Collect. Math. **46**, 1–2 (1995), 57–76

On practical partitions

P. Erdös

Mathematical Institute of the Hungarian Academy of Sciences,
Realtanoda u. 13-15, Pf. 127, H-1364 Budapest, Hungary

J.L. Nicolas*

Institut Girard Desargues, URA 746, Mathématiques, Bat. 101,
Université Claude Bernard (Lyon 1), F-69622 Villeurbanne cedex, France

Dedicated to the memory of Paul Dubreil

Abstract

Let $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_k < \ldots\}$ be an infinite subset of $\mathbb{N}$. A partition of n with parts in $\mathcal{A}$ is a way of writing $n = a_{i_1} + a_{i_2} + \ldots + a_{i_j}$ with $1 \le i_1 \le i_2 \le \ldots \le i_j$. An integer a is said to be represented by the above partition, if it can be written $a = \sum_{r=1}^{j} \varepsilon_r a_{i_r}$ with $\varepsilon_r = 0$ or 1. A partition will be called practical if all $a's$, $1 \le a \le n$ can be represented. When $\mathcal{A} = \mathbb{N}$, it has been proved by P. Erdös and M. Szalay that almost all partitions are practical. In this paper, a similar result is proved, first when $a_k = 2^k$, secondly when $a_k \ge k a_{k-1}$. Finally an example due to D. Hickerson gives a set $\mathcal{A}$ and integers n for which a lot of non practical partitions do exist.

I. Introduction

Let us denote by $\mathbb{N}$ the set of positive integers $\{1, 2, 3, \ldots\}$ and by $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_k < \ldots\}$ an infinite subset of $\mathbb{N}$ containing 1. A partition of an integer

* Research partially supported by CEE fund n° CIPA-CT 92-4022, and by CNRS, Institut Girard Desargues, URA 746.

n with parts in $\mathcal{A}$ will be a solution of the equation:

$$x_1a_1 + x_2a_2 + \ldots + x_ka_k = n \tag{1}$$

where x_i are non negative integers, and $k = k(n)$ is defined by

$$a_k \le n < a_{k+1}\,. \tag{2}$$

Let a be a positive integer. The partition (1) will be said to represent a, if a can be written

$$y_1a_1 + \ldots + y_ka_k = a,\ 0 \le y_i \le x_i, 1 \le i \le k\,.$$

If you have n francs in your purse, with, say x_1 coins of 1 franc, x_2 coins of 2 francs, x_3 coins of 5 francs and x_4 coins of 10 francs, then this partition of n will represent a if you can pay a francs without needing any change.

The partition (1) will be said practical, if it represents all a, $1 \le a \le n$. The number of partitions of n with parts in $\mathcal{A}$ will be denoted by $p(n) = p_{\mathcal{A}}(n)$, while the number of practical, and non practical partitions will be denoted respectively by $\tilde{p}(n) = \tilde{p}_{\mathcal{A}}(n)$ and $M(n) = M_{\mathcal{A}}(n)$, so that we have

$$p(n) = \tilde{p}(n) + M(n)\,. \tag{3}$$

We shall denote by $p(n,m) = p_{\mathcal{A}}(n,m)$ the number of partitions of n in parts in $\mathcal{A}$ and $\le m$, and by $r(n,m) = r_{\mathcal{A}}(n,m)$ the number of partitions of n in parts in $\mathcal{A}$ and $\ge m$. These two functions can be calculated by induction using

$$p(n,m) = p(n,m-1) + p(n-m,m)$$

and

$$r(n,m) = r(n,m+1) + r(n-m,m)$$

for any $\mathcal{A}$, in the same way as for $A = \mathbb{N}$ (cf. [6]). It is convenient to set

$$p(0) = p(0,m) = r(0,m) = \tilde{p}(0) = 1.$$

Let us observe that, as $1 \in \mathcal{A}$, one has for m fixed:

$$p_{\mathcal{A}}(n) \text{ and } p_{\mathcal{A}}(n,m) \text{ are non decreasing in } n\,. \tag{4}$$

Indeed, by adding 1 to any partition of n, a partition of $(n+1)$ is obtained. For more about the increasingness of $p_{\mathcal{A}}(n)$ when $1 \notin \mathcal{A}$, see [1].

