Collect. Math. **46**, 1–2 (1995), 151–157

Products of inverse semigroups

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In memoriam **Paul Dubreil**

Abstract

This paper exhibits, in addition to the direct product, six further associative products of two inverse semigroups, each of which perhaps has a claim to be called a semidirect product; and it poses the problem of whether there are further such products.

1. Introduction

The direct product of two inverse semigroups is an inverse semigroup but, what has been termed the semidirect product of two semigroups, is, when the semigroups are inverse, in general not inverse.

This semidirect product is defined, for arbitrary semigroups, as follows: let S and T be semigroups and let $\theta : S \longrightarrow \text{End } T$ be an antimorphism of S into the endomorphism semigroup of T. (We shall throughout use θ to denote an antimorphism.) If $s \in S$ denote $t(s\theta)$ by t^s . Then the *semidirect product* of T and S, in that order, with *structure map* θ, consists of the set $T \times S$ equipped with the product

$$(t, s)(t_1, s_1) = (tt_1^s, ss_1). \tag{A}$$

This product will be denoted by $T\ {}_\theta\!\times_A\ S$.

In Preston [3], Theorem 6, (following Nico [1], Theorem 2.6) it is shown that $T\ {}_\theta\!\times_A\ S$ is an inverse semigroup if and only if (i) S and T are inverse semigroups and (ii) $S\theta \subseteq \text{Aut } T$, the automorphism group of T.

In Preston [3] we also showed that, from a certain class of possibilities, the only associative product on the set $T \times S$, associative for all choices of S and T and all choices of θ, is the product $T\ {}_\theta{\times}_A\ S$ that we have just defined. We begin by looking at a similar class of possibilities, but this time considering only inverse semigroups S and T. We find six associative products, three of which can be constructed, by a dual procedure, from the other three. We call these products of types A, B, and C, and their duals products of types A', B', and C', respectively. The product defined in equation (A) is the type A product. When θ maps each element of S to the identity mapping of T, then all six types of product reduce to the direct product.

Conditions for a type A product to be regular were established in Preston [4], Proposition 1. A dual result holds for type A' products. It is straightforward to show that for semidirect products of types B and C, and their duals, the resulting semigroup is regular if and only if it is inverse; this can occur if and only if $S\theta \subseteq \mathrm{Aut}\, T$. In this event the products of types A, B, and C coincide, as do the products of types A', B', and C'.

A final comment should be made about the six types of semidirect product introduced. The products arise from solutions to equations (1), (2), and (3), listed in Section 2, below. These are formally the same as the equations (2), (3), and (4) of Preston [3]. Then we were seeking solutions that held in any semigroup. Here we are seeking solutions that hold in any inverse semigroup. In the semigroup situation we found all solutions. In the present inverse semigroup situation I have been unable to ascertain whether there are further solutions yet to be found.

A further comment about this problem will be made at the end of the paper.

2. Some semidirect products of inverse semigroups

We use an approach parallel to that in Preston [3] and consider a class of rules of formation of a product of a pair of inverse semigroups. We find six types of product each of which always results in the product being a semigroup.

Denote by FI_2 the free inverse semigroup on the two element set $\{x_1, x_2\}$, and by FI_2^1 the semigroup resulting from adjoining an identity element 1. If S is any inverse semigroup and $r, s \in S$ and $u = u(x_1, x_2) \in FI_2$, denote by $u(r, s)$ the element of S obtained from u by replacing each occurrence of x_1 in u by r, each occurrence of x_1^{-1} in u by r^{-1}, each occurrence of x_2 in u by s and each occurrence of x_2^{-1} in u by s^{-1}, and evaluating the resulting product in S. Let T be an inverse semigroup and let $\theta : S \longrightarrow \mathrm{End}\, T$ be an antimorphism. If $u \in FI_2$ then, for all t in T, let $t^{u(r,s)}$ denote $t((u(r, s))\theta)$, while for $u = 1$ take t^u to equal t. The antimorphism θ involved, suppressed in this notation, will be clear from the context. Now let $u, v \in FI_2^1$ and define a product on the set $T \times S$ by the rule

$$(t_1, s_1)(t_2, s_2) = (t_1^{u(s_1,s_2)} t_2^{v(s_1,s_2)}, s_1 s_2) \qquad (*)$$

and denote this system by $(T \times S, \theta, u, v)$.

