Collect. Math. **46**, 1–2 (1995), 171–182

Semigroups of cosets of semigroups: variations on a Dubreil theme

Boris M. Schein

Department of Mathematical Sciences, University of Arkansas
Fayetteville, Arkansas 72701, USA

To the memory of Paul Dubreil, whose pioneering work in semigroups was an inspiration for me from my student years

Abstract

In his seminal article of 1941, Paul Dubreil introduced *complexes forts* of semigroups. Strong subsets of a semigroup S form another semigroup under a natural multiplication. Properties of this semigroup are studied and some open problems raised (specially when S is a group or an inverse semigroup). Also, a simple proof of a known result is given: every inverse semigroup can be isomorphically embedded in the semigroup of cosets of a group.

A binary relation $\rho \subset A \times B$ between the elements of sets A and B is called *difunctional* if $\rho \circ \rho^{-1} \circ \rho = \rho$, where $\rho^{-1} = \{(b,a) : (a,b) \in \rho\}$ denotes the binary relation between the elements of B and A that is a *converse* of ρ, and $\circ$ denotes the operation of relative multiplication (if $\rho \subset A \times B$ and $\sigma \subset B \times C$ are binary relations, then $\sigma \circ \rho \subset A \times C$ and also $(a,c) \in \sigma \circ \rho$ if and only if $(a,b) \in \rho$ and $(b,c) \in \sigma$ for some $b \in B$). It is easy to see that $\rho \subset \rho \circ \rho^{-1} \circ \rho$ for every binary relation, and hence ρ is difunctional if $\rho \circ \rho^{-1} \circ \rho \subset \rho$. Difunctional binary relations were introduced by Riguet in [7] and [8].

If $\rho \subset A \times B$ is a binary relation and $H \subset A$ a subset of A, then the *image of H under ρ* is the subset $\rho(H) = \{b : (\exists\, a \in H)\,[(a,b) \in \rho]\}$ of B. If $a \in A$, we define $\rho\langle a\rangle = \rho(\{a\})$. In particular, $\rho \subset A \times B$ is difunctional if and only if $\rho\langle a_1\rangle \cap \rho\langle a_2\rangle \neq \emptyset \Rightarrow \rho\langle a_1\rangle = \rho\langle a_2\rangle$ for all $a_1, a_2 \in A$ or, equivalently, if and only if $\rho^{-1}\langle b_1\rangle \cap \rho^{-1}\langle b_2\rangle \neq \emptyset \Rightarrow \rho^{-1}\langle b_1\rangle = \rho^{-1}\langle b_2\rangle$ for all $b_1, b_2 \in B$ (see [7] and [8]).

Let $(S; o)$ be a *groupoid* (that is, a nonempty set S with a binary operation o on S). A *binary operation* on S is, of course, any mapping $o : S \times S \to S$. For a subset $H \subset S$ define a binary relation $\rho_H = o^{-1}(H) \subset S \times S$. Clearly, $(s,t) \in \rho_H \Leftrightarrow st \in H$. A subset H is called *strong* if ρ_H is difunctional. Thus H is strong if and only if $\rho_H \circ (\rho_H)^{-1} \circ \rho_H \subset \rho_H$ or, equivalently, $(x,y) \in \rho_H \circ (\rho_H)^{-1} \circ \rho_H \Rightarrow (x,y) \in \rho_H$ for any $x, y \in S$. Equivalently, if $(x,v) \in \rho_H, (v,u) \in (\rho_H)^{-1}$ and $(u,y) \in \rho_H$ for some $u, v \in S$, then $(x,y) \in \rho_H$. Thus H is strong if and only if $xv \in H, uv \in H$, and $uy \in H$ imply $xy \in H$ for any $u, v, x, y \in H$.

Strong subsets can be also defined as follows. Let $H. \cdot a = \rho_H\langle a \rangle = \{b : ab \in H\}$ and let $H \cdot . b = (\rho_H)^{-1}\langle b \rangle = \{a : ab \in H\}$. Then $H. \cdot a$ and $H \cdot . b$ are called the *right quotient of* H *by* a, and the *left quotient of* H *by* b, respectively (see [2]). Clearly, H is strong precisely when $H. \cdot a_1 \cap H. \cdot a_2 \neq \emptyset \Rightarrow H. \cdot a_1 = H. \cdot a_2$ for all $a_1, a_2 \in S$. Using left quotients we see that H is strong if and only if $H \cdot . b_1 \cap H \cdot . b_2 \neq \emptyset \Rightarrow H \cdot . b_1 = H \cdot . b_2$ for all $b_1, b_2 \in S$.

The concept of strong subsets in semigroups belongs to Dubreil [2], the alternative definition (H is strong when ρ_H is difunctional) first appeared in [10]. Dubreil called a subset H of a semigroup S *right strong* if $H. \cdot a_1 \cap H. \cdot a_2 \neq \emptyset \Rightarrow H. \cdot a_1 = H. \cdot a_2$ for all $a_1, a_2 \in S$ and *left strong* if $H \cdot . b_1 \cap H \cdot . b_2 \neq \emptyset \Rightarrow H \cdot . b_1 = H \cdot . b_2$ for all $b_1, b_2 \in S$. He proved that a subset is left strong if and only if it is right strong. Strong subsets are one of the most useful concepts for the semigroups of one-to-one (partial) transformations (see [10]). Dubreil introduced strong subsets in a semigroup as an analog of cosets of a subgroup in a group: a nonempty subset of a group is a left (right) coset of some subgroup if and only if it is right (left) strong [2]. However, as shown in [10], strong subsets in monoids (that is, semigroups with identity) are naturally connected not so much with cosets in groups as with cosets in inverse semigroups. Strong subsets as cosets have also been considered in [15]. It turns out that strong subsets of a semigroup form another semigroup under a suitable multiplication.

