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The generalised ellipsoidal wave equation $[0, 3, 1_1]$

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Abstract

Explicit solutions are obtained of the linear differential equation of the second order with three regular singularities and one irregular singularity of the first type. The behavior at the point at infinity is discussed. An important special case is an algebraic form of the ellipsoidal wave equation.

1. Introduction

If the wave equation is separated in ellipsoidal coordinates, the so-called ellipsoidal wave equation which results presents considerable difficulties. See Arscott (1964) Chapter 10. This situation arises from the fact that the equation in question is a confluent form of a second order linear differential equation with five regular singularities. The nature of the singularities of linear differential equations can conveniently be described by means of the Ince symbol $[p, q, r_s]$, where there are p elementary singularities, q regular singularities and r irregular singularities of type s. See Ince (1926) page 497.

The ellipsoidal wave equation in its algebraic form has Ince symbol $[3, 0, 1_1]$ and possesses four singularities, three of which are elementary and the fourth is irregular and of the first type. In this study, the equation with three regular singularities and one irregular singularity is investigated. Its canonical form is taken to be

$$\begin{aligned} x(1-x)(p-x)y'' + & \big[(a+b+1)x^2 - [p(a+b-d+1)+c+d]x + pc\big]y' \\ + & \big(kx^2 + abx + q\big)y = 0 . \quad |p| > 1 . \end{aligned} \tag{1.1}$$

This equation is referred to as the generalised ellipsoidal wave equation because an algebraic form of the ellipsoidal wave equation is recovered from it for special values of the parameters. See Ince (1926) page 502. Unless otherwise stated, it is taken that the variable and parameters are any numbers, real or complex.

From a theoretical point of view, the Frobenius method of solution may be applied to (1.1). In the case of equations of hypergeometric type, this approach gives rise to subsidiary two-term recurrence relations which may be solved completely. Where differential equations of Heun type and the associated confluent forms occur, the recurrence relations then obtained are of three-terms and in the case of (1.1), four-term relations occur. Neither three-term nor four-term recurrences can, in general, be solved explicitly, though, particularly in the case of the former, a considerable amount of information can be deduced from them.

In the case of the Heun equations, another method of attack has been developed. Although this method is completely successful in a number of cases, it is, nevertheless of an *ad hoc* nature, so that any solution obtained, formal at first, must be investigated individually for validity. The method consists of forming a perturbation solution based upon equations of hypergeometric type, leading to explicit solutions by the intermediate use of inhomogeneous hypergeometric functions. For a general discussion and application of this technique, see Exton (1991a) for example.

The perturbation method of solution can be applied to (1.1), but inhomogeneous Heun Functions replace the inhomogeneous hypergeometric functions, leading to an explicit solution of (1.1) valid near its regular singularity at the origin. Solutions of inhomogeneous linear differential equations have been studied in detail by Babister (1967). In what follows, any values of the parameters or variable for which any of the analysis does not make sense are tacitly excluded. Further relevant background can be found in Ronveaux (1995), Arscott, Taylor and Zaha (1983), Descarreau, Maroni et Robert (1978) and Arscott (1956).

2. A formal solution of (1.1)

An elementary perturbation solution of (1.1) in the constant k is now examined. In (1.1), put

$$y = \sum_{r=0}^{\infty}(-k)^r y_r(x)\,, \tag{2.1}$$

where the functions $y_r(x)$ are determined by the system of differential equations

$$\begin{aligned} x(1-x)(p-x)y_0'' + &\ \big[(a+b+1)x^2 - [p(a+b-d+1)+c+d]x + pc\big)y_0' \\ + &\ (abx+q)y_0 = 0 \end{aligned} \tag{2.2}$$

and

$$x(1-x)(p-x)y_r'' + \big[(a+b+1)x^2 - [p(a+b-d+1)+c+d]x + pc\big]y_r'$$

$$+ \ (abx+q)y_r'' = x^2 y_{r-1}\,, \quad \text{for} \quad r = 1, 2, 3, \dots\,. \tag{2.3}$$

The equation (2.2) is Heun's equation, first introduced by Heun (1889) and its standard solution valid near the origin is the Heun function

$$y_0 = F(p, q; a, b, c, d; x) = \sum_{m=0}^{\infty} R_m(p, q; a, b, c, d)x^m\,. \tag{2.4}$$

See also Snow (1952), Babister (1967) and Exton (1993), and Section five of this paper.

