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Ordering the set of antichains of an ordered set

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Abstract

The purpose of this paper is to define, in a natural way, an order relation on the set of antichains of an ordered set. Thus, the set of finite antichains becomes a sup-lattice. If moreover, the given ordered set is a strictly ordered commutative monoid, the finite antichains constitute also an ordered monoid.

In § 3, an ultrametric distance is introduced; it has values in the partially ordered set of finite antichains. This distance appears naturally in the study of rings of generalized power series. It will be used in a forthcoming paper (with Sibylla Prieβ-Crampe) concerning fixed points of contracting maps. Antichains have also been considered in the study of Gröbner bases, where they appear under names like "crowns" and it is likely that the present results will be useful in that context.

By its simplicity, it is also expected that use of these ideas will be found in the theory of ordered sets and their applications.

I

Let $(S, \leq)$ be an ordered set. We denote by $\leq^\circ$ the relation opposite to $\leq$: $s \leq^\circ t$ in S if and only if $t \leq s$. Clearly, $\leq^\circ$ is also an order relation.

Let $A(S, \leq)$ be the set of antichains of $(S, \leq)$, these are the non-empty subsets X of S such that if $s, t \in X$ and $s \leq t$, then $s = t$. Let $F(S, \leq)$ be the set of (non-empty) finite antichains of $(S, \leq)$.

It is clear that $A(S, \leq) = A(S, \leq^\circ), F(S, \leq) = F(S, \leq^\circ)$. For simplicity, we shall use the notations $A(S), F(S)$ for the set of antichains, resp. finite antichains of $(S, \leq)$.

Let $\imath : S \to A(S)$ be the canonical injection $\imath(s) = \{s\}$; it is surjective if and only if the order $\leq$ is total.

We define the relations $\ell \leq$ and $\leq r$ on $A(S)$ as follows. If $X, Y \in A(S)$, let $X\ell \leq Y$ when for every $x \in X$, there exists $y \in Y$ such that $x \leq y$. Similarly, let $X \leq rY$ when for every $y \in Y$, there exists $x \in X$ such that $x \leq y$.

The relations $\ell \leq$ and $\leq r$ on $A(S)$ are related as follows: if $X, Y \in A(S)$, then $X \leq rY$ if and only $X(\ell \leq^\circ)^\circ Y$, where $\leq^\circ$ denotes the opposite order of $\leq$ on S and $(\ell \leq^\circ)^\circ$ the opposite relation of $\ell \leq^\circ$ on $A(S)$.

Thus, it suffices to study the relation $\ell \leq$ on $A(S)$. For simplicity and if there is no ambiguity, we simply write $X \leq Y$ instead of $X\ell \leq Y$.

We establish some properties of the relation $\leq; X, Y, Z$ denote antichains of S.

Proposition 1.1

If $X \subseteq Y$, then $X \leq Y$. In particular, $X \leq X$.

Proposition 1.2

If $X \leq Y$ and $Y \leq X$, then $X = Y$.

Proof. Let $x \in X$, so there exists $y \in Y$ such that $x \leq y$. Similarly, there exists $x' \in X$, such that $y \leq x'$. From $x \leq y \leq x'$, it follows that $x = x'$, hence $x = y \in Y$, showing the inclusion $X \subseteq Y$. Similarly, $Y \subseteq X$. □

Proposition 1.3

$X \leq Y$ and $Y \leq Z$, then $X \leq Z$.

Therefore $(A(S), \leq)$ is an ordered set and the mapping $\imath$ is an immersion of ordered sets.

For each subset X of S, let $\mathrm{Min}(X)$ denote the set of minimal elements of X, so $\mathrm{Min}(X) \in A(S) \cup \{\emptyset\}$. It is not excluded that $\mathrm{Min}(X) = \emptyset$ when $X \neq \emptyset$.

Similarly, let $\mathrm{Max}(X)$ be the set of maximal elements of X; $\mathrm{Max}(X) \in A(S) \cup \{\emptyset\}$ and $\mathrm{Max}(X)$ may be empty when $X \neq \emptyset$.

