Collect. Math. **46**, 3 (1995), 231–238

On the composition operator in $RV_\Phi[a,b]$

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Received December 21, 1993. Revised March 31, 1995

Abstract

We shall denote by F the composition operator generated by a given function $f:\mathbb{R}\to\mathbb{R}$, acting on the space of functions of bounded Riesz Φ-variation. In this paper we prove that the composition operator F maps the space $RV_\Phi[a,b]$ into itself if and only if f satisfies a local Lipschitz condition on $\mathbb{R}$.

Introduction

Some properties of the composition operator F turned out to be important in differential, integral and functional equations, for example, J. Matkowski [8], J. Appell and P. P. Zabrejko [2]. In particular, M. Marcus and V. J. Mizel [7] has proved that the composition operator F maps the space $RVp[a,b]$ into itself if and only if $f:\mathbb{R}\to\mathbb{R}$ satisfies a local Lipschitz condition on $\mathbb{R}$. In the present paper we generalize the above result to the case of the space $RV_\Phi[a,b]$ of functions of bounded Riesz Φ-variation. The particular case corresponding to $\Phi(u)=u$ has proved by M. Josephy [5].

Preliminaries on $RV_\Phi[a,b]$

For a real-function x on $[a,b]$ and for a Φ-function (Φ-functions are positive non-decreasing continuous functions on $\mathbb{R}$ which are 0 only at 0 and $\Phi(u)\to\infty$ as $u\to\infty$), we can define the *Riesz Φ-variation as the number.*

$$V_{\Phi}^{R}(x)=\sup_{\pi}\sum_{k=1}^{m}\Phi\left(\frac{|x(t_k)-x(t_{k-1})|}{t_k-t_{k-1}}\right)(t_k-t_{k-1})\,, \tag{1}$$

where supremum is taken over all partitions $\pi : a=t_0<t_1<...<t_m=b$ of $[a,b]$.

In literature is also well-known the so called Φ-variation

$$V_{\Phi}(x)=\sup_{\pi}\sum_{k=1}^{m}\Phi\left(|x(t_k)-x(t_{k-1})|\right)\,,$$

where supremum is again taken over all partitions π of $[a,b]$.

If Φ is a convex Φ-function, then the space $RV_{\Phi}=RV_{\Phi}[a,b]$ of all real-valued functions on $[a,b]$ such that $V_{\Phi}^{R}(\lambda x)<\infty$ for some $\lambda>0$ is a Banach space with the norm

$$\|x\|_{\Phi}=|x(a)|+\|x\|_{\Phi}^{0}\,,\text{where}\,\|x\|_{\Phi}^{0}=\inf\left\{\epsilon>0:V_{\Phi}^{R}\left(\frac{x}{\epsilon}\right)\leq 1\right\}\,.$$

Is also well known the Banach space $BV_{\Phi}=BV_{\Phi}[a,b]$. Also we consider subspaces $RV_{\Phi}^{0}=\{x\in RV_{\Phi}[a,b]:x(a)=0\}$ and $BV_{\Phi}^{0}=\{x\in BV_{\Phi}:x(a)=0\}$. For the first time space BV_{Φ} and RV_{Φ} appeared in papers [10] and [1], respectively.

When $\Phi(u)=u^{p}(p>1)$, then we have classical space RVp of functions of bounded Riesz p-variation.

Note that the assumption $\limsup_{u\to\infty}\frac{\Phi(u)}{u}=+\infty$, in the case of convex functions Φ, is just $\lim_{u\to\infty}\frac{\Phi(u)}{u}=\infty$. Moreover, as it was observed in [6], pp. 61-62, if Φ is a convex Φ-function and condition (∞_1) is not satisfied, i.e. $\lim_{u\to\infty}\frac{\Phi(u)}{u}=c<\infty$, then $RV_{\Phi}=BV$, where BV means the usual space of functions of finite variation. For a convex Φ-function Φ which satisfies (∞_1) some useful properties of Riesz Φ-variation are stated in the following Lemma.