In [5], it is proved that, when $\mathcal{A} = \mathbb{N}$, almost all partitions are practical, that is to say $M(n) = o(p(n))$ when n goes to infinity. Moreover, it is proved that the number $M(n)$ of non practical partitions is asymptotic to the number of partitions of n without any 1, so that, if a partition is non practical, it is mainly because it contains no 1. One of the authors gave a talk on this subject in Oberwolfach, and M.N. Huxley wrote the following verses:

If a beggar comes crying to you,
Ask Nicolas just what to do.
He proves a fact strange,
That "I don't have the change"
Is the same as "I don't have a sou".

In [4], the above result of [5] is precised: The following asymptotic expansion is given for any b:

$$M(n) = \left(\sum_{a=1}^{b} \alpha_a n^{-a/2} + O(n^{-(b+1)/2}) \right) p(n)$$

with $\alpha_1 = \pi/\sqrt{6}, \alpha_2 = \pi^2/4 - 1$, etc... The following formula is also given

$$M(n) = \sum_{1 \le a \le n/2} \tilde{p}(a-1) r(n-a+1, a+1) \tag{5}$$

Formula (5) is used in [4] to calculate a table of $M(n)$ up to $n = 100$.

It is not difficult to see that (5) still holds when the set of parts $\mathcal{A}$ is any subset of $\mathbb{N}$. Indeed, a partition which represents a, also represents $n - a$, so that for a non practical partition, there is an a, $1 \le a \le n/2$ such that all integers up to a are represented, but a itself is not represented. For a given a, the number of partitions of n for which a is the smallest number non represented can be proved to be equal to

$$\tilde{p}(a-1) r(n-a+1, a+1) \,,$$

exactly in the same way as it was done in [4], lemma 5, when $\mathcal{A} = \mathbb{N}$.

If the set $\mathcal{A}$ would not contain 1, then clearly, $M(n) = p(n)$, all partitions of n are not practical, since 1 is not represented. This is in agreement with (5), which shows by induction that $M(n) = p(n)$, since $r(n, 2) = p(n)$ in that case.

In section 2, we shall deal with binary partitions, that is $\mathcal{A} = \{1, 2, 4, 8, \ldots, 2^k, \ldots\}$ consists of powers of 2. The number $p(n)$ of binary partitions has been studied by Mahler and de Bruijn (cf. [7] and [2]), and $r(n, m)$ can be easily deduced from $p(n)$, so that, from (5), we shall prove:

Theorem 1

Let $\mathcal{A} = \{1, 2, 4, \ldots\}$. Then for all n, almost all partitions are practical. More precisely, $M(n) = 0$ if and only if $n + 1$ is a power of 2 and when n goes to infinity,

$$M(n) = O\left(\frac{\log n}{n} p(n) \right) .$$

In section 4, we shall consider a set of parts such that $a_k \geq ka_{k-1}$. More precisely, the following theorem will be proved:

Theorem 2

Let $\mathcal{A} = \{a_1 = 1, a_2, \ldots, a_k, \ldots\}$ *with* $a_k \geq ka_{k-1}$. *Then, for all* n, *almost all partitions are practical, and moreover:*

$$M_{\mathcal{A}}(n) = O\left(2^{-k(n)} p_{\mathcal{A}}(n)\right). \tag{6}$$

(where $k(n)$ *is defined by (2)).*

In section 5, an example of a set $\mathcal{A}$ will be given for which most of the partitions will not be practical for a sequence of integers tending to infinity. We are very much indebted to Dean Hickerson who provided this example and the proof of (7) below, and who has allowed us to include his work in our paper. Moreover, we should say that Hickerson's example was the starting point of this study, and so we thank him very much.