The products of these systems $(T \times S, \theta, u, v)$ form the class of products we consider.

We wish to discover for which choices of u and v the product $(T \times S, \theta, u, v)$ is a semigroup for all inverse semigroups S and T, for all associated antimorphisms θ. We begin with a lemma, and for this it will be convenient to use Scheiblich's [5] representation of the elements of a free inverse semigroup with identity on a set X. We denote this semigroup by FI_X^1.

The representation is in terms of FG_X, the free group on X, which we take to consist of all reduced finite words in the alphabet $X \cup X^{-1}$, where $X^{-1} = \{x^{-1} | x \in X\}$, where X and X^{-1} are disjoint, and where $x \longmapsto x^{-1}$, $x \in X$, is a bijection of X upon X^{-1}. A word here is said to be reduced if it contains no subwords xx^{-1} or $x^{-1}x$ for x in X. If $g \in FG_X$ then we denote by $\hat{g}$ the set of all initial segments of g, including both the empty word 1 and the word g itself. Thus

$$\hat{g} = \{h |\ g = hk,\ \ h, k \in FG_X \text{ and } \ hk \ \text{ is reduced as it stands}\}.$$

The free inverse semigroup FI_X^1 may then be defined (Scheiblich) as follows. First define the set $\mathcal{X}$ by $\mathcal{X} := \{A |\ A \subseteq FG_X,\ A$ is finite and non-empty, $g \in A$ implies $\hat{g} \subseteq A\}$. Then

$$FI_X^1 := \{(A, g) |\ A \in \mathcal{X}, g \in A\}$$

equipped with the product

$$(A, g)(B, h) = (A \cup gB, gh).$$

The canonical embedding of X in FX_X^1 is then $x \longmapsto (\hat{x}, x)$, $x \in X$. The semilattice of idempotents of FI_X^1 is

$$\{(A, 1) |\ A \in \mathcal{X}\}$$

and

$$(A, g)^{-1} = (g^{-1}A, g^{-1}).$$

We can now state our lemma.

Lemma 2.1

Let $U = FI_X^1$, let E be the semilattice of idempotents of U, and let $T = E \times FG_X$, the direct product. Let $s = (A, g)$ be an element of U and define θ by

$$s\theta : \begin{cases} (e, h) \longmapsto \ (ses^{-1}, ghg^{-1}),\ e \in E \setminus 1,\ h \in FG_X \\ (1, h) \longmapsto \ (1, ghg^{-1}),\ h \in FG_X. \end{cases}$$

Then $\theta : U \longrightarrow End\ T$ is an antimorphism.

Proof. A straightforward calculation suffices. □

Theorem 2.2

$(T \times S, \theta, u, v)$ is a semigroup for all inverse semigroups T and S and all associated θ if and only if u and v satisfy the following equations for all S and for all $s_1, s_2, s_3 \in S$:

$$u(s_1s_2, s_3)u(s_1, s_2) = u(s_1, s_2s_3) \tag{1}$$

$$u(s_1s_2, s_3)v(s_1, s_2) = v(s_1, s_2s_3)u(s_2, s_3) \tag{2}$$

$$v(s_1s_2, s_3) = v(s_1, s_2s_3)v(s_2, s_3) \tag{3}$$

Proof. We easily show that the product in $(T \times S, \theta, u, v)$ is associative if and only if, for all s_1, s_2, s_3 in S and for all t_1, t_2, t_3 in T,

$$\begin{aligned} &t_1^{u(s_1s_2,s_3)\ u(s_1,s_2)} t_2^{u(s_1s_2,s_3)\ v(s_1,s_2)} t_3^{v(s_1s_2,s_3)} \\ &= t_1^{u(s_1,s_2s_3)} t_2^{v(s_1,s_2s_3)\ u(s_2,s_3)} t_3^{v(s_1,s_2s_3)\ v(s_2,s_3)} \end{aligned} \tag{4}$$

It is evident, therefore, that if equations (1), (2) and (3) hold, then $(T \times S, \theta, u, v)$ is associative.