In this paper we concentrate on semigroups of strong subsets. Some results of this paper appeared in [12].

As usual, if S is a semigroup, S^1 denotes the smallest semigroup with identity that contains S (so $S^1 = S$ if S has an identity element; otherwise, S^1 is S with an identity element adjoined).

If T is a subgroupoid of a groupoid S and H a subset of S, then $H \cap T$ is called a *trace* of H on T. Clearly, *a trace of a strong subset of* S *is a strong subset of* T. We call a subset H of a semigroup S *unitarily strong* if H is a trace on S of a strong subset of S^1. Clearly, *unitarily strong subsets are strong.* As was argued in [15], unitarily strong subsets of semigroups are very naturally connected with cosets of inverse semigroups. For example (see [15]), if a semigroup S is embeddable in an inverse semigroup, it can be embedded in such an inverse semigroup T that all unitarily strong subsets of S are traces on S of cosets of T. If $\varphi : S \to T$ is a homomorphism of a groupoid S into a groupoid T and H is a strong subset of T, then it is easy to see that $\varphi^{-1}(H)$ is a strong subset of S. Also, a subset H of a semigroup S is unitarily strong if and only if there exists a homomorphism φ of S into an inverse semigroup T such that $H = \varphi^{-1}(C)$ for some coset C of T. This is one of the reasons why we suggested in [15] to call nonempty unitarily strong subsets of a semigroup *cosets* of this semigroup. We adhere to this terminology here.

It is easily seen that a subset H of a semigroup S is unitarily strong if and only if it satisfies the following conditions:
(1) $xv, uv, uy \in H \Rightarrow xy \in H$,
(2) $v, uv, uy \in H \Rightarrow y \in H$,
(3) $xv, uv, u \in H \Rightarrow x \in H$,
(4) $xv, v, y \in H \Rightarrow xy \in H$,
(5) $x, u, uy \in H \Rightarrow xy \in H$ for any $u, v, x, y \in H$.

In other words, condition (1) is satisfied for all $u, v, x, y \in S^1$ for which the products xv, uv, uy and xy make sense (that is, these products are not the identity element of S^1 in the case S has no identity element).

Example 1: (1) Let $\mathbb{N}$ be the additive semigroup of positive integers. Consider a subset $K = \{n \in \mathbb{N} : n > 1\}$. If $x + v, u + v, u + y \in K$ for some $u, v, x, y \in \mathbb{N}$, then $x + y \in K$ because $x + y > 1$, and so K is strong. However, $0 + 2, 1 + 2, 1 + 1 \in K$, but $0 + 1 \notin K$, and hence K is not unitarily strong. Thus there exist strong but not unitarily strong subsets. Such subsets exist even in inverse semigroups as part (2) of our example shows.

(2) Let $\mathbf{I}$ be a free inverse semigroup generated by a single element a. Each element of $\mathbf{I}$ has the form $a^{-s}a^{n}a^{-d}$, where $n \geq 1$, $s, d \geq 0$, $s \leq n$, and $d \leq n$ (see [3]). If $K = \{a, a^{-1}\}$, then $a^{-1} \cdot 1 = a^{-1} \in K$, $a \cdot 1 = a \cdot a^{-1}a = a \in K$, but $a^{-1} \cdot a^{-1}a \notin K$. Thus K is not unitarily strong. Let $xv, uv, uy \in K$ for some $u, v, x, y \in \mathbf{I}$. Without loss of generality suppose that $xv = a$. Then either $x = a$ and $v = a^{-1}a$, or $x = aa^{-1}$ and $v = a$ (see [13]). In both cases $uv \neq a^{-1}$, and hence $uv = a$. If $v = a^{-1}a$, then $ua^{-1}a = uv = a$, which implies $u = a = x$. If $v = a$, then $ua = uv = a$, and so $u = aa^{-1} = x$. Thus $xy = uy \in K$, which shows that K is a strong but not unitarily strong subset if $\mathbf{I}$.

Let $\mathbf{C}(S)$ denote the set of all cosets of a groupoid S and let $\mathbf{F}(S)$ be the set of all nonempty strong subsets of S. Of course, $\mathbf{C}(S) \subset \mathbf{F}(S)$. Neither $\mathbf{C}(S)$ nor $\mathbf{F}(S)$ are closed under the ordinary multiplication of subsets in S : if G and H

are strong subsets (or cosets) of S, then GH need not be strong. Obviously, an intersection of any (finite or infinite) family of strong (unitarily strong) subsets is strong (unitarily strong), and thus there exist two closure operators $\not c : \mathcal{P}^*(S) \to \mathbf{C}(S)$ and $f : \mathcal{P}^*(S) \to \mathbf{F}(S)$, where $\mathcal{P}^*(S)$ is the set of all nonempty subsets of S. If $H \subset S$, then $\not c(H)$ is the least unitarily strong subset of S that contains H (that is, $\not c(H)$ is the intersection of all unitarily strong subsets containing H), while $f(H)$ is the least strong subset of S that contains H (that is, $f(H)$ is the intersection of all strong subsets that contain H). Define a multiplication $\bullet$ in $\mathbf{C}(S)$ as follows: for $G, H \in \mathbf{C}(S)$, $G \bullet H = \not c(GH)$. In other words, $G \bullet H$ is the smallest coset of S that contains the ordinary product GH of G and H. Analogously, define a multiplication $\diamond$ in $\mathbf{F}(S)$: if $G, H \in \mathbf{F}(S)$ then $G \diamond H = f(GH)$.

