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Periodic solutions of a non coercive hamiltonian system

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Abstract

In this paper, we study the existence of periodic solutions of hamiltonian systems:

$$\dot{x} = J\,H'(t,x)$$

where the hamiltonian H is non coercive of the type

$$H(t,r,p) = f\big(|p - Ar|\big) + h(t)\cdot(r,p)\,.$$

1. Introduction

Let be given a relativistic particle with charge e and mass at rest m_0 submitted to a constant uniform magnetic field B and a uniform electric field $E(t)$, its movement is governed by the hamiltonian equations:

$$\dot{r} = \frac{\partial H}{\partial p}(t,r,p)\,,\ \dot{p} = -\frac{\partial H}{\partial r}(t,r,p)$$

where H is the particle energy given in terms of the time t, the position r and the impulsion p by the formula:

$$H\big(t,r,p\big) = c\Big[m_0^2\,c^2 + \big|p - \frac{e}{2c}\,B\wedge r\big|^2\Big]^{1/2} - e\,E(t)\cdot r$$

where c is the light speed.

This leads us to study the existence of periodic solutions of the hamiltonian systems:

$$(\mathcal{H}) \qquad \dot{x} = JH'(t,x)$$

where the hamiltonian H is non coercive of the type

$$H\big(t,r,p\big) = f\big(|p - Ar|\big) + h(t)\cdot(r,p)$$

with A a matrix of order n, $f:\mathbb{R}_+ \longrightarrow \mathbb{R}$ is a non decreasing convex continuously differentiable function, $h:\mathbb{R}\longrightarrow\mathbb{R}^{2n}$ is a forcing term and J is the antisymmetric matrix:

$$J = \begin{pmatrix} 0 & I_{\mathbb{R}^n} \\ -I_{\mathbb{R}^n} & 0 \end{pmatrix}.$$

In this case the dual least action principle seems to provide the best results in the simplest way.

2. Autonomous case

In this section we take $h = 0$ and we assume that the matrix A is non symmetric and that the function f satisfies the assumptions:

$$(1) \qquad \forall\, t \in \mathbb{R}_+^*\,,\ 0 = f(0) < f(t) \quad \text{and} \quad f'(0) = 0\,;$$

$$(2) \qquad \exists\, k > 0\,,\ \exists\, c \geq 0 : \forall\, t \in \mathbb{R}_+\,,\ f(t) \leq \frac{k}{2}\,t^2 + c\,;$$

$$(3) \qquad \exists\, \alpha > 0\,,\ \exists\, K \in \Big]\frac{k}{2\tau}\big(1 + \|A\|^2\big)\,,\ +\infty\Big[\quad \text{such that}$$

$$\forall\, t \in [o,\alpha]\,,\ \frac{k}{2}\,t^2 \leq f(t)\,;$$

where

$$\tau = \sup\big\{b\cdot(A^* - A)\,a\,;\ a^2 + b^2 = 1\,;\ a,\, b \in \mathbb{R}^n\big\}$$

and A^* is the adjoint of A. The corresponding hamiltonian H is then continuously differentiable and we obtain:

Theorem 1

For all $T \in \big]\frac{\pi}{K\tau}\,, \frac{2\pi}{k(1+\|A\|^2)}\big[$, the hamiltonian system $(\mathcal{H})$ has a non trivial periodic solution with minimal period T.

Proof. We proceed by proving successive lemmas.

We denote by f^* the Legendre transformation of f:

$$f^*(s) = \sup \left\{ st - f(t)\,;\; t \in \mathbb{R} \right\}.$$

From the assumptions (2) and (3) we deduce easily the following lemma:

Lemma 1

f^* satisfies

$$\forall\, t \in \mathbb{R}\,,\; f^*(t) \geq \frac{1}{2k}\, t^2 - c\,, \tag{4}$$

$$\exists\, r > 0\,;\; \forall\, |t| \leq r\,,\; f^*(t) \leq \frac{1}{2K}\, t^2\,. \tag{5}$$

It is easy to show that the function H is convex and its Legendre transformation H^* is given for $(s, q) \in \mathbb{R}^n \times \mathbb{R}^n$ by:

$$H^*(s,q) = \begin{cases} f^*(|q|) & \text{if} \quad s + A^*q = 0 \\ +\infty & \text{elsewhere}\,. \end{cases}$$