Hence,

$$x(1-x)(p-x)y_1'' + \big[(a+b+1)x^2 - [\,p(a+b-d+1)+c+d\,]x + pc\big]y_1'$$

$$+ \ (abx+q)y_1 = \sum_{m_0=0}^{\infty} R_{m_0}(p, q; a, b, c, d;)x^{m_0+2}\,. \tag{2.5}$$

By virtue of the linearity of the system,

$$y_1 =_{m_0=0} R_{m_0}(p.q; a, b, c, d)f_{m_0+3}(p, q; a, b, c, d; x)\,, \tag{2.6}$$

where the inhomogeneous Heun function f_s is given by

$$f_s(p, q; a, b, c, d; x) = \frac{x^s}{ps(s+c-1)} \sum_{n=0}^{\infty} Q_n(p, q; a, b, c, d; s)x^n \tag{2.7}$$

See Babister (1967) page 287. The coefficients $R_m(p, q; a, b, c, d)$ and $Q_n(p, q; a, b, c, d; s)$ remain to be determined.

It thus follows that

$$y_1 = \frac{x^3}{3p(2+c)} \sum_{m_0, m_1=0}^{\infty} \frac{(3, m_0)(2+c, m_0)}{(4, m_0)(3+c, m_0)}$$

$$\times R_{m_0}(p, q; a, b, c, d)Q_{m_1}(p, q; a, b, c, d; 3+m_0)x^{m_0+m_1}\,. \tag{2.8}$$

As, usual,

$$(a,m) = a(a+1)(a+2)(a+2)\dots(a+m-1);\,(a,0) = 1\,. \tag{2.9}$$

If this process is repeated several times, it is clear that

$$\begin{aligned} y_r = {} & \frac{x^{3r}}{(9p)^r r!(c/3+\frac{2}{3},r)} \sum_{m_0,\dots,m_{r-1}=0}^{\infty} \frac{(3,m_0)(2+c,m_0)}{(4,m_0)(3+c,m_0)} \\ \times {} & \frac{(6,m_0+m_1)(5+c,m_0+m_1)}{(7,m_0+m_1)(6+c,m_0+m_1)} \times \dots \times \\ \times {} & \frac{(3r,m_0+\dots+m_{r-1})(c+3r-1,m_0+\dots+m_{r-1})}{(3r+1,m_0+\dots+m_{r-1})(c+3r,m_0+\dots+m_{r-1})} \\ \times {} & R_{m_0}(p,q;a,b,c,d)Q_{m_1}(p,q;a,b,c,d;3+m_0) \\ \times {} & R_{m_2}(p,q;a,b,c,d;6+m_0+m_1) \times \dots \times \\ \times {} & Q_{m_{r-1}}(p,q;a,b,c,d;3r-3+m_0+\dots+m_{r-2}) \\ \times {} & \sum_{m_r=0}^{\infty} Q_{m_r}(p,q;a,b,c,d;3r+m_0+\dots+m_{r-1})x^{m_0+\dots+m_r}\,. \end{aligned} \tag{2.10}$$

For convenience, we also write

$$\begin{aligned} y_r &= \frac{x^{3r}}{(9p)^r\, r!(c/3+\frac{2}{3},r)}\; S_r(p,q;a,b,c,d;x) \\ &= \frac{1}{(9p)^r\, r!(c/3+\frac{2}{3},r)} \sum D_{m_0,\dots,m_r}(p,q;a,b,c,d) \\ &\quad \times\; x^{m_0+\dots+m_r+3r}\,. \end{aligned} \tag{2.11}$$

The indices of summation run over all the non-negative integers.