Observe that $\mathcal{P}^*(S)$ is an inclusion ordered groupoid called the *global* groupoid of S. It is conditionally complete in the sense that, for every subset of $\mathcal{P}^*(S)$ with a nonempty intersection, this set-theoretical intersection is the infimum, while its supremum is its union. Analogously, both $(\mathbf{C}(S); \bullet)$ and $(\mathbf{F}(S); \diamond)$ are inclusion ordered groupoids, which are conditionally complete in the sense that, for every subset of either $\mathbf{C}(S)$ or $\mathbf{F}(S)$, its infimum is its set-theoretical intersection, if nonempty, while its supremum is the $\not c$-closure (respectively, f-closure) of its set-theoretical union. If S is a semigroup, then $\mathcal{P}^*(S)$ is called the *global semigroup* of S.

Theorem 1

The closure operation $\not c$ maps the global semigroup $\mathcal{P}^*(S)$ homomorphically onto the groupoid $(\mathbf{C}(S); \bullet)$. Thus $(\mathbf{C}(S); \bullet)$ is a semigroup. If both $\mathcal{P}^*(S)$ and $(\mathbf{C}(\mathbf{S}); \bullet)$ are considered as ordered by their set-theoretical inclusion, then the homomorphism $\not c$ preserves suprema. Analogously, the closure operation f is a suprema-preserving homomorphism of the global semigroup $\mathcal{P}^*(S)$ onto the groupoid $(\mathbf{F}(S); \diamond)$, so that $(\mathbf{F}(S); \diamond)$ is a semigroup.

Proof. We prove theorem for unitarily strong subsets only. For strong subsets the proof is analogous. If H and F are two subsets of S, then $H.\cdot F = \{s \in S : Fs \subset H\}$, while $H \cdot . F = \{s \in S : sF \subset H\}$. If H is (unitarily) strong, then both $H. \cdot F$ and $H \cdot . F$ are (unitarily) strong. Indeed, if $xv, uv, uy \in H. \cdot F$ for some $u, v, x, y \in S$ (or $\in S^1$), then $Fxv \subset H$, $Fuv \subset H$, and $Fuy \subset H$. Thus, $(fx)v, (fu)v, (fu)y \in H$ for every $f \in F$. Since H is (uniformly) strong, $(fx)y \in H$, and hence $Fxy \subset H$, so that $xy \in H . \cdot F$ and $H . \cdot F$ is (unitarily) strong. Analogously, $H \cdot . F$ is (unitarily) strong.

Now let $G, H \in \mathcal{P}^*(S)$. For every subset F of $S, F \subset \not c(F)$. Thus $GH \subset \not c(G) \bullet \not c(H)$, and hence $\not c(GH) \subset \not c(G) \bullet \not c(H)$. Also $GH \subset \not c(GH) \Leftrightarrow G \subset \not c(GH) \cdot . H \Leftrightarrow$

$\not{c}(G) \subset \not{c}(GH) \cdot . H \Leftrightarrow \not{c}(G)H \subset \not{c}(GH) \Leftrightarrow H \subset \not{c}(GH) . \cdot \not{c}(G) \Leftrightarrow \not{c}(H) \subset \not{c}(GH) . \cdot \not{c}(G) \Leftrightarrow \not{c}(G)\not{c}(H) \subset \not{c}(GH) \Leftrightarrow \not{c}(G) \bullet \not{c}(H) \subset \not{c}(GH)$. Using $GH \subset \not{c}(GH)$, we obtain $\not{c}(G) \bullet \not{c}(H) \subset \not{c}(GH)$, and so $\not{c}(GH) = \not{c}(G) \bullet \not{c}(H)$, that is, $\not{c}$ is a homomorphism of $\mathcal{P}^*(S)$ onto $\mathbf{C}(S)$. Clearly, this homomorphism is onto, because $\not{c}(H) = H$ for every unitarily strong subset H of S. If $I \neq \emptyset$ and $(H_i)_{i \in I}$ is an indexed family of subsets of S, then $H_i \subset \not{c}(H_i)$, and hence $\bigcup H_i \subset \bigcup \not{c}(H_i)$ and $\not{c}(\bigcup H_i) \subset \not{c}(\bigcup \not{c}(H_i))$. Also, $H_i \subset \bigcup H_i$, and hence $\not{c}(H_i) \subset \not{c}(\bigcup H_i)$, so that $\bigcup \not{c}(H_i) \subset \not{c}(\bigcup H_i)$. Therefore, $\not{c}(\bigcup H_i) = \not{c}(\bigcup \not{c}(H_i)) = \bigvee \not{c}(H_i)$, where $\bigvee$ is the symbol of supremum. Thus $\not{c}$ preserves suprema. □

Problem 1. In which semigroups $\mathbf{C}(S) = \mathbf{F}(S)$? A semigroup S is called *globally idempotent* if $S^2 = S$. Obviously, S^2 is a strong subset of S, but it is unitarily strong only if S is globally idempotent. It follows that a necessary condition for $\mathbf{C}(S) = \mathbf{F}(S)$ is global idempotence of S. However, this condition is not sufficient because every inverse semigroup is globally idempotent, but, as Example 1 shows, there exist inverse semigroups S with $\mathbf{C}(S) \neq \mathbf{F}(S)$.

Remark. Semigroups in which every subsemigroup is a strong subset and semigroups in which every subset is strong were characterized in [5] and [6].