Proposition 1.4

Let $X, Y \in A(S)$.

a) For every $z \in X \cup Y$, there exists $t \in \mathrm{Max}(X \cup Y)$ such that $z \leq t$.
b) For every $z \in X \cup Y$, there exists $t \in \mathrm{Min}(X \cup Y)$ such that $z \geq t$.

Proof. a) Let $z \in X \cup Y$, say $z \in X$. If $z \notin \mathrm{Max}(X \cup Y)$, there exists $y \in X \cup Y$ such that $z < y$; hence $y \notin X$, so $y \in Y$. If $y \notin \mathrm{Max}(X \cup Y)$, there exists $x \in X \cup Y$ such that $y < x$, hence $x \notin Y$, so $x \in X$. Therefore $z < y < x$, with $z, x \in X$, which is impossible.

b) The proof is similar. □

Proposition 1.5

$(A(S), \leq)$ *is a sup-lattice and* $(F(S), \leq)$ *is a sub-sup-lattice.*

Proof. Let $X, Y \in A(S)$; we show that $\sup\{X, Y\}$ exists and it is equal to $\mathrm{Max}(X \cup Y) \in A(S)$.

First, $X \leq \mathrm{Max}(X \cup Y)$ and also $Y \leq \mathrm{Max}(X \cup Y)$, as follows from (1.4). If $X \leq Z$ and $Y \leq Z$ and $t \in \mathrm{Max}(X \cup Y) \subseteq X \cup Y$, then necessarily there exists $z \in Z$ such that $t \leq z$. This shows that $(A(S), \leq)$ is a sup-lattice, where the sup is given as indicated.

If $X, Y \in F(S)$ then $\mathrm{Max}(X \cup Y) \in F(S)$, so $(F(S), \leq)$ is a sub-sup-lattice of $(A(S), \leq)$. □

We note that if $X \in F(S)$, then $X = \sup\{\{x\} \mid x \in X\}$.

Now we prove the following universal property.

Proposition 1.6

Let $(L, \leq)$ *be a sup-lattice and* $\varphi : S \to L$ *an order-homomorphism. Then, there exists a unique order-homomorphism* $\psi : F(S) \to L$, *such that* $\psi \circ \imath = \varphi$ *and* $\psi(\sup\{X, Y\}) = \sup\{\psi(X), \psi(Y)\}$.

Proof. We define $\psi : F(S) \to L$ as follows: $\psi(X) = \sup\{\varphi(x) \mid x \in X\}$; we note that since X is a finite set, the above sup exists.

If $X \leq Y$, for each $x \in X$ there exists $y_x \in Y$ such that $x \leq y_x$; then $\varphi(x) \leq \varphi(y_x)$ and $\psi(X) = \sup\{\varphi(x) \mid x \in X\} \leq \sup\{\varphi(y_x) \mid x \in X\} \leq \sup\{\varphi(y) \mid y \in Y\} = \psi(Y)$.

We have also $\psi \circ \imath(s) = \psi(\{s\}) = \varphi(s)$ for every $s \in S$.

Clearly $\sup\{\psi(X), \psi(Y)\} \leq \psi(\sup\{X, Y\})$. If $z \in L$ and $\psi(X) \leq z, \psi(Y) \leq z$, then $\varphi(x) \leq z$ for every $x \in X$ and $\varphi(y) \leq z$ for every $y \in Y$. So $\varphi(t) \leq z$ for every $t \in X \cup Y$, hence also for every $t \in \mathrm{Max}(X \cup Y) = \sup\{X, Y\}$. By definition, $\psi(\sup\{X, Y\}) = \sup\{\varphi(t) \mid t \in \sup\{X, Y\}\} \leq z$.