With these forms of S and D, the series (2.1) is a formal solution (1.1). The convergence of this development must now be discussed. The above type of treatment is an extension of the method which has been successfully applied to equations of Heun type. See Exton (1991a) and (1991b).

3. The convergence of the series (2.1). The standard solution of (1.1)

From the properties of the Heun function, the series representation (2.11) converges absolutely and uniformly for all values of its parameters if $|x| < 1$. When $x = 1$, the same holds if $Re(c+d-a-b) > 0$. See Snow (1952) page 90 and Babister (1967) page 258. If we put

$$T_r = (-k)^r \, y_r(x) \,, \tag{3.1}$$

then

$$T_{r+1}/T_r = \frac{-kx^3}{9p(1+r)(c/3+\frac{2}{3}+r)} \, S_{r+1}/S_r \,. \tag{3.2}$$

With the above restrictions, for sufficiently large values of r, the ratio S_{r+1}/S_r is bounded, so that

$$\lim_{r\to\infty} \big(T_{r+1}/T_r\big) = 0 \,. \tag{3.3}$$

The series solution (2.1) is then absolutely and uniformly convergent and can be re-arranged in any way. This solution is thus valid in the neighborhood of the regular singularity of (1.1) at the origin and is taken to be the standard solution there relative to the zero exponent. It is denoted by the symbol

$$E(p,q;a,b,c,d;k;x) \,. \tag{3.4}$$

4. Further solutions of the generalised ellipsoidal wave equation $[0,3,1_1]$

If we replace y by $x^{1-c}y$ in (1.1), its form remains unchanged, so that a solution corresponding to

$$E\big(p,q;a,b,c,d;k;x\big) \tag{4.1}$$

is found to be

$$x^{1-c}E(p,q-(1-c)[d+p(1+a+b-c-d)];1+a-c,1+b-c,2-c,d;k;x) \,. \tag{4.2}$$

Similarly, on replacing x by $1-x$ in (1.1), we may deduce the solution

$$E\left(1-p,-q-k-ab;\frac{a+b+\sqrt{(a-b)^2-8k}}{2},\frac{a+b-\sqrt{(a-b)^2-8k}}{2},\right.$$
$$\left.1+a+b-c-d,d;-k;1-x\right), \tag{4.3}$$

and the replacement of x by px gives

$$E\big(1/p, q/p; a, b, c, 1+a+b-c-d; kp; x/p\big)\,. \tag{4.4}$$

Hence, if $Z(p, q; a, b, c, d; k; x)$ (I)
is a solution of (1.1), then so also are

$$x^{1-c}Z(p, q-(1-c)[d+p(1+a+b-c-d)]; 1+a-c, 1+b-c, 2-c, d; k; x) \tag{II}$$

and

$$Z\left(1-p, -q-k-ab; \frac{a+b+\sqrt{(a-b)^2-8k}}{2}, \frac{a+b-\sqrt{(a-b)^2-8k}}{2},\right.$$
$$\left. 1+a+b-c-d, d; -k; 1-x\right). \tag{III}$$

When these results are suitably combined, further solutions arise:

$$x^{1-c}Z(1-p, -q-(1-c)[d+p(1+a+b-c-d)]-k-(1+a-c)(1+b-c)\,;$$
$$1-c+\frac{a+b+\sqrt{(a-b)^2-8k}}{2},\ 1-c+\frac{a+b-\sqrt{(a-b)^2-8k}}{2},$$
$$1+a+b-c-d, d; -k; 1-x)\,, \tag{IV}$$