Theorem 3

Let $f(0) \geq 1, f(1), \ldots, f(n), \ldots$ *a non decreasing sequence of integers tending to infinity with* n. *There exists a set* $\mathcal{A} = \{a_1 = 1, a_2, \ldots\}$ *with* $a_k = O(kf(k)2^k)$ *such that*

$$\underline{\lim}\, \tilde{p}_{\mathcal{A}}(n)/p_{\mathcal{A}}(n) = 0 \tag{7}$$

and

$$\overline{\lim}\, \tilde{p}_{\mathcal{A}}(n)/p_{\mathcal{A}}(n) = 1. \tag{8}$$

In the Hickerson example of section 5, $\mathcal{A}$ is the union of ranges of consecutive integers far away of each other. A great deal of partitions of n are shown to be non practical because they have a part $> n/2$.

In section 3, we shall recall classical estimations for $p_{\mathcal{A}}(n)$. These estimations are rough, but good enough when $k = k(n)$ (defined by (2)) tends to infinity rather slowly, and they will be used together with (5) to prove Theorems 2 and 3.

We did not succeed in finding a characterization of $\mathcal{A}$ such that almost all partitions are practical. Let us formulate two conjectures:

Conjecture 1: Let $\mathcal{A} = \{a_1 = 1, a_2, \ldots, a_k, \ldots\}$ with $a_k \leq 2^k$. Then $M_{\mathcal{A}}(n) = o(p_{\mathcal{A}}(n))$, that is almost all partitions are practical.

Conjecture 2: Let $\mathcal{A}$ any subset of $\mathbb{N}$ containing 1. There exists a sequence n_r such that $M_{\mathcal{A}}(n_r) = o(p_{\mathcal{A}}(n_r)$.

2. Binary partitions

In this section, $\mathcal{A} = \{1, 2, 4, 8, \ldots\}$ is the set of powers of 2. We shall need the following results:

Proposition 1

Let $p(n) = p_{\mathcal{A}}(n)$ the number of binary partitions of n. For x real ≥ 0, let us set $p(x) = p(\lfloor x \rfloor)$. The following asymptotic expansion holds when x goes to infinity:

$$\log p(2x) = \frac{1}{2\log 2}\left(\log\left(\frac{x}{\log x}\right)\right)^2 + \left(\frac{1}{2} + \frac{1}{\log 2} + \frac{\log\log 2}{\log 2}\right)\log x - \left(1 + \frac{\log\log 2}{\log 2}\right)\log\log x + O(1)\,. \tag{9}$$

Moreover, there exist two positive real numbers α and β such that for all $x \geq 2$.

$$\frac{\alpha x}{\log x} \leq \frac{p(2x)}{p(x)} \leq \frac{\beta x}{\log x}\,. \tag{10}$$

Proof. When x is an integer, formula (10) has been proved by de Bruijn (cf. [2]) improving preceding results of Mahler (cf. [7]). It is easy to extend it whenever x is a positive real number.

One deduces from (9):

$$\log p(2x) - \log p(x) = \log x - \log\log x + O(1)\,,$$

which proves (10). □

Proof of Theorem 1. We shall start from (5)

$$M(n) = \sum_{a=1}^{n/2} \tilde{p}(a-1) r(n-a+1, a+1)\,. \tag{11}$$

Let us define $k = k(n)$ by (2):

$$2^k \leq n < 2^{k+1}$$

and the binary expansion of $n+1$:

$$n+1=\sum_{i=0}^{k+1} c_i 2^i, \quad c_i=0,1.$$

Note that c_{k+1} is always 0 unless $n+1=2^{k+1}$. It will be convenient to use, for $0 \le t \le k$:

$$N_t=\sum_{i=t+1}^{k+1} c_i 2^i, \quad n_t=\sum_{i=0}^{t} c_i 2^i,$$

so that $n+1=N_t+n_t$. We can rewrite (11) as:

$$\begin{aligned} M(n)=&\sum_{t=0}^{k-2} \sum_{2^t \le a<2^{t+1}} \tilde{p}(a-1) r(n-a+1, 2^{t+1}) \\ &+\sum_{2^{k-1} \le a \le n/2} \tilde{p}(a-1) r(n-a+1, 2^k). \end{aligned} \tag{12}$$