It remains to show the uniqueness of ψ. If $\psi' : F(S) \to L$ is an order-homomorphism satisfying the same properties as ψ, then $\psi'(X) = \psi'(\sup\{\{x\} \mid x \in X\}) = \sup\{\psi'(\{x\}) \mid x \in X\} = \sup\{\varphi(x) \mid x \in X\} = \psi(X)$. □

An ordered set $(S, \leq)$ is *noetherian* if every strictly ascending chain in S is finite. It is equivalent to say that if T is any non-empty subset of S, then $\mathrm{Max}(T) \neq \emptyset$. In particular, if $s \in S$ there exists $t \in \mathrm{Max}(S)$ such that $s \leq t$. We also agree that the empty set is noetherian.

An ordered set is *artinian* if every strictly descending chain in S is finite. $(S, \leq)$ is artinian if and only if $(S, \leq^\circ)$ is noetherian. We also say that the empty set is artinian.

Proposition 1.7

Assume that $(S, \leq)$ is noetherian and has a smallest element. Then $(A(S), \leq)$ is a complete lattice: if $(X_i)_{i \in I}$ is any non-empty family of elements of $A(S)$, then

$$\sup\{X_i \mid i \in I\} = \mathrm{Max}\left(\bigcup_{i \in I} X_i\right)$$

$$\inf\{X_i \mid i \in I\} = \mathrm{Max}\big(\{t \in S \mid \textit{for every } i \in I\,. \;\; \textit{there exists } x_i \in X_i \textit{ such that } t \leq x_i\}\big)\,.$$

Moreover, the last element of $A(S)$ is $\mathrm{Max}(S)$, and the first element of $A(S)$ is $\{0\}$.

Proof. The last assertion is obvious because S is noetherian and it has the smallest element 0.

We first determine the sup. It is clear that $X_j \leq \mathrm{Max}\big(\cup_{i \in I} X_i\big)$ for every $j \in I$, because S is noetherian, so for every $x \in \cup_{i \in I} X_i$ there exists $t \in \mathrm{Max}\big(\cup_{i \in I} X_i\big)$ such that $x \leq t$.

Moreover, if $X_i \leq T$ for every $i \in I$, if $x \in \mathrm{Max}\big(\cup_{i \in I} X_i\big) \subseteq \cup_{i \in I} X_i$, there exists $t \in T$ such that $x \leq t$; this shows that $\mathrm{Max}\big(\cup_{i \in I} X_i\big) \leq T$ and determines the sup. Let $T = \{t \in S \mid \text{ for every } i \in I \text{ there exists } x_i \in X_i \text{ such that } t \leq x_i\}$. The smallest element of S belongs to T. Since S is noetherian, then $\mathrm{Max}(T) \neq \emptyset$, so $\mathrm{Max}(T) \in A(S)$.

Clearly $\mathrm{Max}(T) \leq T \leq X_i$ for every $i \in I$.

If $Z \in A(S)$ and $Z \leq X_i$ for every $i \in I$, then $Z \subseteq T$, so for every $z \in Z$ there exists $t \in \mathrm{Max}(T)$ such that $z \leq t$. This shows that $Z \leq \mathrm{Max}(T)$ and determines the inf. $\square$

We prove now a universal property fpr $A(S)$:

Proposition 1.8

Assume that $(S, \leq)$ is noetherian and has a smallest element. Let $(L, \leq)$ be a complete sup-lattice and $\varphi : S \to L$ an order-homomorphism. Then, there exists a unique order-homomorphism $\psi : A(S) \to L$, such that $\psi \circ \imath = \varphi$ and $\psi(\sup\{X_i \mid i \in I\}) = \sup\{\psi(X_i) \mid i \in I\}$.

Proof. We define $\psi : A(S) \to L$ as follows: $\psi(X) = \sup\{\varphi(x) \,|\, x \in X\}$. The proof is similar to that of (1.6) and may therefore be omitted. □

Let $(S, \leq)$ be an ordered set. A subset X of S is *narrow* if all its antichains are finite. In particular, the empty set is narrow.