Lemma 1

Let Φ be a convex Φ-function.
(a) (Musielak-Orlicz [10]) If $x\in RV_{\Phi}$ and $\|x\|_{\Phi}^{0}>0$, then

$$V_{\Phi}^{R}\left(\frac{x}{\|x\|_{\Phi}^{0}}\right)\leq 1\,.$$

(b) (Maligranda-Orlicz [6]) If $x\in RV_{\Phi}$, then x is bounded on $[a,b]$ and

$$\sup_{t\in[a,b]}|x(t)|\leq C_{\Phi}(h)\,\|x\|_{\Phi}^{0} \tag{2}$$

where $C_\Phi(h) = \max\left\{\min\left[\frac{1}{\Phi(1)}, \frac{1}{h\Phi\left(\frac{1}{h}\right)}\right], \frac{h}{\Phi^{-1}\left(\frac{1}{h}\right)}\right\}$, $h = b - a$.

Moreover, if additionally Φ *satisfies condition* $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$, *then:*
(c) (Medvedev [9]) $V_\Phi^R(x) < \infty$ *if and only if* x *is absolutely continuous on* $[a,b]$ *and* $\int_a^b \Phi(|x\prime(t)|)dt < \infty$. *In this case we also have equality* $V_\Phi^R(x) = \int_a^b \Phi(|x\prime(t)|)dt$.
(d) (Cybertowicz - Matuszewska [4]) If $x \in RV_\Phi$, *then:*

$$\|x\|_\Phi^0 = \inf\left\{\varepsilon > 0 : \int_a^b \Phi\left(\left|\frac{|x'(t)|}{\varepsilon}\right|\right) dt \le 1\right\}.$$

The purpose of this paper is to solve the superposition problem for spaces RV_Φ, that is, when for function $f:\mathbb{R}\to\mathbb{R}$ the composition operator F generated by f maps the space RV_Φ into itself. Before presenting our main result (Theorem) below, we briefly review what is known in the literature:

1° Josephy [5] proved that $F: BV \to BV$ if and only if f is a locally Lipschitz function on $\mathbb{R}$. Recall that function $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz function on $\mathbb{R}$ if for every $r>0$ there exists $L = L(r) > 0$ such that $|f(s) - f(t)| \le L\,|s-t|\,(s,t \in [-r,r])$.

2° Marcus - Mizel [7] generalized the Josephy result to spaces $RVp, 1 < p < \infty$. We have $F: RVp \to RVp$ if and only if f is a locally Lipschitz function on $\mathbb{R}$.

3° Ciemnoczolowski - Orlicz [3] generalized result of Josephy to BV_Φ^0 spaces. Let Φ be a strictly increasing Φ-function such that $\Phi \in \delta_2$, $\Phi^{-1} \in \delta_2$ ($\Phi \in \delta_2$ is there exist constants $c > 1$ and T_0 such that $\Phi(2t) \le c\,\Phi(t)$ for all $0 < t \le T_0$).

For $f:\mathbb{R}\to\mathbb{R}$, $f(0) = 0$ we have $F: BV_\Phi^0 \to BV_\Phi^0$ if and only if f is a locally Lipschitz function $\mathbb{R}$. Then Prus-Wisniowski [11] showed that the assumption $\Phi \in \delta_2$ may be dropped.

An example take book [2] pag 173, where it was given the context of BV-spaces, is now presented in the context of RV_Φ- spaces, in order to show that assumption $f \in RV_\Phi$ is not enough for $F(RV_\Phi) \subset RV_\Phi$.