Example 2: In the additive semigroup $\mathbb{N}$ of positive integers the subsets $\mathbb{N}$ and $K = \{n \in \mathbb{N} : n > 1\}$ are strong, but only $\mathbb{N}$ is unitarily strong (see Example 1). Now, $\mathbb{N}+\mathbb{N} = K$ implies $\mathbb{N} \diamond \mathbb{N} = K$ and $\mathbb{N} \bullet \mathbb{N} = \mathbb{N} \neq K$. Thus $\mathbf{C}(S)$ is not necessarily a subsemigroup of $\mathbf{F}(S)$, although $\mathbf{C}(S) \subset \mathbf{F}(S)$.

Problem 2. When is $\mathbf{C}(S)$ a subsemigroup of $\mathbf{F}(S)$? An obvious necessary condition is that S is globally idempotent because $S \diamond S = S^2$ and $S \bullet S = S$.

A quasi order relation ζ (that is, a reflexive and transitive binary relation) on a semigroup S is called *steady* if $\zeta\langle z \rangle$ is unitarily strong for every $z \in S$. In other words, ζ is steady if and only if $z \leq xv$, $z \leq uv$, and $z \leq uy$ imply $z \leq xy$ for any $u, v, x, y \in S^1$ for which the inequalities make sense (here $X \leq Y$ stands for $(X, Y) \in \zeta$). Each semigroup possesses steady quasi order relations ($\zeta = S \times S$ is one of them). Let $\hat{\zeta}$ be the least steady quasi order on S (that is, $\hat{\zeta}$ is the intersection of all steady quasi orders). It is called *the strong quasi order relation* of S. A semigroup is isomorphically embeddable in an inverse semigroup if and only if its strong quasi order is antisymmetric (that is, it is an order relation). Equivalently, a semigroup is embeddable in an inverse semigroup if and only if it possesses a strong order relation. A proof is nontrivial and lies beyond the scope of this paper (see [10], and also [9] and [14]).

Theorem 2

The converse of the set-theoretical inclusion relation on $\mathbf{C}(S)$ *is steady.*

Proof. Let $U, V, X, Y, Z \in \mathbf{C}(S)^1$, $Z \supset X \bullet V, Z \supset U \bullet V$, and $Z \supset U \bullet Y$. Then $Z \supset XV$, $Z \supset UV$, and $Z \supset UY$. Let $x \in X$ and $y \in Y$. For $u \in U$ and $v \in V$ we see that $xv, uv, uy \in Z$, and hence $xy \in Z$. It follows that $XY \subset Z$, whence $X \bullet Y \subset Z$, or, equivalently, $Z \supset X \bullet Y$. Therefore, $\supset$ is a steady order on $\mathbf{C}(S)$. □

Problem 3. Is $\supset$ the strong order relation of $\mathbf{C}(S)$ for every semigroup S?

Corollary

The semigroup $\mathbf{C}(S)$ of cosets of any semigroup S is isomorphically embeddable in an inverse semigroup.

Define two mappings $\varphi : S \to \mathbf{C}(S)$ and $\psi : S \to \mathbf{F}(S)$ as follows: $\varphi(s) = \not{c}(\{s\})$ and $\psi(s) = f(\{s\})$ for all $s \in S$. It follows from Theorem 1 that φ and ψ are homomorphisms of S into $\mathbf{C}(S)$ and $\mathbf{F}(S)$, respectively.

Theorem 3

The following three properties are equivalent for any semigroup S:

(1) φ is an isomorphic embedding of S into its semigroup $\mathbf{C}(S)$ of cosets.

(2) S is isomorphically embeddable into its semigroup $\mathbf{C}(S)$ of cosets;

(3) S is isomorphically embeddable in an inverse semigroup;

Proof. (1) $\Rightarrow$ (2) is trivial and (2) $\Rightarrow$ (3) follows from Corollary to Theorem 2.

(3) $\Rightarrow$ (1). Since S is embeddable in an inverse semigroup, $\hat{\zeta}$ is antisymmetric, that is, $y \in \hat{\zeta}\langle x\rangle$ and $x \in \hat{\zeta}\langle y\rangle$ imply $x = y$. As proved in [10], $y \in \hat{\zeta}\langle x\rangle$ and $x \in \hat{\zeta}\langle y\rangle$ is equivalent to $\varphi(x) = \varphi(y)$. Thus φ is one-to-one, and hence (1) holds. □

Theorem 4

If S is an inverse semigroup, then $\mathbf{C}(S)$ is an inverse semigroup and φ is an isomorphic embedding of S into $\mathbf{C}(S)$. The canonical (i.e., natural) order relation of $\mathbf{C}(S)$ is the converse of the set-theoretical inclusion.

Proof. Let S be an inverse semigroup. It is known (see [10] and so also [14]) that a subset $H \subset S$ is unitarily strong if and only if it is *majorantly closed* (that is, $\omega(H) = H$, where ω is the canonical, or natural, order relation on S) and H is a generalized subgroud of S (that is, $HH^{-1}H = H$, where $H^{-1} = \{h^{-1} : h \in H\}$). These two properties mean that $x \in H$ and $x \leq y$ imply $y \in H$, and $x, y, z \in H$ imply $xy^{-1}z \in H$ for all $x, y, z \in S$. The mapping $H \to H^{-1}$ is an involution in $\mathbf{C}(S)$, that is, $(F \bullet H)^{-1} = H^{-1} \bullet F^{-1}$ and $(H^{-1})^{-1} = H$ for all $F, H \in \mathbf{C}(S)$. Moreover, $HH^{-1}H = H$ implies that $H \to H^{-1}$ is an inverting involution, i.e., H^{-1}