Now, let $T \in \left]\frac{\pi}{K\tau}\,,\, \frac{2\pi}{k(1+\|A\|^2)}\right[$. We consider the functional Φ in the space E_0 defined by

$$E_0 = \left\{ \big(-A^*v, v \big)\,;\; v \in L^2\big(0, T; \mathbb{R}^n\big)\,,\; \int_0^T v(t)\, dt = 0 \right\},$$

$$\Phi(w) = \frac{1}{2} \int_0^T \langle Jw\,,\; \pi w \rangle\, dt + \int_0^T H^*(w)\, dt\,,$$

where πw is the primitive of w with mean value zero:

$$\frac{d}{dt}\,(\pi w) = w\,,\; \int_0^T (\pi w)(t)\, dt = 0\,.$$

Lemma 2

Let, for $w \in L^2\big(0, T; \mathbb{R}^{2n}\big)$

$$g(w) = \int_0^T H^*(w)\, dt\,.$$

The subdifferential of $g_{|E_0}$ at a point $w \in E_0$, where g is finite, is given by:

$$\bar{\partial} g(w) = \left\{ u \in L^2(0, T; \mathbb{R}^{2n});\, \exists\, x \in E_0^{\perp},\; u(t) + x(t) \in \partial H^*\big(u(t)\big)\ a.e \right\}$$

where $\bar{\partial}$ designates the subdifferential in E_0.

Proof. It is clear that

$$\bar{\partial}g(w) = \partial\big(g + \delta_{E_0}\big)\,(w)$$

where

$$\delta_{E_0}(w) = \begin{cases} 0 & \text{if} \quad w \in E_0 \\ +\infty & \text{elsewhere.} \end{cases}$$

We have $\delta^*_{E_0} = \delta_{E_0^\perp}$ and

$$E_0^\perp = \mathbb{R}^{2n} + \big\{(v, Av);\ v \in L^2(0,T;\,\mathbb{R}^n)\big\}\,.$$

Consequently, for $u \in L^2(0,T;\,\mathbb{R}^{2n})$, we have

$$\big(g^*\nabla\delta^*_{E_0}\big)(u) = \inf_{x\in\mathbb{R}^{2n}} \int_0^T H(u+x)\,dt\,.$$

So, by the assumption (2), there exist $\alpha, \beta > 0$ such that

$$0 \le \big(g^*\nabla\delta^*_{E_0}\big)(u) \le \alpha\|u\|^2_{L^2} + \beta$$

and, since $g^*\nabla\delta^*_{E_0}$ is convex, then it is continuous. Noting $u = (r,p)$, we have

$$\big(g^*\nabla\delta^*_{E_0}\big)(u) = \inf_{\xi\in\mathbb{R}^n} \int_0^T f\big(|p - Ar + \xi|\big)\,dt\,.$$

Therefore, by assumptions (2), (3) and the convexity of f, the function $g^*\nabla\delta^*_{E_0}$ is exactly.

In the other hand, g and δ_{E_0} are convex, l.s.c. and proper. Then for all $w \in E_0$, where g is finite, we have

$$\partial\big(g + \delta_{E_0}\big)(w) = \partial g(w) + \partial\delta_{E_0}(w)\,.$$

The result follows then from the equalities:

$$\partial\delta_{E_0}(w) = E_0^\perp \text{ and } \partial g(w) = \big\{u \in L^2(0,T;\mathbb{R}^{2n});\ u(t) \in \partial H^*(w(t))\ a.e.\big\} \quad \square$$

Lemma 3

The function Φ has a global minimum on E_0:

$$\exists\,\bar{w} \in E_0\,, \quad \min_{E_0}\Phi = \Phi(\bar{w})\,.$$

Proof. By the inequality (4) we have, for all $w \in E_0$;

$$\int_0^T H^*(w)dt \geq \frac{1}{2k(1+\|A\|^2)}\,\|w\|^2_{L^2} - c\,T\,.$$

By application of the Wirtinger inequality, we obtain

$$\Phi(w) \geq \frac{1}{2}\Big\{\frac{1}{k[1+\|A\|^2]} - \frac{T}{2\pi}\Big\}\|w\|^2_{L^2} - c\,T$$

and since $T < \frac{2\pi}{k(1+\|A\|^2)}$ and the space E_0 is reflexive the minimum of Φ on E_0 is achieved. □

Lemma 4

We have

$$\min_{E_0} \Phi < 0\,.$$

Proof. By definition, we have

$$T > \frac{\pi}{K\tau} = \frac{\pi}{K}\ \inf\Big\{\frac{a^2+b^2}{b\cdot(A^*-A)a}\,;\ b\cdot(A^*-A)a > 0\Big\}\,,$$

so there exist $a,\, b \in \mathbb{R}^n$ such that

$$b\cdot(A^*-A)a > 0\,,\ a^2+b^2 \leq r^2 \quad \text{and} \quad T > \frac{\pi(a^2+b^2)}{K\,b\cdot(A^*-A)\,a}\,.$$