Example: RVp is not closed under composition. For 1< p < 2 and $[a,b] = [-1,1]$ let $f:\mathbb{R}\to\mathbb{R}$,, be defined by:

$$f(t) := \begin{cases} 1 & \text{if } -\infty < t \le -1, \\ \sqrt{|t|} & \text{if } -1 \le t \le 1, \\ 1 & \text{if } 1 \le t < \infty. \end{cases}$$

It is easily verified that $V_p^R(f) = \frac{4}{2^p(2-p)}$, so that $1 \le p < 2$. Also $x \in RV_p$, where $x(s) = s^2 \sin^2\,(\frac{1}{s})$ for $s \in [-1,1] - \{0\}$ and $x(0) = 0$ This follows since x is absolutely continuous on $[-1,1]$ an has bounded derivative. On the other hand, the composition $(f \circ x)(s) = \left|s \sin(\frac{1}{s})\right|$ does not have a finite variation and hence $(f \circ x) \notin RV_p$.

As will be seen from the next theorem, the situation above exemplified results from the fact that f does not satisfy a Lipschitz condition at $t = 0$.

Theorem

Let Φ be a convex Φ-function which satisfies condition $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. *For $f : \mathbb{R} \to \mathbb{R}$ we have $F : RV_\Phi \to RV_\Phi$ if and only if f is a locally Lipschitz function on $\mathbb{R}$. Moreover, if F maps RV_Φ into RV_Φ, then the mapping is bounded and the following inequality holds:*

$$||Fx||_\Phi \le \Big\{1 \;+\; 2L\;[c_\Phi(b-a)\|\|x\|_\Phi^0\;]\Big\}\,\|x\|_\Phi\,(x \in RV_\Phi)\,.$$

Proof. Without loss of generality, we can assume that $[a,b] = [0,1]$. Let x be a function in $RV_\Phi[0,1]$. By Lemma 1 (b), we have that there exists a non-negative constant $c_\Phi(1)$ such that

$$\sup_{t\in[0,1]} |x(t)| \le c_\Phi(1)\,\|x\|_\Phi^0$$

Since f satisfies a local Lipschitz condition on $\mathbb{R}$, then the following can be obtained

$$|f(t) - f(s)| \le L(c_\Phi(1)\,\|x\|_\Phi^0)\,|t-s| \quad \forall s,t \in [-c_\Phi(1)\,\|x\|_\Phi^0\,, c_\Phi(1)\,\|x\|_\Phi^0],$$

and

$$|f(t)| \le L(c_\Phi(1)\,\|x\|_\Phi^0)\,|t| + |f(0)| \quad \forall s,t \in [-c_\Phi(1)\,\|x\|_\Phi^0\,,\, c_\Phi(1)\,\|x\|_\Phi^0]\,.$$

Then, we have the inequality $\|Fx\|_\Phi \le \left\{1 + 2L\;[c_\Phi(1)\;\;\|x\|_\Phi^0]\right\}\|x\|_\Phi$.

Now let $f : \mathbb{R} \to \mathbb{R}$ be such that the composition operator F maps the space $RV_\Phi[0,1]$ into itself. For the function $x_0(t) = t$, we obtain that the composition $f(x_0(t)) = f(t)$ belongs to the space $RV_\Phi[0,1]$, hence f is bounded on [0,1], with a bounded M. Without loss of generality, we can assume that $M = \frac{1}{2}$.

Suppose that f does not satisfy a local Lipschitz condition on $\mathbb{R}$, hence there exists $r > 0$ such that $\frac{|f(u)-f(s)|}{|u-s|}$ is unbounded for $|u|,\,|s| \le r(u \ne s)$. Without loss of generality, we can assume that $r = 1$. Given the sequence $\{k_n\}_{n=1}^{\infty}$ defined by $k_n = 2n(n+1)(n = 1, 2, ..,)$, there exist sequences $\{u_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ in [0,1] such that

$$k_n\,|u_n - s_n| < |f(u_n) - f(s_n)| \le 1 \tag{3}$$

Note that $u_n - s_n \to 0$ as $n \to \infty$, by considering subsequences, if necessary, we may assume that $u_n \to t^*$ as $n \to \infty$. The analysis can be reduced to the following two cases:

(i) t^* belongs only to finitely many intervals $[u_n\,,\,s_n]$.

(ii) t^* belongs to infinitely many intervals $[u_n\,,\,s_n]$.