The subsets which are artinian and narrow (also called *quasi-well-ordered* sets) have been extensively studied. We recall the following characterization:

Proposition 1.9

$(S, \leq)$ is artinian and narrow if and only if for every sequence $\{s_1, s_2, \ldots\}$ of elements of S, there exist indices $i < j$ such that $s_i \leq s_j$.

It is clear that if $(S, \leq)$ is artinian (or noetherian), then $F(S)$ need not to have the same property. For example, taking $S = \{s_1, s_2, \ldots s_n, \ldots\}$ with the trivial order, $S_i = \{s_1, s_2, \ldots, s_i\}, S'_i = S \backslash S_i$, then $S'_1 > S'_2 > S'_3 > \ldots$ and $S_1 < S_2 < S_3 < \ldots$).

However, we show:

Proposition 1.10

If $(S, \leq)$ is artinian and narrow, then $F(S)$ is also artinian and narrow.

Proof. According to the characterization of artinian and narrow sets quoted in (1.9), we need to show that if $\{X_1, X_2, \ldots\}$ is any sequence in $F(S)$, there exist indices $i < j$ such that $X_i \leq X_j$. Assume this is not true, so if $1 < j$ then $X_1 \not\leq X_j$. Hence there exists $x_{1j} \in X_1$ such that $\{x_{1j}\} \not\leq X_j$. Since X_1 is finite, there exists an infinite sequence $j_2 < j_3 < \ldots$ (with $1 < j_2$) such that $x_{1j_2} = x_{1j_3} = \ldots$, say equal to some element $x_1 \in X_1$.

Consider the subsequence $\{X_{j_2}, X_{j_3}, \ldots\}$ and for simplicity label these sets as $X_{j_k} = X_k$ (for $k \geq 2$). Repeating the same argument, there exists an element $x_2 \in X_2$ and an infinite subsequence $j_3 < j_4 < \ldots$ (with $2 < j_3$) such that $\{x_2\} \not\leq X_{j_k}$ and also $\{x_1\} \not\leq X_{j_k}$ (for $j_3 \leq k$). This process may be repeated leading to an infinite subset $\{x_1, x_2, \ldots\}$ of S, such that if $i < j$ then $x_i \not\leq x_j$. This is however impossible, because $(S, \leq)$ is artinian and narrow. □

II

Let S be an additive commutative monoid with zero element 0. If $X, Y \subseteq S$, let $X + Y = \{x + y \,|\, x \in X,\ y \in Y\}$.

Assume that $\leq$ is a compatible order relation on S, that is, if $s, t, u \in S$ then $s \leq t$ implies $s + u \leq t + u$.

Thus $(S, \leq)$ is a commutative *ordered monoid.*

The order $\leq$ is *strict* whenever $s < t$ implies $s + u < t + u$ (for $s, t, u \in S$); then $(S, \leq)$ is called a *strictly ordered monoid.*

If $(S, \leq)$ is an ordered monoid, we define an operation on the set $F(S)$ of finite antichains of S, as follows:

$$X \underset{a}{+} Y = \mathrm{Max}(X + Y)\,.$$

We note that $X \underset{a}{+} Y \in F(S)$ because $\mathrm{Max}(X + Y) \neq \emptyset$. Now we describe the properties of this operation. Let $X, Y, Z \in F(S)$.

Proposition 2.1

$X \underset{a}{+} Y = Y \underset{a}{+} X$ *and* $X \underset{a}{+} \{0\} = X$.

Proposition 2.2

If $(S, \leq)$ *is a strictly ordered monoid, then*

$$X \underset{a}{+} (Y \underset{a}{+} Z) = (X \underset{a}{+} Y) \underset{a}{+} Z\,.$$

Proof. Let $a \in (X \underset{a}{+} Y) \underset{a}{+} Z = \mathrm{Max}(\mathrm{Max}(X + Y) + Z)$, so $a = b + z$, with $b \in \mathrm{Max}(X + Y)$, $z \in Z$. Then, $b = x + y$, with $x \in X$, $y \in Y$, hence $a = x + y + z$.