Now, $r(n, 2^t)$ is 0 unless n is a multiple of 2^t where it is $p(n/2^t)$, so, in (12) in order that $r(n-a+1, 2^{t+1})$ does not vanish, we must have

$$a \equiv n+1 \equiv n_t \mod 2^{t+1}$$

and a will be $\ge 2^t$ if and only if $c_t=1$. In the last term of (12), we should have

$$a \equiv n_{k-1} \bmod 2^k$$

and c_{k-1} must be 1 to get $a \ge 2^{k-1}$. On the other hand, $n_{k-1}<2^k \le N_{k-1}$, so that $2n_{k-1}<n_{k-1}+N_{k-1}=n+1$, and

$$n_{k-1} \le n/2\,, \tag{13}$$

so (12) can be rewritten as

$$M(n)=\sum_{t=0}^{k-1} c_t \tilde{p}(n_t-1) p(N_t/2^{t+1})\,. \tag{14}$$

From (14), it follows that whenever $n+1$ is a power of 2, all the $c_t's$ vanish, and $M(n)=0$. If $n+1$ is different of a power of 2, at least one of the c_t does not vanish, and since the partition of n made with n $1's$ is certainly practical, $\tilde{p}(n) \geq 1$ for all n, so that (14) yields $M(n) \neq 0$.

From (13) and (14), it follows that

$$M(n) \leq 2p(n/2) + \sum_{t=1}^{k-2} p(2^{t+1})p(n/2^{t+1})\,. \tag{15}$$

To estimate the above sum, it is convenient to cut it into three parts : from $t = 1$ to $t = t_0 = \frac{\log n}{3\log 2}$, from t_0 to $t_1 = 2t_0$, and from t_1 to $k-1$. Let us set $y_t = p(2^{t+1})p(n/2^{t+1})$. From (10) one has, for $1 \leq t \leq k-3$,

$$\frac{y_{t+1}}{y_t} \leq \frac{\beta 2^{t+1}\log(n/2^{t+2})}{(t+1)\log 2 \quad \alpha n/2^{t+2}} \ll \frac{2^{2t}\log n}{n} \tag{16}$$

and similarly, since $t \leq k = O(\log n)$:

$$\frac{y_{t+1}}{y_t} \gg \frac{2^{2t}}{n\log n}\,. \tag{17}$$

It follows from (16) that for $1 \leq t \leq t_0$,

$$\frac{y_{t+1}}{y_t} \ll \frac{2^{2t_0}\log n}{n} = (\log n)n^{-1/3}$$

and so, far n large enough, $y_{t+1}/y_t \leq 1/2$. Therefore,

$$S_1 = \sum_{t=1}^{t_0} y_t = O(y_1) = O(p(n/4)) = O\left(\frac{\log^2 n}{n^2}p(n)\right) \tag{18}$$

by (10).

In the same way, from (17), one deduces that

$$S_3 = \sum_{t_1 \leq t \leq k-2} y_t = O(y_{k-2}) = O(p(2^{k-1})) = O(p(n/2))\,. \tag{19}$$

It remains to estimate S_2. For $t_0 < t < t_1$, the number of terms is $O(\log n)$, and each term satisfies:

$$y_t \leq p(2^{t_1+1})p(n/2^{t_0+1}) \leq (p(2n^{2/3}))^2\,. \tag{20}$$

But, from (9),

$$\log p(2x) = \frac{1}{2\log 2}(1+o(1))\log^2 x \tag{21}$$

holds, so that, for n large enough

$$\left(p(2n^{2/3})\right)^2 \le \exp\left(\frac{1+o(1)}{2\log 2}\frac{8}{9}\log^2 n\right) \le p(2n)/n^3\,. \tag{22}$$