is an inverse of H in $\mathbf{C}(S)$. Let H be an idempotent of $\mathbf{C}(S)$, that is, $H \bullet H = H$. Then $HH \subset H$, whence $H \subset H. \cdot H$. Let $x \in H. \cdot H$, that is, $Hx \subset H$. If $h \in H$, then $(hh^{-1})h = h \in H$, $hh \in H$, and $hx \in H$. Since H is strong, $hh^{-1}x \in H$. However, $hh^{-1}x \leq x$, and $x \in H$. It follows that $H. \cdot H \subset H$, whence $H. \cdot H = H$. Analogously, $H \cdot .H = H$. It follows from $HH^{-1}H = H$ that $H^{-1} \subset (H. \cdot H) \cdot .H = H$, i.e. $H = H^{-1}$. Therefore, the idempotents of $\mathbf{C}(S)$ are fixed points of the involution $H \to H^{-1}$. Thus [9] (also [1], where this characterization of inverse semigroups is attributed to W. D. Munn), $\mathbf{C}(S)$ is an inverse semigroup.

It follows from Theorem 2 that $\supset$ is a stable (compatible with multiplication) steady order relation on $\mathbf{C}(S)$, so it coincides with the canonical (also called natural) order of $\mathbf{C}(S)$ because the canonical order is the only stable and steady order on an inverse semigroup (see [11]). □

Problem 4. Study the semigroup $\mathbf{F}(S)$ for an inverse semigroup S. Clearly, $\mathbf{F}(S)$ does not have to be inverse (see Example 1, where K has no inverse in $\mathbf{F}(\mathbf{I})$). Also, $KK^{-1}K \neq K$, and hence the mapping $H \to H^{-1}$ is an involution which is not inverting. Elements of $\mathbf{F}(S)$ are majorantly closed, for if $x \in H \in \mathbf{F}(S)$ and $x \leq y$, then $yy^{-1}x = yy^{-1}xx^{-1}x = y(xx^{-1}y)^{-1}x = yx^{-1}x = x = xx^{-1}x = xx^{-1}y$, and $yy^{-1} \cdot x, xx^{-1} \cdot x, xx^{-1} \cdot y \in H$, so that $y = yy^{-1} \cdot y \in H$. As we have seen in the proof of Theorem 4, $H = H^{-1}$ for idempotents $H \in \mathbf{F}(S)$, so $HH^{-1}H = HHH \subset H$, and hence H is a majorantly closed inverse subsemigroup of S, and a unitarily strong subset of $\mathbf{F}(S)$.

Let G be a group. It contains an identity element, and hence $\mathbf{C}(G) = \mathbf{F}(G)$. Strong subsets of a group are precisely its cosets [2]. Every group is an inverse semigroup, thus the semigroup $\mathbf{C}(G)$ of cosets of a group is inverse. The idempotents of $\mathbf{C}(G)$ are the subgroups of G, and the canonical order relation on $\mathbf{C}(G)$ is the converse of the set-theoretical inclusion relation. Therefore the lattice of idempotents of $\mathbf{C}(G)$ is dually isomorphic to the subgroup lattice of G. Obviously, central idempotents of $\mathbf{C}(G)$ are the normal subgroups of G, and the lattice of central idempotents of $\mathbf{C}(G)$ ordered by the canonical order relation is dually isomorphic to the lattice of normal subgroups of G and hence modular.

Problem 5. Do central idempotents of $\mathbf{C}(S)$ from a modular lattice for an inverse semigroup S?

Obviously, $\not{c}(g) = \{g\}$ for every $g \in G$. If $\{g\}$ is identified with g, then G becomes a subgroup of $\mathbf{C}(G)$ where the elements of G are maximal elements of $\mathbf{C}(G)$. Thus the set of all maximal elements of $\mathbf{C}(G)$ forms a subgroup, and each element of $\mathbf{C}(G)$ is the infimum (the greatest lower bound) of a set of its maximal elements. A subset B of a (partially) ordered set A is called a *minorant basis* if each nonempty subset C of B possesses an infimum $\bigwedge C$ in A and each element $a \in A$ is the infimum of a nonempty subset of A. Thus G is minorant basis of $\mathbf{C}(G)$ with respect to the canonical order.

The following theorem provides an abstract characterization of $\mathbf{C}(G)$.

Theorem 5

An inverse semigroup Γ is isomorphic to the inverse semigroup $\mathbf{C}(G)$ of all cosets of a group G if and only if Γ satisfies the following three conditions:

(1) The set of all maximal elements of Γ forms a subgroup isomorphic to G, and this set is a minorant basis for Γ;

(2) If $\bigwedge H$ denotes the infimum of a subset H of Γ, then

$$\left(\bigwedge H_1\right)\left(\bigwedge H_2\right) = \bigwedge H_1H_2$$

for any two subsets H_1 and H_2 of Γ (that is, multiplication distributes over infima);

(3) Every inverse semigroup that satisfies (1) and (2) is a homomorphic image of Γ.