Let

$$v(t) = a\ \cos\Big(\frac{2\pi}{T}t\Big) + b\ \sin\Big(\frac{2\pi}{T}t\Big)\,,\ w(t) = \big(-A^*v(t), v(t)\big)\,,$$

we have by the inequality (5) and easy calculation

$$\int_0^T f^*(|v|)\,dt \leq \frac{1}{2K}\|v\|^2_{L^2} = \frac{a^2+b^2}{4K}\,,$$

$$\int_0^T \langle Jw, \pi w\rangle\,dt = -\frac{T}{2\pi}\,b\cdot(A^*-A)a\,,$$

consequently

$$\Phi(w) \leq \frac{1}{4\pi}\,b\cdot(A^*-A)a\Big[\frac{\pi(a^2+b^2)}{K\,b\cdot(A^*-A)a} - T\Big] < 0$$

and so $\inf_{E_0} \Phi < 0$.

Now, let w be a point of E_0 where the minimum of Φ is achieved, then we have

$$0 \in -J\pi w + \bar{\partial} g(w)\,.$$

From Lemma 2, there exist $\xi \in \mathbb{R}^{2n}$ and $h \in L^2(0,T;\mathbb{R}^n)$ such that

$$J(\pi w)(t) + \xi + \big(h(t),\, Ah(t)\big) \in \partial H^*\big(w(t)\big) \ a.e.$$

By setting $x = J\pi w + \xi$, we obtain by the Fenchel reciprocity $\dot{x} = JH'(x)$. It is clear that x is T-periodic and, by lemma 4, x is non constant.

It remains to show that T is the minimal period of x. Assume that x and then w are $\frac{T}{m}$- periodic with $m \geq 2$. Let $y(t) = w\big(\frac{t}{m}\big)$, then $(\pi y)(t) = m(\pi m)\big(\frac{t}{m}\big)$. This point y belongs to E_0 and verifies

$$\begin{aligned}\Phi(y) &= \frac{m}{2}\int_0^T \langle Jw,\, \pi w\rangle dt + \int_0^T H*(w)\, dt \\ &\leq m\,\Phi(w)\,.\end{aligned}$$

So, by lemma 4, we have $\Phi(y) < \inf_{E_0} \Phi$ which is a contradiction, then T is the minimal period of x. □

3. Non autonomous case

Here we assume that f is not constant and there exist $k > 0$ and $a \geq 0$ such that:

$$\forall\, t \in \mathbb{R}_+\,, \quad 0 \leq f(t) \leq \frac{k}{2}t^2 + a \quad \text{and} \quad f'(0) = 0\,, \tag{6}$$

and the function h is continuous, periodic with minimal period $T > 0$ and mean value zero.

Theorem 2

For all $T \in \big]0, \frac{\pi}{k[1+\|A\|^2]}\big[$, the hamiltonian system ($\mathcal{H}$) has a periodic solution with minimal period T.

Proof. We proceed as in section 2, so we omit some details. We consider the functional Φ over the space E defined by

$$E = E_0 + h$$
$$\Phi(w) = \frac{1}{2}\int_0^T \langle Jw,\, \pi w\rangle dt + \int_0^T H^*(t,w)\,dt$$

where E_0 is defined as in section 2.

For $w = \big(-A^*v, v\big) + h \in E,$ we have

$$\Phi(w) = \frac{1}{2}\int_0^T \langle Jw,\, \pi w\rangle dt + \int_0^T f^*(|v|)\,dt\,.$$

From the assumption (6), we have

$$\int_0^T f^*(|v|)\,dt \geq \frac{1}{2k}\|v\|_{L^2}^2 - aT,$$

it follows then, by Hölder inequality, that there exists a constant c such that

$$\Phi(w) \geq \frac{1}{4\pi}\Big[\frac{\pi}{k[1+\|A\|^2]} - T\Big]\|w\|_{L^2}^2 - c\,,$$

and then the global minimum of Φ over E is achieved at a point w. Therefore, there exists $\xi \in \mathbb{R}^{2n}$ and $r \in L^2(0,T;\mathbb{R}^n)$ such that

$$0 \in -J\pi w + \xi + (r, Ar) + \partial H^*(t,w) \quad a.e.$$

By using the Fenchel reciprocity, it is clear that the function $x = J\pi w + \xi$ is a periodic solution of $(\mathcal{H})$ with minimal period T. □

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