We show that $y + z \in \mathrm{Max}(Y + Z)$. If $y' \in Y$, $z' \in Z$ and $y + z \leq y' + z'$, then $a = x + y + z \leq x + y' + z'$. Since $X + Y$ is finite, there exists $x_1 \in X$, $y_1 \in Y$, such that $x + y' \leq x_1 + y_1 \in \mathrm{Max}(X + Y)$. Then $a \leq x_1 + y_1 + z'$ and since $a \in \mathrm{Max}(\mathrm{Max}(X + Y) + Z)$, then $a = x_1 + y_1 + z'$, hence $x + y + z = x + y' + z'$. From $y + z \leq y' + z'$ and the fact that the order is strict, then $y + z = y' + z'$. So $a \in X + \mathrm{Max}(Y + Z)$.

If $x'' \in X$, $y'' \in Y$, $z'' \in Z$ are such that $a = x + y + z \leq x'' + y'' + z''$, with $y'' + z'' \in \mathrm{Max}(Y + Z)$, then $(x + y) + z \leq (x'' + y'') + z''$. But $X + Y$ is finite, so there exists $x_2 \in X$, $y_2 \in Y$ such that $x_2 + y_2 \in \mathrm{Max}(X + Y)$ and $x'' + y'' \leq x_2 + y_2$. Therefore $a = (x + y) + z \leq (x_2 + y_2) + z''$.

Again, since $a \in \mathrm{Max}(\mathrm{Max}(X+Y)+Z)$, then $a = (x_2+y_2)+z'' = x''+(y''+z'') \in \mathrm{Max}(X + \mathrm{Max}(Y + Z)) = X \underset{a}{+} (Y \underset{a}{+} Z)$.

The proof of the converse is similar. □

Thus, if $(S,\leq)$ is a strictly ordered monoid, then $F(S)$ is a commutative monoid with the operation $\underset{a}{+}$ with zero element $\{0\}$. The canonical injection $\imath : S \to F(S)$ is a monoid-homomorphism.

We shall henceforth assume that $(S,\leq)$ is strictly ordered.

Proposition 2.3

The order on $F(S)$ is compatible with the operation $\underset{a}{+}$, so $(F(S),\leq)$ is a commutative ordered monoid.

Proof. Let $X, Y, Z \in F(S)$ be such that $X \leq Y$; we show that $X \underset{a}{+} Z \leq Y \underset{a}{+} Z$.

Let $x + z \in \mathrm{Max}(X + Z)$, with $x \in X,\ z \in Z$. So there exists $y_1 \in Y$ such that $x \leq y_1$, hence $x + z \leq y_1 + z$. Since $Y + Z$ is finite, there exist $y' \in Y,\ z' \in Z$, such that $y_1 + z \leq y' + z' \in \mathrm{Max}(Y + Z)$. This shows the statement. □

We have also:

Proposition 2.4

If $X, Y, Z \in F(S)$, then $\sup\{X \underset{a}{+} Z,\ Y \underset{a}{+} Z\} = \sup\{X, Y\} \underset{a}{+} Z$.

Proof. By the preceding result, $\sup\{X \underset{a}{+} Z, Y \underset{a}{+} Z\} \leq \sup\{X, Y\} \underset{a}{+} Z$.

Conversely, we show that it $T \in F(S)$ and $X \underset{a}{+} Z \leq T, Y \underset{a}{+} Z \leq T$, then $\sup\{X, Y\} \underset{a}{+} Y \leq T$, and this proves the statement.

Let $v \in \mathrm{Max}(X \cup Y),\ z \in Z$ and $v + z \in \mathrm{Max}(\mathrm{Max}(X \cup Y) + Z)$. Say, $v \in X$, so $v + z \in X + Z$. Since $X + Z$ is finite, there exist $x' \in X,\ z' \in Z$, such that $x' + z' \in \mathrm{Max}(X + Z)$ and $v + z \leq x' + z'$.