Proof. **Necessity.** (1) We have already observed that G is the subgroup of all maximal elements of $\mathbf{C}(G)$. The canonical order relation in $\mathbf{C}(G)$ is the converse inclusion For every nonempty subset H of G, $g \in H \Rightarrow g \in \not c(H) \in \mathbf{C}(G) \Rightarrow \{g\} \subset \not c(H) \Rightarrow \not c(H) \leq \{g\}$. Suppose that $F \in \mathbf{C}(G)$ and $F \leq \{g\}$ for all $g \in H$. Then $g \in F$, and so $H \subset F$. It follows that $\not c(H) \subset F$, that is, $F \leq \not c(H)$. Thus $\not c(H) = \bigwedge H$, and hence $\bigwedge H$ exists. Also, $H \leq \{g\}$ for any $H \in \mathbf{C}(G)$ and $g \in H$. If $F \in \mathbf{C}(G)$ and $F \leq \{g\}$ for all $g \in H$, then $\{g\} \subset F$, so that $g \in F$. It follows that $H \subset F$, and hence $F \leq H$. Thus $H = \bigwedge \{\{g\} : g \in H\}$.

(2) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be subsets of $\mathbf{C}(G)$. By Theorem 1,

$$\begin{aligned}
\bigwedge \mathcal{H}_i = \not c\left(\bigcup \mathcal{H}_i\right) = \not c\left(\bigcup \not c(H) : H \in \mathcal{H}_i\right), \quad \text{and hence} \quad \left(\bigwedge \mathcal{H}_1\right) \bullet \left(\bigwedge \mathcal{H}_2\right) = \\
&= \not c\left(\not c\left(\bigcup \not c(H) : H \in \mathcal{H}_1\right) \not c\left(\bigcup \not c(H) : H \in \mathcal{H}_2\right)\right) \\
&= \not c\left(\not c\left(\bigcup \not c(H) : H \in \mathcal{H}_1\right) \not c\left(\bigcup \not c(H) : H \in \mathcal{H}_2\right)\right) \\
&= \not c\left(\left(\bigcup \not c(H) : H \in \mathcal{H}_1\right)\left(\bigcup \not c(H) : H \in \mathcal{H}_2\right)\right) \\
&= \not c\left(\left(\bigcup H : H \in \mathcal{H}_1\right)\left(\bigcup H : H \in \mathcal{H}_2\right)\right) \\
&= \not c\left(\bigcup FH : F \in \mathcal{H}_1,\ H \in \mathcal{H}_2\right) = \not c\left(\bigcup \mathcal{H}_1\mathcal{H}_2\right) = \bigwedge \mathcal{H}_1\mathcal{H}_2,
\end{aligned}$$

and so (2) holds.

(3) Let S be an inverse semigroup that satisfies conditions (1) and (2). For convenience sake assume that the set of all maximal elements of S coincides with G. Let $\varphi : \mathbf{C}(G) \to S$ be defined as follows. For $H \in \mathbf{C}(G)$, $\varphi(H) = \bigwedge H$, where H in the right-hand of the equality is the subset of G, and hence of S, and $\bigwedge H$ is the infimum of H is S. By (1), φ is an onto mapping. To prove that φ is a homomorphism, we need to show that $(\bigwedge H_1)(\bigwedge H_2) = \bigwedge(H_1 \bullet H_2)$ for any $H_1, H_2 \in \mathbf{C}(G)$. By (2), $(\bigwedge H_1)(\bigwedge H_2) = \bigwedge H_1H_2$, and hence there remains to prove that $\bigwedge H_1H_2 = \bigwedge H_1 \bullet H_2$. To this end we need a technical lemma.

Lemma

If $H \subset G$, let $H^{[2n-1]}$ denote $H(H^{-1}H)^{2n-2}$. Then $\not{c}(H) = \bigcup_{n=1}^{\infty} H^{[2n-1]}$.

Proof. $H = H^{[1]} \subset \bigcup_{n=1}^{\infty} H^{[2n-1]}$. Also, $\big(\bigcup_{n=1}^{\infty} H^{[2n-1]}\big)\big(\bigcup_{n=1}^{\infty} H^{[2n-1]}\big)^{-1}\big(\bigcup_{n=1}^{\infty} H^{[2n-1]}\big) = \bigcup_{n=2}^{\infty} H^{[2n-1]} \subset \bigcup_{n=1}^{\infty} H^{[2n-1]}$, and hence $\bigcup_{n=1}^{\infty} H^{[2n-1]} \in \mathbf{C}(G)$. Thus $\not{c}(H) \subset \bigcup_{n=1}^{\infty} H^{[2n-1]}$. Also, $\not{c}(H)\not{c}(H)^{-1}\not{c}(H) = \not{c}(H)$, so that $\not{c}(H)^{[2n-1]} = \not{c}(H)$ for every n. It follows from $H \subset \not{c}(H)$ that $H^{[2n-1]} \subset \not{c}(H)^{[2n-1]} = \not{c}(H)$, and hence $\bigcup_{n=1}^{\infty} H^{[2n-1]} \subset \not{c}(H)$. □

$H_1 \bullet H_2$ is the least coset of G that contains H_1H_2. Replacing H by H_1H_2 in Lemma, we obtain $H_1 \bullet H_2 = \bigcup_{n=1}^{\infty}(H_1H_2)^{[2n-1]}$, and hence

$$(\bigwedge H_1)(\bigwedge H_2) = ((\bigwedge H_1)(\bigwedge H_2))^{[2n-1]} = (\bigwedge H_1H_2)^{[2n-1]} = \bigwedge(H_1H_2)^{[2n-1]},$$

for every n. Therefore,

$$(\bigwedge H_1)(\bigwedge H_2) = \bigwedge \big\{ \bigwedge (H_1H_2)^{[2n-1]} : n \in \mathbb{N}\big\} = \bigwedge H_1 \bullet H_2 .$$

Sufficiency. Let Γ be an inverse semigroup that satisfies conditions (1)-(3). By (3), there exists a homomorphism h of Γ onto $\mathbf{C}(G)$. Homomorphisms preserve canonical order relations of inverse semigroups, and so maximal elements of Γ are mapped into maximal elements of $\mathbf{C}(G)$, and every maximal element of $\mathbf{C}(G)$ is so obtained. Therefore, without loss of generality, assume that G is the set of all maximal elements of Γ and h induces the identity automorphism on G.