$$(1-x)^{c+d-a-b}Z(1-p, -q-k-ab-(c+d-a-b)[d+c(1-p)];$$
$$c+d-\frac{a+b+\sqrt{(a-b)^2-8k}}{2}, c+d-\frac{a+b-\sqrt{(a-b)^2-8k}}{2},$$
$$c+d-a-a+1, d; -k; 1-x)\,, \tag{V}$$

$$x^{1-c}(1-x)^{c+d-a-b}Z(1-p, -q-k-ab; \frac{2-a-b+\sqrt{(a-b)^2-8k}}{2},$$
$$\frac{2-a-b-\sqrt{(a-b)^2-8k}}{2}, c+d-a-b+1, d; -k; 1-x)\,, \tag{VI}$$

$$(1-x)^{c+d-a-b}Z(p, q+2k-\Big(\frac{1}{2}a-\frac{1}{2}b\Big)^2$$
$$+pc(a+b-c-d); c+d-a, c+d-b, c, d; k; x\big) \tag{VII}$$

and

$$x^{1-c}(1-x)^{c+d-a-b}Z(p, q; 1-a, 1-b, 2-c, d; k; x)\,. \tag{VIII}$$

A further solution may be deduced from (4.4), that is

$$Z\big(1/p, q/p; a, b, c, 1+a+b-c-d; kp; x/p\big)\,. \tag{IX}$$

From the above results, it is evident that if $Z \equiv E$,

$$\begin{aligned} &E\big(p, q; a, b, c, d; k; x\big) \\ &= (1-x)^{c+d-a-b}E\Big(p, q+2k-\big(\frac{1}{2}a-\frac{1}{2}b\big)^2 \\ &\qquad + pc(a+b-c-d); c+d-a, c+d-b, c, d; k; x\Big) \\ &= E\big(1/p, q/p; a, b, c, 1+a+b-c-d; kp; x/p\big) \\ &= (p-x)^{1-d}E\Big(1/p, q/p+2kp-\big(\frac{1}{2}a-\frac{1}{2}b\big)^2 \\ &\qquad + c(d-1)/p; 1+a-d, 1+b-d, c, 1+a+b-c-d; kp; x/p\Big)\,. \end{aligned} \tag{4.5}$$

This formula generalises an Euler transformations of the Gauss hypergeometric function. It should also be noted that, by symmetry,

$$Z\big(p, q; b, a, c, d; k; x\big)\,, \tag{4.6}$$

is a solution of (1.1)

Many more solutions of (1.1) may be obtained by similar means, and the use of general combinations of solutions of a differential system is extremely effective in deducing its general solution in certain cases. This approach has been employed successfully in connection with particular hypergeometric partial differential systems by Olsson (1977) and Exton (1992) for example.

5. Explicit formulae for the coefficients R_n and Q_m

By applying a similar method to that used by Exton (1991a), for example, to Heun's equation (2.2), we obtain the following closed form representation of the Heun function:

$$
\begin{aligned}
&F(p,q;a,b,c,d;x)\\
&=\sum_{r,m_0,\ldots,m_r=0}^{\infty}\frac{H^r\left(\frac{1}{2}a,r\right)\left(\frac{1}{2}b,r\right)}{\left(\frac{1}{2}c+\frac{1}{2},r\right)r!}\\
&\quad\times\frac{(A,m_0)(B,m_0)(a+1,m_0)(b+1,m_0)(c+1,m_0)(2,m_0)}{(A+2,m_0)(B+2,m_0)(a,m_0)(b,m_0)(c,m_0)(1,m_0)}\times\ldots\\
&\quad\times\frac{(A+2,m_0+m_1)(B+2,m_0+m_1)(a+3,m_0+m_1)}{(A+4,m_0+m_1)(B+4,m_0+m_1)(a+2,m_0+m_1)}\\
&\quad\times\frac{(b+3,m_0+m_1)(c+3,m_0+m_1)(4,m_0+m_1)}{(b+2,m_0+m_1)(c+2,m_0+m_1)(3,m_0+m_1)}\\
&\quad\times\ldots\\
&\quad\times\frac{(A+2r-2,m_0+\ldots+m_{r-1})(B+2r-2,m_0+\ldots+m_{r-1})}{(A+2r,m_0+\ldots+m_{r-1})(B+2r,m_0+\ldots+m_{r-1})}\\
&\quad\times\frac{(a+2r-1,m_0+\ldots+m_{r-1})}{(a+2r-2,m_0+\ldots+m_{r-1})}\\
&\quad\times\frac{(b+2r-1,m_0+\ldots+m_{r-1})(c+2r-1,m_0+\ldots+m_{r-1})}{(b+2r,m_0+\ldots+m_{r-1})(c+2r-2,m_0+\ldots+m_{r-1})}\\
&\quad\times\frac{(2r,m_0+\ldots+m_{r-1})}{(2r-1,m_0+\ldots+m_{r-1})}\\
&\quad\times\frac{(A+2r,m_0+\ldots+m_r)(B+2r,m_0+\ldots+m_r)}{(c+2r,m_0+\ldots+m_r)(1+2r,m_0+\ldots+m_r)}z^{m_0+\ldots+m_r+2r}\,, \qquad (5.1)
\end{aligned}
$$

where $A+B+1=[a+b-d+1-(c+d)/p]p/(p+1)\,,$

$$AB=-q(p+1)\,,\quad H=-p/(p+1)^2\quad\text{and}\quad z=(p+1)x/p\,. \qquad (5.2)$$

See Exton (1993).

Put $m_0+\ldots+m_r+2r=n$, when the coefficient of x^n in (5.1) is found to be

$$
\begin{aligned}
&R_n(p,q;a,b,c,d)\\
&= \sum_{r,m_0,\ldots,m_r=0} \frac{(\frac{1}{2}a,r)(\frac{1}{2}b,r)(A,n-m_1-\ldots-m_r-2r)}{(\frac{1}{2}c+\frac{1}{2},r)r!(A+2,n-m_1-\ldots-m_r-2r)}\\
&\quad\times \frac{(B,n-m_1-\ldots-m_r-2r)}{(B+2,n-m_1-\ldots-m_r-2r)}\\
&\quad\times \frac{(a+1,n-m_1-\ldots-m_r-2r)(b+1,n-m_1-\ldots-m_r-2r)}{(a,n-m_1-\ldots-m_r-2r)(b,n-m_1-\ldots-m_r-2r)}\\
&\quad\times \frac{(c+1,n-m_1-\ldots-m_r-2r)}{(c,n-m_1-\ldots-m_r-2r)}\\
&\quad\times \frac{(2,n-m_1-\ldots-m_r-2r)(A+2,n-m_2-\ldots-m_r-2r)}{(1,n-m_1-\ldots-m_r-2r)(A+4,n-m_2-\ldots-m_r-2r)}\\
&\quad\times \frac{(B+2,n-m_2-\ldots-m_r-2r)}{(B+4,n-m_2-\ldots-m_r-2r)}\\
&\quad\times \frac{(a+3,n-m_2-\ldots-m_r-2r)(b+3,n-m_2-\ldots-m_r-2r)}{(a+2,n-m_2-\ldots-m_r-2r)(b+2,n-m_2-\ldots-m_r-2r)}\\
&\quad\times \frac{(c+3,n-m_2-\ldots-m_r-2r)}{(c+2,n-m_2-\ldots-m_r-2r)}\\
&\quad\times \frac{(4,n-m2-\ldots-m_r-2r)}{(3,n-m2-\ldots-m_r-2r)}\times\ldots\\
&\quad\times \frac{(A+2r-2,n-m_r-2r)(B+2r-2-2,n-m_r-2r)}{(A+2r,n-m_r-2r)(B+2r,n-m_r-2r)}\\
&\quad\times \frac{(a+2r-1,n-m_r-2r)(b+2r-1,n-m_r-2r)}{(a+2r-2,n-m_r-2r)(b+2r-2,n-m_r-2r)}\\
&\quad\times \frac{(c+2r-1,n-m_r-2r)(2r,n-m_r-2r)}{(c+2r-2,n-m_r-2r)(2r-1,n-m_r-2r)}\\
&\quad\times \frac{(A+2r,n-2r)(B+2r,n-2r)}{(c+2r,n-n-2r)(1+2r,n-2r)}\\
&\quad\times H^{1/2(n-m_1-\ldots-m_r)}\big[(p+1)/p\big]^n\,.
\end{aligned}
\tag{5.3}
$$