Therefore, there exists $t \in T$ such that $v + z \leq x + z \leq t$, concluding the proof. □

Proposition 2.5

Assume that $0 \leq s$ for every $s \in S$.

If $X, Y \in F(S)$, then

$$\sup\{X, Y\} \leq X \underset{a}{+} Y.$$

Proof. Let $t \in \sup\{X, Y\} = \operatorname{Max}(X \cup Y)$, say $t \in X$; let $y \in Y$ so $0 \leq y$, hence $t \leq t + y \in X + Y$; hence there exist $x' \in X$, $y' \in Y$ such that $t + y \leq x' + y' \in \operatorname{Max}(X + Y) = X \underset{a}{+} Y$. This proves the statement. □

III

In this section we shall define an ultrametric distance.

Let $(S, \leq)$ be an ordered set, let R be an abelian additive group, with zero, element 0. For each $f : S \to R$, we define the *support of* f by

$$\operatorname{supp}(f) = \{s \in S \mid f(s) \neq 0\}\,.$$

The zero-mapping has empty support.

Let $A = \{f : S \to R \mid \operatorname{supp}(f) \text{ is artinian and narrow}\}$.

With pointwise addition, we see that $\operatorname{supp}(f + g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g)$; hence A is an abelian additive group.

We consider the set $F(S)$ of finite antichains of S endowed with order $\ell \leq^\circ = (\leq r)^\circ$, which we denote by $\triangleleft$ for simplicity. Explicitly, $X \triangleleft Y$ means that for every $x \in X$ there exists $y \in Y$ such that $y \leq x$. It follows from (1.5) that $(F(S), \triangleleft)$ is a sup-lattice, with $\sup_\triangleleft\{X, Y\} = \operatorname{Max}_{\leq^\circ}(X \cup Y) = \operatorname{Min}_{\leq}(X \cup Y)$.

We adjoin to $F(S)$ an element, namely the empty set $\emptyset$, and we put $\emptyset \triangleleft X$ for every $X \in F(S)$.

Now we define $d : A \times A \to (F(S) \cup \{\emptyset\}, \triangleleft)$ by letting $d(f, g) = \operatorname{Min}_{\leq}(\operatorname{supp}(f - g))$. We note that since f, g have artinian and narrow support, then $d(f, g) \in F(S) \cup \{\emptyset\}$.

Proposition 3.1

d is an ultrametric distance on A, compatible with the addition, that is, for $f, g, h, \in A$ we have:

1) *$d(f, g) = \emptyset$ if and only if $f = g$.*
2) *$d(f, g) = d(g, f)$.*
3) *$d(f, g) \triangleleft \sup_\triangleleft\{d(f, h),\, d(g, h)\}$.*
4) *$d(f + h, g + h) = d(f, g)$.*

Proof. The assertions (1), (2), (4) are trivial. We show (3) and we may assume $f \neq g$, so $d(f,g) \neq \emptyset$. Let $t \in d(f,g) = \mathrm{Min}_{\leq}(\mathrm{supp}(f-g))$, so $t \in \mathrm{supp}(f-g) \subseteq \mathrm{supp}(f-h) \cup \mathrm{supp}(g-h)$, say $t \in \mathrm{supp}(f-h)$. Hence there exists $u \in \mathrm{Min}_{\leq}(\mathrm{supp}(f-h)) = d(f,h) \subseteq d(f,h) \cup d(g,h)$ such that $u \leq t$. Again, there exists $v \in \mathrm{Min}_{\leq}(d(f,h) \cup d(g,h)) = \sup_{\lhd}\{d(f,h), d(g,h)\}$, such that $v \leq u \leq t$; this shows the statement (3). □

We deduce the following special case. Let $(S, \leq)$ be an ordered set, let $W(S)$ be the set of all artinian and narrow subsets of S. We define $d : W(S) \times W(S) \to (F(S) \cup \{\emptyset\}, \lhd)$, by $d(X,Y) = \mathrm{Min}_{\leq}(X \oplus Y)$ (the *symmetric difference* $X \oplus Y$ is the set of all $z \in X \cup Y$ which are not in $X \cap Y$).