From (20), (22), and (10), one has

$$S_2 = \sum_{t_0<t<t_1} y_t \le \frac{\log n}{n^3}p(2n) = O(p(n)/n^2)\,,$$

which together with (15), (18), (19) and (10) completes the proof of Theorem 1. □

At the end of this paper, a table of $p(n)$ and $M(n)$ up to $n = 100$ will be found. In fact it has been calculated up to $n = 1000$, by using (11). As shown by formula (14), large values of $M(n)$ are obtained whenever c_0 or c_1 or c_2 are equal to 1, or on the other hand when c_{k-1} or c_{k-2} are 1. For the computation of $p(n)$, see [3].

3. Upper and lower bounds for partition functions

Proposition 2

Let $a_1, a_2, \ldots, a_k$, k positive real numbers, and $A_k = a_1 + a_2 + \ldots + a_k$. The number $N_k(z)$ of solutions of the inequality

$$x_1a_1 + x_2a_2 + \ldots + x_ka_k \le z$$

satisfies:

$$\frac{z^k}{k!a_1a_2\ldots a_k} \le N_k(z) \le \frac{(z+A_k)^k}{k!a_1a_2\ldots a_k} \tag{23}$$

and

$$N_k(z) \le \frac{z^k}{k!a_1a_2\ldots a_k}\exp\left(\frac{kA_k}{z}\right)\,. \tag{24}$$

Proof. This Proposition is a classical one. For instance, a proof of (23) can be found in [8], p. 401. The upper bound (24) can be deduced from (23) by

$$\left(1+\frac{A_k}{z}\right)^k = \exp\left(k\log\left(1+\frac{A_k}{z}\right)\right) \le \exp\left(\frac{kA_k}{z}\right)\,.\ \square \tag{25}$$

Corollary 1

Let $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_j < \ldots\}$ *be a subset of* $\mathbb{N}, n$ *a positive integer, and* $A_j = a_1 + a_2 + \ldots + a_j$*. Then the following inequality holds:*

$$\frac{n^{j-1}}{(j-1)!a_2 a_3 \ldots a_j} \leq p(n, a_j) \leq \frac{n^{j-1} \exp(jA_j/n)}{(j-1)!a_2 a_3 \ldots a_j} . \tag{26}$$

Proof. For $j = 1$, (26) in obvious. For $j \geq 2$, there is a one to one correspondence between the partitions of n in parts $a_1, a_2, \ldots, a_j$ and the solutions of the inequality:

$$x_2 a_2 + x_3 a_3 + \ldots + x_j a_j \leq n$$

and so, (26) follows from (23) and (24). □

Corollary 2

Let $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_k < \ldots\}$ *be a subset of* $\mathbb{N}$*,* n *a positive integer;* $k = k(n)$ *is defined by (2), and* $A_k = a_1 + a_2 + \ldots + a_k$*. For* j *such that* $0 \leq j \leq k-2$*, let us introduce the set* $\mathcal{R}(n)$ *of partitions in parts* $a_{j+1}, a_{j+2}, \ldots, a_k$ *of* $n, n-1, \ldots, n - a_{j+1} + 1$*, and*

$$R(n) = card\ \mathcal{R}(n) = \sum_{a=0}^{a_{j+1}-1} r(n-a, a_{j+1}).$$

Then the following inequality holds:

$$\frac{n^{k-j-1}}{(k-j-1)!a_{j+2}a_{j+3}\ldots a_k} \leq R(n) \leq \frac{n^{k-j-1}\exp(kA_k/n)}{(k-j-1)!a_{j+2}a_{j+3}\ldots a_k} . \tag{27}$$