Suppose that we are in case (i) and that infinitely many intervals not containing t^* lie to left of t^*. Let us define a subsequence of these intervals having the following property:

$$u_n < s_n < u_{n+1} < s_{n+1} < \quad t^* \quad (n = 1, 2, ...)\,.$$

For each interval $I_n = [u_n,\, s_n]\,(n = 1, 2, ...)$, we define a partition π_n in the following way:

$$\pi_n : u_n = t_0^n < t_1^n < ..... < t_{\alpha(n)}^n = s_n\,,$$

where $t_k^n - t_{k-1}^n = \frac{(s_n - u_n)}{\alpha(n)}$ $(k = 1, ..., \alpha(n))$ and $\{\alpha(n)\}$ is a sequence of suitably odd numbers.

Define the function x on [0,1] in the following way:
$x(0) = 0, x(t) = t^*$ if $t^* \le t \le 1$, $x(t) = t$ if $t \notin \cup_{n=1}^{\infty}[u_n, s_n]$, while on the other intervals is defined by:

$$x(t) := \begin{cases} \dfrac{s_n - u_n}{t_k^n - t_{k-1}^n}(t - t_{k-1}^n) + u_n & \text{if} \quad t_{k-1}^n \le t \le t_k^n,\ k = 1, 3, ...\,\alpha(n)\,, \\[2ex] \dfrac{u_n - s_n}{t_k^n - t_{k-1}^n}(t - t_{k-1}^n) + u_n & \text{if} \quad t_{k-1}^n \le t \le t_k^n,\ k = 2, 4, ...\,\alpha(n) - 1\,. \end{cases}$$

We claim that $x \in RV_\Phi[0,1]$, but $f \circ x \notin RV_\Phi$ [0,1]. Indeed, from inequality (3) and Lemma 1 (c), the following two estimates can be obtained

$$V_\Phi^R\ (x; [0,1]) \le \Phi(1) + \sum_{n=1}^{\infty}\ \Phi\big(\alpha(n)\big)\,|s_n - u_n|\,, \tag{4}$$

and

$$V_\Phi^R\big(f \circ x; [0,1]\big) \ge \sum_{n=1}^{\infty}\ \Phi\big(2\alpha(n)\big)\ \frac{k_n}{2}\ |s_n - u_n|\,. \tag{5}$$

We shall find a sequence $\{\alpha(n)\}_{n=1}^{\infty}$ of odd numbers such that the series (4) is convergent and the series (5) is divergent.

Let $K>1$ be an arbitrary constant, of course we have

$$\frac{K+1}{n^2\,|s_n-u_n|}+\frac{K-1}{n^2\,|s_n-u_n|}=\frac{2K}{n^2\,|s_n-u_n|}\,,\ (n=1,2,...)\,.$$

Since $k_n \geq n^2 (n=1,2,...)$, from the inequality (3) we obtain

$$\frac{1}{n^2\,|s_n-u_n|}>1(n=1,2,...)\,.$$

Since Φ^{-1} is a concave function and by the above identity we have

$$\Phi^{-1}\Big(\frac{K}{n^2\,|s_n-u_n|}\Big)-\frac{1}{2}\Phi^{-1}\Big(\frac{K-1}{n^2\,|s_n-u_n|}\Big)\geq\frac{1}{2}\Phi^{-1}\Big(\frac{K+1}{n^2\,|s_n-u_n|}\Big)\geq\frac{1}{2}\Phi^{-1}(K-1)\,.$$

Taking K sufficiently large, we choose the sequence $\{\alpha(n)\}_{n=1}^{\infty}$ of odd numbers such that

$$\frac{1}{2}\Phi^{-1}(K-1)\leq\frac{1}{2}\Phi^{-1}\Big(\frac{K-1}{n^2\,|s_n-u_n|}\Big)\leq\alpha(n)\leq\Phi^{-1}\Big(\frac{K}{n^2\,|s_n-u_n|}\Big)\ (n=2,3,4,....).$$