Proposition 3.2

The mapping $d : W(S) \times W(S) \to (F(S) \cup \{\emptyset\}, \lhd)$ defined by $d(X,Y) = Min_{\leq}(X \oplus Y)$, is an ultrametric distance, compatible with the symmetric difference.

Proof. Consider the set A of all the mappings $f : S \to \mathbb{Z}$ having artinian and narrow support. For each $X \in W(S)$, let $\omega_X : S \to \mathbb{Z}$ be the characteristic function of X, namely

$$\omega_X(s) = \begin{cases} 1 & \text{if} \quad s \in X \\ 0 & \text{if} \quad s \notin X\,. \end{cases}$$

so $\omega_X \in A$ for each $X \in W(S)$.

If $X, Y \in W(S)$ then $X \cup Y \in W(S)$, hence also $X \oplus Y \in W(S)$ and $X \oplus Y = \mathrm{supp}(\omega_X - \omega_Y)$. It follows that $d(X,Y) = \mathrm{Min}_{\leq}(X \oplus Y) = \mathrm{Min}_{\leq}(\mathrm{supp}(\omega_X - \omega_Y)) = d(\omega_X, \omega_Y)$. According to (3.1), d is an ultrametric distance on $W(S)$, with values in $(F(S) \cup \{\emptyset\}, \lhd)$. Clearly, $d(X \oplus Z, Y \oplus Z) = d(X,Y)$, for any $X, Y, Z \in W(S)$. □

Here is a particular case. If $(S, \leq)$ is totally ordered, then $W(S)$ is the set of all well-ordered subsets of S. Now, if $X, Y \in W(S)$, then $d(X,Y)$ is the smallest element of the set $X \oplus Y$.

The following result generalizes a noteworthy property of the usual ultrametric distances:

Proposition 3.3

Let $f, g, h \in A$ and assume that $d(f,h) \cap d(g,h) = \emptyset$ and $d(f,h) \lhd d(g,h)$. Then $d(f,g) = d(g,h)$.

Proof. By (3.1), $d(f,g) \lhd \sup_\lhd \{d(f,h), d(g,h)\} = d(g,h)$ and we need to show that $d(g,h) \lhd d(f,g)$.

Let $s \in d(g,h) = \mathrm{Min}_\leq(\mathrm{supp}(g-h))$, hence $s \in \mathrm{supp}(g-h)$. If $(f-g)(s) = 0$, then $s \in \mathrm{supp}(f-h)$, so there exists $t \in \mathrm{Min}_\leq(\mathrm{supp}(f-h)) = d(f,h)$ such that $t \leq s$. Again, by hypothesis, there exists $u \in d(g,h)$ such that $u \leq t \leq s$ and necessarily, $u = t = s \in d(f,h) \cap d(g,h) = \emptyset$, which is a contradiction.

Thus $s \in \mathrm{supp}(f-g)$ and there exists $v \in \mathrm{Min}_\leq(\mathrm{supp}(f-g))$ such that $v \leq s$. This proves that $d(g,h) \lhd d(f,g)$, as required. □

In particular, If $(S, \leq)$ is totally ordered, if $d(f,h) \leq^\circ d(g,h)$, then $d(f,g) = d(g,h)$, and this is a known property for ultrametric distances with values in a totally ordered set.

We define now the norm $\|f\|$ for every $f \in A$:

$$\|f\| = d(f,0)\,.$$

So we have at once:

Proposition 3.4

If $f, g \in A$:

1) $\|f\| = \emptyset$ *if and only if* $f = 0$.
2) $\|f+g\| \lhd \sup_\lhd\{\|f\|, \|g\|\}$.
3) *If* $\|f\| \lhd \|g\|$ *and* $\|f\| \cap \|g\| = \emptyset$, *then* $\|f+g\| = \|g\|$.

Proof. This follows routinely from (3.1) and (3.3).