Proof. To every solution of the inequality

$$x_{j+2}a_{j+2} + x_{j+3}a_{j+3} + \ldots + x_k a_k \leq n \tag{28}$$

with non negative integers x_i, one can associate exactly one integer $x_{j+1} \geq 0$ such that

$$n - a_{j+1} + 1 \leq \sum_{i=j+1}^{k} x_i a_i \leq n,$$

and so, there is a one to one correspondence between the solutions of (28) and $\mathcal{R}(n)$. Then, (27) follows from (23) and (24), by observing that $(k-j-1)A_{k-j-1} \leq kA_k$. □

4. Proof of Theorem 2

Let $\mathcal{A} = \{a_1 = 1 < a_2 < \ldots < a_k < \ldots\}$ and $a_k \geq k a_{k-1}$. One has

$$A_k = a_1 + a_2 + \ldots + a_k \leq a_k \left(1 + \frac{1}{k} + \frac{1}{k(k-1)} + \ldots + \frac{1}{k!}\right)$$

$$\leq a_k \left(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{k!}\right) < e a_k\,. \tag{29}$$

For a given $n, k = k(n)$ is defined by (2), and from $n \geq a_k \geq k!$ and Stirling's formula, one deduces

$$k = O(\log n / \log\log n). \tag{30}$$

Further, from

$$n \geq a_k \geq k a_{k-1} \geq k(k-1) a_{k-2} \geq \ldots \geq k(k-1)\ldots(j+1) a_j,$$

one has

$$a_j \leq \frac{n}{k(k-1)\ldots(j+1)}\,. \tag{31}$$

Now, we want to prove that if $m = \lfloor n/2 \rfloor$, and $k = k(n)$ is defined by (2), one has:

$$p(m) = O(p(n)/2^k)\,. \tag{32}$$

First, we shall suppose that

$$(k-2) a_k \leq n < a_{k+1}\,. \tag{33}$$

From (26), we get

$$p(m) = p(m, a_k) \leq \frac{m^{k-1}}{(k-1)!\, a_2 a_3 \ldots a_k} \exp\left(\frac{k A_k}{m}\right)$$

$$= O\left(\frac{n^{k-1}}{(k-1)!\, a_2 a_3 \ldots a_k} 2^{-k}\right) \tag{34}$$

since (29) and (33) yield $\exp(k A_k / m) = O(1)$.

On the other hand, (26) gives

$$p(n) = p(n, a_k) \geq \frac{n^{k-1}}{(k-1)!\, a_2 a_3 \ldots a_k} \tag{35}$$

which, with (34), proves (32) whenever (33) is satisfied.

Let us suppose now that

$$a_k \le n < (k-2)a_k \tag{36}$$

and let us define $t = \lfloor m/a_k \rfloor$. Clearly, for $k \ge 4$, t satisfies

$$0 \le t \le k-3\,, \tag{37}$$

and one has

$$\begin{aligned} p(m) &= \sum_{u=0}^{t} p(m-ua_k, a_{k-1}) \\ &\le 2p(m, a_{k-1}) + \sum_{u=1}^{t-1} p(m-ua_k, a_{k-1}) \end{aligned}$$

by (4). Further, by (26), we get:

$$p(m) \le \frac{m^{k-2}}{(k-2)!a_2a_3\dots a_{k-1}} \left[2\exp\left(\frac{(k-1)A_{k-1}}{m}\right) + \sum_{u=1}^{t-1}\left(1-\frac{ua_k}{m}\right)^{k-2}\exp\left(\frac{(k-1)A_{k-1}}{m-ua_k}\right)\right]. \tag{38}$$

But, for $0 \le u \le t-1$, one has from (29):

$$\frac{(k-1)A_{k-1}}{m-ua_k} \le \frac{kea_{k-1}}{a_k} \le e\,.$$

We also have, by (37):

$$\begin{aligned} \left(1-\frac{ua_k}{m}\right)^{k-2} &= \exp\left((k-2)\log\left(1-\frac{ua_k}{m}\right)\right) \le \exp\left(-(k-2)\frac{ua_k}{m}\right) \\ &\le \exp\left(-\frac{(k-2)u}{t+1}\right) \le \exp(-u). \end{aligned}$$