Hence

$$V_{\Phi}^{R}\big(x;[0,1]\big)\leq\Phi(1)+\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty\,,$$

and

$$V_{\Phi}^{R}\big(f\circ x;[0,1]\big)\geq\sum_{n=1}^{\infty}(K-1)=\infty\,.$$

Thus $x\in RV_{\Phi}[0,1]$, and $f\circ x\notin RV_{\Phi}[0,1]$, which is contradiction. Now consider case (ii). We define a subsequence of intervals $[u_n,s_n]$ having the following properties:

$$\big[u_{n+1},s_{n+1}\big]\subset\big[u_n,s_n\big]\,,\ (n=1,2,..)\quad\text{and}\quad\bigcap_{n=1}^{\infty}\big[u_n,s_n\big]=\{t^*\}\,.$$

Taking the sequence $k_n=2n(n+1)$ in the inequality (3) , we have.

$$2n(n+1)\,|s_n-u_n|<|f(u_n)-f(s_n)|\leq 1(n=1,2,...)\,. \qquad (6)$$

Hence we have

$$\frac{1}{2n(n+1)|s_n - u_n|} > 1(n = 1, 2, ...) .$$

Let us define the numbers m_n and m'_n by:

$$m_n = \frac{1}{2n(n+1)|s_n - u_n|}\,, \quad \text{and } m'_n = [m_n]$$

where [.] denotes as usual the integral part.

For each $n = 1, 2, ...,$ we define a partition π_n by:

$$\pi_n : \frac{1}{n+1} = t_0^n < t_1^n < ... < t_{2m'_n}^n < t_{2m'_n+1}^n = \frac{1}{n}\,,$$

where

$$t_k^n = \frac{1}{n+1} + \frac{k}{2}\,|s_n - u_n|\ ,\ (k = 1, 2, 3, ..., 2m'_n)\,.$$

Define the function x on [0,1] in the following way: $x(0) = t^*, x(1) = u_1,$ while in the interval (0,1) we prescribe x by:

$$x(t) = \begin{cases} \dfrac{u_n - u_{n+1}}{t_1^n - t_0^n}(t - t_1^n) + u_n & \text{if} \quad t_0^n \le t \le t_1^n\,, \\ \dfrac{s_n - u_n}{t_{k+1}^n - t_k^n}(t - t_k^n) + u_n & \text{if} \quad t_k^n \le t \le t_{k-1}^n\,,\ k = 1, 3, ..., 2m'_n - 1\,, \\ \dfrac{u_n - s_n}{t_{k+1}^n t_k^n}(t - t_k^n) + s_n & \text{if} \quad t_k^n \le t \le t_{k-1}^n\,,\ k = 2, 4, ..., 2m'_n\,. \end{cases}$$

We claim that $x \in RV_\Phi[0,1]$, but $f\circ x \notin RV_\Phi[0,1]$. Indeed, from inequality (6), Lemma 1 (c), (d), and the definitions of m_n and m'_n, the following two estimate can be obtained

$$V_\Phi^R\big(x; [0,1]\big) \le \Phi(4) \sum_{n=1}^{\infty} 2m'_n\,|s_n - u_n| \le 2\Phi(4) \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \tag{7}$$

and

$$V_1\big(f\circ x;\ [0,1]\big) \ge \sum_{n=1}^{\infty} 2m'_n\,|f(s_n) - f(u_n)| \ge \sum_{n=1}^{\infty} \frac{n(n+1)}{n(n+1)}\,\frac{|s_n{-}u_n|}{|s_n - u_n|} = \sum_{n=1}^{\infty} 1\,. \tag{8}$$

Hence the series (7) is convergent and the series (8) is divergent. Thus $x \in RV_\Phi[0,1]$, and $f\circ x \notin RV_\Phi[0,1]$, which is a contradiction. □

Acknowledgement. I would like to thank Dr. F. Szigeti for his helpful suggestions during the preparation of this work.

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