For example,

$$\begin{aligned}\|f+g\| &= d(f+g,0) \lhd \sup{}_\lhd\{d(f+g,g), d(g,0)\}\\ &= \sup{}_\lhd\{d(f,0),\, d(g,0)\} = \sup{}_\lhd\{\|f\|,\, \|g\|\}\,. \;\square\end{aligned}$$

Let $\alpha : (F(S) \cup \{\emptyset\}, \lhd) \to (F(S) \cup \{\emptyset\}, \leq r)$ be the identity mapping; we recall that $\leq r$ is the opposite order to $\lhd = (\ell \leq^\circ)$.

We define $\pi(f) = \alpha(\|f\|) \in (F(S) \cup \{\emptyset\}, \leq r)$ and we note the properties:

Proposition 3.5

If $f, g \in S$:

1) $\pi(f) = \emptyset$ *if and only if* $f = 0$.
2) $\inf_{\leq r}\{\pi(f), \pi(g)\} \leq r\, \pi(f+g)$.
3) *If* $\pi(g) \leq r\, \pi(f)$ *and* $\pi(f) \cap \pi(g) = \emptyset$, *then* $\pi(f+g) = \pi(g)$.

We shall now assume that $(S, \leq)$ is a strictly ordered abelian additive monoid and R is a commutative ring with unit element. Then A is a commutative ring, with multiplication.

$$(f * g)(s) = \sum_{t+u=s} f(t)g(u)$$

(see [1]), called the ring of generalized power series with exponents in $(S, \leq)$ and coefficients in R.

From the definition, $\operatorname{supp}(f * g) \subseteq \operatorname{supp}(f) + \operatorname{supp}(g)$.

In this situation, we note the following property of the distance:

Proposition 3.6

If $f, g, h \in A$, then $d(f * h, g * h) \lhd d(f, g) \underset{a}{+} \|h\|$.

Proof. Let $s \in d(f*h, g*h) = \operatorname{Min}_{\leq}(\operatorname{supp}(f*h - g*h))$, so $s \in \operatorname{supp}(f*h - g*h) = \operatorname{supp}((f-g)*h) \subseteq \operatorname{supp}(f-g) + \operatorname{supp}(h)$. Hence there exists $t \in \operatorname{supp}(f-g), u \in \operatorname{supp}(h)$ such that $s = t + u$. Then there exist $t' \in \operatorname{Min}(\operatorname{supp}(f-g)) = d(f,g)$ and $u' \in \operatorname{Min}(\operatorname{supp}(h)) = \|h\|$ such that $t \geq t',\ u \geq u'$, so $t' + u' \leq s$. Again, there exist $v \in \operatorname{Min}_{\leq}(d(f,g) + \|h\|) = d(f,g) \underset{a}{+} \|h\|$, such that $v \leq t' + u' \leq s$. This shows the statement. □

In terms of the norm, we have:

Proposition 3.7

If $f, g \in A$, then

$$\|f * g\| \leq \|f\| \underset{a}{+} \|g\| .$$

Again, we have:

Proposition 3.8

If $f, g \in A$, then

$$\pi(f) + \pi(g) \leq r\, \pi(f * g) .$$

In the special case when $(S, \leq)$ is totally ordered, $\pi(f)$ is the smallest element of $\operatorname{supp}(f)$ and we have:

$$\pi(f + g) \geq \inf \{\pi(f), \pi(g)\} ,$$
$$\pi(f * g) \geq \pi(f) + \pi(g) .$$

If moreover, R is a domain, then A is a domain and $\pi(f * g) = \pi(f) + \pi(g)$.

Therefore, π may be extended to a valuation of the field of fractions of A (see [2]).

References

1. P. Ribenboim, *Generalized power series rings,* in "Lattices, Semigroups and Universal Algebra" (edited by J. Almeida, G. Bordalo and P. Dwinger), Plenum, New York (1990).
2. P. Ribenboim, Rings of generalized power series: nilpotent elements, *Abh. Math. Sem. Univ. Hamburg* **61** (1991), 1–19.