Let $\gamma_1, \gamma_2 \in \Gamma$, $h(\gamma_1) = h(\gamma_2)$, and $H_i = \{g \in G : \gamma_i \le g\}$, $(i = 1, 2)$. By (1) and (2), $H_iH_i^{-1}H_i = H_i$, that is, H_i is a coset of G. Therefore, H_1 and H_2 are elements of $\mathbf{C}(G)$. By condition (1), $\bigwedge H_i = \gamma_i$, and so $h(\gamma_i) = \bigwedge\big(\{h\} : h \in H_i\big) = H_i$. Thus $H_1 = H_2$, which implies $\gamma_1 = \gamma_2$. It follows that h is an isomorphism. □

Theorem 6 is more general than Theorem 5 and is given without proof, because its proof is analogous to our proof of Theorem 5.

Theorem 6

An inverse semigroup Γ is isomorphic to the inverse semigroup $\mathbf{C}(S)$ of all cosets of an inverse semigroup S if and only if Γ satisfies the following three conditions:

(1) *Γ contains a majorantly closed inverse subsemigroup S_0 isomorphic to S, and S_0 is a minorant basis of Γ;*

(2) *$(\bigwedge H_1)(\bigwedge H_2) = \bigwedge H_1H_2$ for any two subsets H_1 and H_2 of Γ (that is, multiplication distributes over infima);*

(3) *Every inverse semigroup that satisfies (1) and (2) is a homomorphic image of Γ.*

Remark. Conditions (2) and (3) coincide with conditions (2) and (3) of Theorem 5. Analogously, we can state and prove a theorem that characterizes semigroups isomorphic to $\mathbf{C}(S)$, where S is an arbitrary semigroup. Instead of the canonical order relation on inverse semigroups we have to consider the strong quasi order relation on an arbitrary semigroup S.

Theorem 7 [4]

Every inverse semigroup is embeddable in the inverse semigroup of cosets of a suitable group.

Proof. (Outline). Every inverse semigroup S is isomorphic to an inverse semigroup Γ of one-to-one partial transformations of a set A. Let B be a set that contains A, where every one-to-one partial transformation of A can be extended to a bijection of B onto itself. For example, this is true if $B \neq A$ and, if A is infinite, the complement of A in B has the same cardinality as A. Let $\mathbf{G}_B$ be the group of all bijections of B onto itself. For every $\gamma \in \Gamma$ define $\varphi(\gamma) = \{g \in \mathbf{G}_B : \gamma \subset \alpha\}$, where $\gamma \subset \alpha$ means that α is an extension of γ (γ is a subset of α if γ and α are considered as binary relations). Clearly, $\varphi(\gamma) \neq \emptyset$ and $\varphi(\gamma)$ is a coset of $\mathbf{G}_B$. It is no less clear that $\varphi(\gamma) = \varphi(\delta) \Rightarrow \gamma = \delta$ for all $\gamma, \delta \in \Gamma$, that is, φ is one-to-one.

Let $\gamma, \delta \in \Gamma$. If $\alpha \in \varphi(\gamma)$ and $\beta \in \varphi(\delta)$ for $\alpha, \beta \in \mathbf{G}_B$ (that is, $\gamma \subset \alpha$ and $\delta \subset \beta$), then $\delta \circ \gamma \subset \beta \circ \alpha$, and hence $\beta \circ \alpha \in \varphi(\delta \circ \gamma)$. Therefore, $\varphi(\delta) \circ \varphi(\gamma) \subset \varphi(\delta \circ \gamma)$, and hence $\varphi(\gamma) \bullet \varphi(\delta) = \not{c}(\varphi(\delta) \circ \varphi(\gamma)) \subset \varphi(\delta \circ \gamma)$. Here we write factors for $\circ$ and $\bullet$ from the right to the left and from the left to the right, respectively.