To prove the converse we make a special choice of T, S and θ, namely as in Lemma 1 with $|X| > 1$, and show that, with this choice, equations (1), (2), and (3) hold. Because, in Lemma 1, U is any free inverse semigroup, it follows that if u and v are chosen so that (1), (2) and (3) hold in U, then they also hold in any inverse semigroup S.

First take $t_2 = t_3 = 1$. Equation (4) then reduces to

$$t_1^{u(s_1s_2,s_3)u(s_1,s_2)} = t_1^{u(s_1,s_2s_3)}. \tag{5}$$

Since $U = FI_X^1$, we may write $u(s_1s_2, s_3)u(s_1, s_2) = (C_1, g_1)$, and $u(s_1, s_2s_3) = (C_2, g_2)$, say. Take $t_1 = ((B, 1), h)$. Then (5) is the same as

$$\begin{aligned} &((C_1, g_1)\ (B, 1)\ (C_1, g_1)^{-1},\ g_1hg_1^{-1}) \\ &= ((C_2, g_2)\ (B, 1)\ (C_2, g_2)^{-1},\ g_2hg_2^{-1}). \end{aligned}$$

So, equating components,

$$((C_1 \cup g_1B, 1) = (C_2 \cup g_2B, 1) \tag{6}$$

and

$$g_1hg_1^{-1} = g_2hg_2^{-1} \tag{7}$$

Since (5) holds for all t_1 , equation (7) holds for all h. Thus $g_1^{-1}g_2$ belongs to the centre of FG_X. Since $|X| > 1$, the centre of FG_X is trivial. Hence $g_1 = g_2$.

Now take $B = \{1\}$, and recall that in the Scheiblich representation, $g_1 \in C_1$ and $g_2 \in C_2$. Thus, from (6), we get $C_1 = C_2$. Hence equation (1) holds with U any free inverse semigroup FI_X^1, with $|X| > 1$.

The arguments to show that equations (2) and (3) hold are similar. □

I have been unable to find a complete solution to equations (1), (2), and (3). By straightforward verification we easily check that six pairs of solutions listed in the next theorem are, in fact, solutions.

Theorem 2.3

The following set of seven choices for the ordered pairs (u, v) *of words* $u = u(x_1, x_2)$ *and* $v = v(x_1, x_2)$ *satisfy the equations (1), (2), and (3).*

(A)	$(1, x_1)$	*(A′)*	$(x_2^{-1}, 1)$
(B)	$(x_1x_2x_2^{-1}x_1^{-1}, x_1)$	*(B′)*	$(x_2^{-1}, x_2^{-1}x_1^{-1}x_1x_2)$
(C)	$(x_1x_2x_2^{-1}x_1^{-1}, x_1x_2x_2^{-1})$	*(C′)*	$(x_2^{-1}x_1^{-1}x_1, x_2^{-1}x_1^{-1}x_1x_2)$
(D)	(1,1)	*(D′)*	(1.1).

The solutions have been listed in the above fashion in order to point to a duality. The solution (D), corresponding to the direct product, is the same as (D'): (D) is self dual. The solutions (A), (B), and (C) have as duals, in a sense made precise below, the solutions (A'), (B'), and (C'), respectively; and vice versa.

We define the dual of a pair of words $(u(x_1, x_2),\ v(x_1, x_2))$ to be the pair of words $(v(x_2^{-1},\ x_1^{-1}),\ u(x_2^{-1}, x_1^{-1}))$. Each pair is then the dual of the other pair. In this sense, as already stated, (A) and (A'), (B) and (B'), (C) and (C'), are dual pairs of words.

The connection between dual pairs of solutions of equations (1), (2), and (3) is elucidated in the following theorem. In particular it shows that any solution pair is always accompanied by a dual solution.