Applying a similar analysis to the inhomogeneous Heun equation in the form given by Babister (1967) page 287, we consider

$$z(1-z)y'' + \big[c-(A+B+1)z\big]y' - ABy = H\big[z^3y'' + (a+b+1)z^2y' + abzy\big] + \big[pz/(p+1)\big]^{s-1}. \qquad (5.4)$$

In this instance, from Exton (1993), it may be shown that

$$\begin{aligned} & f_s\big(p,q;a,b,c,d;x\big) \\ &= \frac{z^s}{s(s+c-1)} \sum_{r,m_0,\ldots,m_r=0} \frac{H^r\big(\frac{1}{2}s+\frac{1}{2}a,r\big)\big(\frac{1}{2}s+\frac{1}{2}s+\frac{1}{2}b,r\big)}{\big(\frac{1}{2}s+1,r\big)\big(\frac{1}{2}s+\frac{1}{2}c+\frac{1}{2},r\big)} \\ &\quad \times \frac{(A+s,m_0)(B+s,m_0)(a+1+s,m_0)}{(A+2+s,m_0)(B+2+s,m_0)(a+s,m_0)} \\ &\quad \times \frac{(b+1+s,m_0)(c+1+s,m_0)(2+s,m_0)}{(b+s,m_0)(c+s,m_0)(1+s,m_0)} \\ &\quad \times \frac{(A+2+s,m_0+m_1)(B+2+s,m_0+m_1)}{(A+4+s,m_0+m_1)(B+4+s,m_0+m_1)} \\ &\quad \times \frac{(a+3+s,m_0+m_1)(b+3+s,m_0+m_1)}{(a+2+s,m_0+m_1)(b+2+s,m_0+m_1)} \\ &\quad \times \frac{(c+3+s,m_0+m_1)(4+s,m_0+m_1)}{(c+2+s,m_0+m_1)(3+s,m_0+m_1)} \times \ldots \\ &\quad \times \frac{(A+s+2r-2,m_0+\ldots+m_{r-1})(B+s+2r-2,m_0+\ldots+m_{r-1})}{(A+s+2r,m_0+\ldots+m_{r-1})(B+s+2r,m_0+\ldots+m_{r-1})} \\ &\quad \times \frac{(a+s+2r-1,m_0+\ldots+m_{r-1})}{(a+s+2r-2,m_0+\ldots+m_{r-1})} \\ &\quad \times \frac{(b+s+2r-1,m_0+\ldots+m_{r-1})(c+s+2+2r-1,m_0+\ldots+m_{r-1})}{(b+s+2r-2,m_0+\ldots+m_{r-1})(c+s+2r-2,m_0+\ldots+m_{r-1})} \\ &\quad \times \frac{(s+2r,m_0+\ldots+m_{r-1})}{(s+2r-1,m_0+\ldots+m_{r-1})} \\ &\quad \times \frac{(A+s+2r,m_0+\ldots+m_r)(B+s+2r,m_0+\ldots+m_r)}{(c+s+2r,m_0+\ldots+m_r)(1+s+2r,m_0+\ldots+m_r)} z^{3r+m_0+\ldots+m_r} \end{aligned} \qquad (5.5)$$

in which the relations (5.2) apply. $Q_n(p, q; a, b, c, d)$ may be deduced by putting $m_0 = n - m_1 - \ldots - m_r - 2r$, after restoring the variable x and the taking the coefficient of x^n.

