\ u_{j,j} &= 0, \quad \text{on} \quad \Omega_0 \\ u_i &= 0, \quad \text{on} \quad \partial\Omega_0 \end{aligned} \tag{22}$$

where $\varpi_\alpha = \frac{(\rho_1 n^2 + \epsilon_\alpha m - m)}{(2\kappa + \xi(1+\epsilon_\alpha) + 2\gamma\epsilon_\alpha)}$.

Problem (22) has been studied by Dafermos [8], who proved that, generically, it has no solution. He also obtained solutions for the case of the circle. Falqués [10] has also obtained solutions for the cylinder.

Let us consider the particular case $\xi = 0$ and $\rho_1 = \rho_2$. We may take $\epsilon = 1$ and system (22) becomes

$$\begin{aligned} u_{i,jj} + \frac{\rho_1 n^2}{2(\kappa+\gamma)} u_i &= 0, \\ u_{j,j} &= 0, \qquad \text{on} \quad \Omega_0, \\ u_i &= 0, \qquad \text{on} \quad \partial\Omega_0. \end{aligned}$$

In this case we may take an infinite collection of constants n such that, for the case of circle and cylinder the system admits a non trivial solution.

We have seen that there exist processes which behave asymptotically as isothermal undamped oscillations of the form $\mathbf{u} = \epsilon \mathbf{w}$. □

5. One dimensional case

The object of this Section is to study some solutions to the system (20) for one dimensional homogeneous isotropic bodies.

First we may identify our reference configuration with an interval $[0, L]$. The energy of the system is

$$\int_0^L \left[\rho_1 v^2 + \rho_2 z^2 + a\theta^2 + \alpha d^2\right] dx + \int_0^L \left[a_1 (u')^2 + 2a_2 u'w' + a_3 (w')^2\right] dx.$$

Assumptions $(i) - (iv)$ are satisfied whenever:

(a) $\rho_1,\ \rho_2,\ a,\ \alpha,\ a_1,\ a_3$ are positives.
(b) $a_2^2 < a_1 a_3$.

System (20) can be expressed as

$$\begin{aligned} a_1 u'' + (\rho_1 n^2 - \alpha) u + 2a_2 w'' + \alpha w &= 0, \\ 2a_2 u'' + \alpha u + a_3 w'' + (\rho_2 n^2 - \alpha) w &= 0, \\ \beta u' + \alpha w' &= 0, \qquad \text{on} \quad [0, L], \\ u = w &= 0, \qquad \text{for} \quad x = 0, L. \end{aligned} \tag{23}$$

The last two equations imply

$$w = -\frac{\beta}{\alpha} u \equiv \epsilon u,$$

and system (23) can be reduced to

$$\begin{aligned}(a_1 + 2\epsilon a_2)u'' + (\rho_1 n^2 + \alpha\epsilon - \alpha)u &= 0, \\ (2a_2 + \epsilon a_3)u'' + (\rho_2 n^2 \epsilon + \alpha - \alpha\epsilon)u &= 0, \qquad \text{on} \quad [0, L], \\ u &= 0, \qquad \text{for} \quad x = 0, L.\end{aligned}$$

Both equations agree if

$$(a_1 + 2\epsilon a_2)(\rho_2 n^2 \epsilon + \alpha - \alpha\epsilon) = (2a_2 + \epsilon a_3)(\rho_1 n^2 + \alpha\epsilon - \alpha). \tag{24}$$

Now our equations have nontrivial solutions if

$$\frac{\rho_1 n^2 + \alpha\epsilon - \alpha}{a_1 + 2\epsilon a_2} = \left(\frac{k\pi}{L}\right)^2, \tag{25}$$

where k is a natural number.

Is easy to show with an example that assumptions (a), (b), and equations (24), (25) are consistent. If we choose $\epsilon = 1$ and $\rho_2 = 2\rho_1$ then for fixed n, L, k, ρ_1 the system (a), (b), (24), (25) is consistent if

$$\begin{aligned}a_1 &> \frac{2\sqrt{3} - 1}{3}\rho_1\left(\frac{nL}{k\pi}\right)^2, \\ a_2 &= \frac{1}{2}\rho_1\left(\frac{nL}{k\pi}\right)^2 - \frac{1}{2}a_1, \\ a_3 &= a_1 + \rho_1\left(\frac{nL}{k\pi}\right)^2.\end{aligned}$$

Thus, we can conclude that in the one dimensional case, there exist certain materials and geometries with non trivial solutions.

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