Theorem 2.4

Let S *and* T *be inverse semigroups, let* $\theta : S \longrightarrow \operatorname{End} T$ *and let* (u, v) *and* (u', v') *be dual pairs of words. Denote the products determined by equation* $(*)$ *for* $(T \times S, \theta, u, v)$ *and* $(T \times S, \theta, u', v')$ *by* $\circ$ *and* $\circ'$*, respectively. If* $(t, s) \in T \times S$ *denote* (t^{-1}, s^{-1}) *by* $(t, s)'$*. Then for* $A, B \in T \times S$,

$$A \circ B = (B' \circ' A')'.$$

Hence $\circ$ *is associative if and only if its dual* $\circ'$ *is associative.*

Proof. Let $A = (t_1, s_1)$ and $B = (t_2, s_2)$. Then

$$\begin{aligned} A \circ B &= (t_1^{u(s_1,s_2)} t_2^{v(s_1,s_2)},\ s_1 s_2) \\ &= (t_1^{v'(s_2^{-1},s_1^{-1})}\ t_2^{u'(s_2^{-1},s_1^{-1})}, s_1 s_2) \end{aligned}$$

from the definition of dual pairs,

$$= (((t_1^{-1})^{v'(s_2^{-1},s_1^{-1})})^{-1}\ ((t_2^{-1})^{u'(s_2^{-1},s_1^{-1})})^{-1},\ s_1 s_2),$$

since inverses are mapped to inverses under inverse semigroup morphisms,

$$\begin{aligned} &= (((t_2^{-1})^{u'(s_2^{-1},s_1^{-1})}\ (t_1^{-1})^{v'(s_2^{-1},s_1^{-1})})^{-1},\ s_1 s_2) \\ &= ((t_2^{-1})^{u'(s_2^{-1},s_1^{-1})}\ (t_1^{-1})^{v'(s_2^{-1},s_1^{-1})},\ s_2^{-1} s_1^{-1})' \\ &= ((t_2^{-1}, s_2^{-1})\ \circ'\ (t_1^{-1}, s_1^{-1}))' \\ &= (B' \circ' A')'. \end{aligned}$$

Suppose that $\circ$ is associative. Then

$$\begin{aligned} (A' \circ B') \circ' C' = (B \circ A)' \circ' C' &= (C \circ (B \circ A))' \\ &= ((C \circ B) \circ A)' \\ &= A' \circ' (B' \circ' C'). \end{aligned}$$

so $\circ'$ is associative. Similarly when $\circ'$ is associative then so also is $\circ$. $\square$

3. Context of problem

The interest in finding associative products of the kind discussed here arose from looking at possible definitions of wreath products for inverse semigroups, as used, for example, in a special case, by Mario Petrich [2], p.226. I was looking for a framework that would enable a wreath product to be defined abstractly, without working with representations of the inverse semigroups involved, and presented a survey of the situation in a lecture at the Second Australian Mathematics Convention, Sydney University, 11-15 May, 1981. A possible candidate was the subset $M = \{(t^{ss^{-1}}, s) \mid (t, s) \in T \times S\}$ of $T_\theta \times S$, endowed with the product $\circ_B$ or the product $\circ_C$, which happen to coincide on the set M, and make M an inverse subsemigroup of both $T\,_\theta \times_B S$ and $T\,_\theta \times_C S$. The mapping $(t, s) \longmapsto (t^{ss^{-1}}, s)$ from $T\,_\theta \times_A S$, is a morphism onto M, with the above product, while for $T\,_\theta \times_B S$ and $T\,_\theta \times_C S$, the morphism is a retraction. The existence of dual associative products was discovered later and discussed in a seminar at Monash University in May 1982. In 1988, at the Argonne National Laboratory, Illinois, with the generous help of Dr Ewing L. Lusk, I set up a program to check by computer whether a pair of words $u(x, y)$ and $v(x, y)$ satisfied equations (1), (2) and (3) of Theorem 2.2. The program allowed me to check systematically all possible choices for u and v of lengths ≤ 2, ≤ 3, etc. Unless a slip was made, the results showed that no choices of u, v other than those listed in Theorem 2.2, for both u and v of length less than 12, would work.

References

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