6. The behavior near the singularity at infinity

Since the singularity at infinity is irregular, any direct attempt to apply the technique used in Section 2 yields results in terms of logarithms and their powers which are difficult to interpret, so that a different approach is necessary. First of all, since the irregular singularity is of the first type, we put $x = z^{-2}$ and $y = \exp(1/z)Y$. After some reduction, we obtain the result

$$\begin{aligned}
&z^4(z^2-1)(pz^2-1)Y'' + \\
&+ \begin{bmatrix} p(3-2c)z^7 - 2lz^6 + \big[p(2a+2b-2d-1) + 2c - 2d - 3\big]z^5 \\ -2l(p+1)z^4 - \big(1-2a-2b\big)z^3 - 2lz^2 \end{bmatrix} Y' \\
&+ \begin{bmatrix} l(2-3p+2cp)z^5 + (4q+l^2)z^4 - l\big[p(2a+2b-2d+1) + 2c + 2d - 1\big]z^3 \\ +\big[4ab - l^2(p+1)\big]z^2 + l(2a+ab+1)z + l^2 + 4k \end{bmatrix} Y = 0\,.
\end{aligned} \tag{6.1}$$

Put $l^2 + 4k = 0$, when the indicial equation is $t = a + b + \frac{1}{2}$. An aggregate of subnormal solutions exists of the form

$$y \sim \exp\big(\pm 2ik^{1/2}x^{1/2}\big)x^{-1/4-1/2a-1/2b}\big[1 + 0(x^{-1/2})\big]\,. \tag{6.2}$$

The same approach as that used in Exton (1991) in connection with the confluent Heun equation $[0, 2, 1_1]$ is now adopted. From (2.11), consider a Neuman series expansion of

$$x^{3r}S_r(p, q; a, b, c, d; x) = \sum D_{m_0,\ldots,m_r}(p, q; a, b, c, d)x^{m_0+\ldots+m_r+3r}\,. \tag{6.3}$$

Write $(gx)^{m+3r}S_r(p, q; a, b, c, d; gx)$

$$\begin{aligned}
&= (2g)^m \sum D_{m_0,\ldots,m_r}(p, q; a, b, c, d; gx)(2g)^{m_0+\ldots+m_r+3r} \\
&\quad \times \Big(\sqrt{\frac{1}{2}x}\Big)^{2m_0+\ldots+2m_r+6r}\,,
\end{aligned} \tag{6.4}$$

where g and m remain to be determined.

We note that

$$\Big(\sqrt{\frac{1}{2}x}\Big)^{2m+2m_0+\ldots+2m_r+6r}$$
$$\times \sum_{s=0}^{\infty} \frac{(2m+2m_0+\ldots+2m_r+6r+2s)(2m+2m_0+\ldots+2m_r+6r+s-1)!}{s!}$$
$$\times J_{2m+2m_0+\ldots+2m_r+6r+2s}\big(x^{\frac{1}{2}}\big)\,. \tag{6.5}$$

See Watson (1948) page 138.

Hence, $(gx)^{m+3r}S_r(p,q;a,b,c,d;gx)$

$$= (4g)^m \sum D_{m_0,\ldots,m_r}(p,q;a,b,c,d)(4g)^{m_0+\ldots+m_r+3r}$$

$$\left[\sum_{s=0}^{\infty} \frac{(2m+2m_0+\ldots+2m_r+6r+2s)(2m+2m_0+\ldots+2m_r+6r+s-1)!}{s!} \atop J_{2m+2m_0+\ldots+2m_r+6r+2s}(x^{1/2})\right], \tag{6.6}$$

Replace x by $-4kx$, and let $g=-(4k)^{-1}$, $m=\frac{1}{2}a+\frac{1}{2}b$ and $N=m_0+\ldots+m_r+3r+s$.