Let $\alpha \in \varphi(\delta \circ \gamma)$, that is, $\delta \circ \gamma \subset \alpha$. Then $\gamma, \delta^{-1} \circ \delta$, and $\gamma \circ \gamma^{-1}$ can be extended to some $\rho, \sigma, \tau \in \mathbf{G}_B$. More than a single choice for ρ, σ, and τ can be possible, and we claim they can chosen so that $\tau \circ \sigma \circ \rho$ extends $\delta^{-1} \circ \alpha$, that is, $\delta^{-1} \circ \alpha \subset \tau \circ \sigma \circ \rho$, and hence $\delta^{-1} \subset \tau \circ \sigma \circ \rho \circ \alpha^{-1}$. We call our entire proof an "outline" because we skip a not very enlightening construction for ρ, σ, and τ. Even with full details, this proof is shorter than the original proof given in [4]. Then $\delta^{-1} \subset \tau \circ \sigma \circ \rho \circ \alpha^{-1}$, so that $\delta \circ \gamma = (\delta \circ \gamma) \circ (\delta \circ \gamma)^{-1} \circ (\delta \circ \gamma) = \delta \circ \gamma \circ \gamma^{-1} \circ \delta^{-1} \circ \delta \circ \gamma \subset (\tau \circ \sigma \circ \rho \circ \alpha^{-1})^{-1} \circ \tau \circ \sigma \circ \rho = \alpha$. It follows that $\alpha \in \varphi(\delta) \circ \varphi(\gamma \circ \gamma^{-1}) \circ \varphi(\delta^{-1} \circ \delta) \circ \varphi(\gamma)$. Observe that $\varphi(\gamma)^{-1} = \varphi(\gamma)^{-1}$ and $\varphi(\gamma \circ \gamma^{-1}) \subset \varphi(\gamma) \circ \varphi(\gamma^{-1})$ (analogously $\varphi(\delta^{-1} \circ \delta) \subset \varphi(\delta^{-1}) \circ \varphi(\delta)$). The equality means that $\gamma^{-1} \subset \pi \Leftrightarrow \gamma \subset \pi^{-1}$ for any $\pi \in \mathbf{G}_B$, and the inclusion means that if $\gamma \circ \gamma^{-1} \subset \pi$ for some $\pi \in \mathbf{G}_B$, then there are $\mu, \nu \in \mathbf{G}_B$ for which $\gamma \subset \mu, \gamma \subset \nu$, and $\mu \circ \nu^{-1} = \pi$, pretty obvious propositions. Therefore, $\varphi(\delta) \circ \varphi(\gamma \circ \gamma^{-1}) \circ \varphi(\delta^{-1} \circ \delta) \circ \varphi(\gamma) \subset \varphi(\delta) \circ \varphi(\gamma) \circ \varphi(\gamma^{-1}) \circ \varphi(\delta^{-1}) \circ \varphi(\delta) \circ \varphi(\gamma) = \big(\varphi(\delta) \circ \varphi(\gamma)\big)\big(\varphi(\delta) \circ \varphi(\gamma)\big)^{-1}\big(\varphi(\delta) \circ \varphi(\gamma)\big) = \big(\varphi(\delta) \circ \varphi(\gamma)\big)^{[3]} \subset \varphi(\gamma) \bullet \varphi(\delta)$. The last inclusion follows from the inclusion $\varphi(\delta) \circ \varphi(\gamma) \subset \varphi(\gamma) \bullet \varphi(\delta)$ and the fact that $\varphi(\gamma) \bullet \varphi(\delta)$ is a coset. Thus $\alpha \in \varphi(\gamma) \bullet \varphi(\delta)$, and hence $\varphi(\delta \circ \gamma) \subset \varphi(\gamma) \bullet \varphi(\delta)$.

Therefore $\varphi(\delta \circ \gamma) = \varphi(\gamma) \bullet \varphi(\delta)$ and φ is an isomorphic embedding of S into $\mathbf{C}(\mathbf{G}_B)$. □

Problem 6. Let $\mathbf{K}$ be a class of groups. Which inverse semigroups are embeddable in $\mathbf{C}(G)$ for $G \in \mathbf{K}$? How are properties of groups in K and of the embeddable inverse semigroups connected? For example, $\mathbf{C}(G)$ is commutative for every abelian group G. Is every commutative inverse semigroup embeddable in $\mathbf{C}(G)$ for a suitable abelian group G?

References

1. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups, I*. Amer. Math. Soc., Providence, R.I., 1961.
2. Paul Dubreil, Contribution à la théorie des demi-groupes, *Mém. Acad. Sci. Inst. France* **63**:3 (1941), 1–52.
3. G. G. Dyadchenko and Boris M. Schein, Monogenic inverse semigroups, *Algebra and Number Theory, Kabardino-Balkar. State Univ., Nalchik* **1** (1973), 3–26 [Russian].
4. D. B. McAlister, Embedding inverse semigroups in coset semigroups, *Semigroup Forum* **20**:3 (1980), 255–267.
5. Mario Petrich, Sur certaines classes de demi-groupes I, *Bull. Cl. Sci. Acad. Royale Belg.* (5) **49**:8 (1963), 785–797.
6. Mario Petrich, Sur certaines classes de demi-groupes III, *Bull. Cl. Sci. Acad. Royale Belg.* (5) **53**:2 (1967), 60–73.

7. J. Riguet, Relations binaires, fermetures, correspondances de Galois, *Bull. Soc. Math. France* **76**:1–4 (1948), 114–155.
8. J. Riguet , Fondéments de la théorie des relations binaires. Thèse, Fac. Sci. Paris, 1951.
9. Boris M. Schein, A system of axioms for semigroups embeddable in inverse semigroups, *Doklady Akad. Nauk SSSR* **134**:5 (1960), 1030–1034 [Russian; English translation in *Soviet Math. Doklay* **1**:5 (1960), 1180–1183].
10. Boris M. Schein, Embedding semigroups in inverse semigroups, *Matem. Sbornik* **55**:97 (1961), 379–400 [Russian; English translation in *Amer. Math. Society Translations* (2) **139** (1988), 93–116].
11. Boris M. Schein, On the theory of inverse semigroups, *Doklady Akad. Nauk SSSR* **153**:2 (1963), 296–299.
12. Boris M. Schein, Semigroups of strong subsets. *Volzhskĭi Matem. Sbornik, Kuĭbyshev* **6** (1966), 180–186 [Russian].
13. Boris M. Schein, Free inverse semigroups are not finitely presentable, *Acta Math. Acad. Sci. Hungar.* **26**:1–4 (1975) 41–52.
14. Boris M. Schein, Embedding semigroups in inverse semigroups, *Algebra and Number Theory, Kabardino-Balkar. State Univ., Nalchik* **2** (1977), 147–163 [Russian].
15. Boris M. Schein, Cosets in groups and semigroups, *Proc. Conf. on Semigroups with Appl.* (Oberwolfach, 1991), World Scientific Publ. Co., Singapore, (1992), 205–221.