Thus, $x^{3r}S_r\big(p,q;a,b,c,d;x\big)$

$$= (kx)^{-1/2a-1/2b}\sum_{N=0}^{\infty}\Big[\sum k^{-m_0-\ldots-m_r-3r}D_{m_0,\ldots,m_r}(p,q;a,b,c,d)$$
$$\times \frac{\Gamma(a+b+m_0+\ldots+m_r+3r+N)}{(N-m_0-\ldots-m_r-3r)!}\Big]$$
$$\times (-1)^{1/2a+1/2b}(a+b+2N)I_{a+b+2N}\big(2\sqrt{kx}\big)\,. \tag{6.7}$$

From Watson (1984) page 203, we have, for $-\frac{1}{4}\pi < \arg(kx) < \frac{3}{4}\pi$,

$$I_{a+b+2N}\big(2\sqrt{kx}\big)$$

$$\sim (8kx)^{-1/4}\Big[\exp\big(2\sqrt{kx}\big)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N, \frac{1}{2}-a-b-2N; -; \big(4\sqrt{kx}\big)^{-1}\Big)$$
$$+ (-1)^{a+b+1/2}\exp\big(-2\sqrt{kx}\big)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N, \frac{1}{2}-a-b-2N; -; -\big(4\sqrt{kx}\big)^{-1}\Big)\Big] \tag{6.8}$$

and for $-\frac{3}{4}\pi < \arg(kx) < \frac{1}{4}\pi$.

$$I_{a+b+2N}\big(2\sqrt{kx}\big)$$

$$\sim (8kx)^{-1/4}\Big[\exp\big(2\sqrt{kx}\big)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N,\frac{1}{2}-a-b-2N;-;\big(4\sqrt{kx}\big)^{-1}\Big)$$

$$+(-1)^{a+b+1/2}\exp\big(-2\sqrt{kx}\big)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N,\frac{1}{2}-a-b-2N;-;-\big(4\sqrt{kx}\big)^{-1}\Big)\Big]. \tag{6.9}$$

Replace k by $-k$ when, apart from constant multipliers, (6.7) reduces effectively to two parts:

$$x^{-1/4-1/2a-1/2b}\exp\big(2i\sqrt{kx}\big)\sum_{N=0}^{\infty}\Big[\sum(-k)^{-m_0-\ldots-m_r-3r}D_{m_0,\ldots,m_r}(p,q;a,b,c,d)$$

$$\times\frac{\Gamma(a+b+m_0+\ldots+m_r+3r+N)}{(N-m_0-\ldots-m_r-3r)!}\Big]$$

$$\times(a+b+2N)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N,\frac{1}{2}-a-b-2N;-;-i\big(4\sqrt{kx}\big)^{-1}\Big) \tag{6.10}$$

and

$$x^{-\frac{1}{4}-\frac{1}{2}a-\frac{1}{2}b}\exp\big(2i\sqrt{kx}\big)\sum_{N=0}^{\infty}\Big[\sum(-k)^{-m_0-\ldots-m_r-3r}D_{m_0,\ldots,m_r}(p,q;a,b,c,d)$$

$$\times\frac{\Gamma(a+b+m_0+\ldots+m_r+3r+N)}{(N-m_0-\ldots-m_r-3r)!}\Big]$$

$$\times(a+b+2N)\,{}_2F_0\Big(\frac{1}{2}+a+b+2N,\frac{1}{2}-a-b-2N;-;-i\big(4\sqrt{kx}\big)^{-1}\Big). \tag{6.11}$$

The aggregate of subnormal solutions which constitutes a formal asymptotic representation of the fundamental system of integrals of (1.1) relative to the irregular singularity at infinity may be obtained by inserting in turn (6.10) and (6.11) into (2.11). On employing the solutions of the generalised ellipsoidal wave equation listed in Section 4, the above results give a complete integration of this equation near to all of its singular points.

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