Then (38) becomes:

$$p(m) \le \frac{e^e m^{k-2}}{(k-2)!a_2a_3\dots a_{k-1}}\left[2+\sum_{u=1}^{t-1}\exp(-u)\right],$$

and so,

$$p(m) = O\left(\frac{n^{k-2}2^{-k}}{(k-2)!a_2a_3\dots a_{k-1}}\right). \tag{39}$$

On the other hand, by (26), one has

$$p(n) \geq p(n, a_{k-1}) \geq \frac{n^{k-2}}{(k-2)!a_2a_3\dots a_{k-1}},$$

which, with (39), completes the proof of (32).

Now, let us define $h = h(n)$ by

$$a_h \leq n/2 < a_{h+1}. \tag{40}$$

From (5), we have:

$$M(n) = \sum_{j=1}^{h-1} \sum_{a_j \leq a < a_{j+1}} \tilde{p}(a-1)r(n-a+1, a+1)$$
$$+ \sum_{a_h \leq a \leq n/2} \tilde{p}(a-1)r(n-a+1, a+1). \tag{41}$$

But, for $a_j \leq a < a_{j+1}$, one has, by (4):

$$r(n-a+1, a+1) = r(n-a+1, a_{j+1})$$

$$\tilde{p}(a-1) \leq p(a-1) \leq p(a) = p(a, a_j) \leq p(a_{j+1}, a_j)$$

and (41) becomes, with $m = \lfloor n/2 \rfloor$:

$$M(n) \leq \sum_{j=1}^{h-1} p(a_{j+1}, a_j) \sum_{a_j \leq a < a_{j+1}} r(n-a+1, a_{j+1})$$
$$+ p(m) \sum_{a_h \leq a \leq m} r(n-a+1, a_{h+1}). \tag{42}$$

We now have to consider two cases: First, let us suppose that $a_k = a_k(n)$ satisfies:

$$n/2 < a_k \leq n. \tag{43}$$

As there is at most one element of $\mathcal{A}$ between $n/2$ and n, one has from (40), $h = k-1$. In the last term of (42), $r(n-a+1, a_{h+1}) = r(n-a+1, a_k)$ vanishes for all a, unless

$n-a+1 = a_k$, where it is 1. Similarly, let us deal with the last term of the first sum in (42), corresponding to $j = h-1 = k-2$. We have $r(n-a+1, a_{j+1}) = r(n-a+1, a_{k-1})$, which is the number of partitions of $n-a+1$ in parts a_{k-1} or a_k. But, clearly, in such a partition, a_k can occur at most once (from (43)). If a_k does not appears, $n-a+1$ must be a multiple of a_{k-1}, and as $a < a_{j+1} = a_{k-1}$, this will happen at most once. In the same way, if a_k appears once, $n - a_k - a + 1$ must be a multiple of a_{k-1}.

In conclusion,

$$p(a_h, a_{h-1}) \sum_{a_{h-1} \le a < a_h} r(n-a+1, a_h) + p(m) \sum_{a_h \le a \le m} r(n-a+1, a_{h+1})$$
$$\le 2p(a_h, a_{h-1}) + p(m) \le 2p(a_h) + p(m) \le 3p(m)\,. \tag{44}$$

The second case, when (43) does not hold is easier. We then have $h = k$, the last term of (42) obviously vanishes, and $r(n-a+1, a_k) = 0$ except for when $n-a+1$ is a multiple of a_k, so that (44) still holds, even with $p(m)$ instead of $3p(m)$ on the right hand side.

We now have to deal with the first terms of (42). Let us set:

$$S_{h-2} = \sum_{j=1}^{h-2} p(a_{j+1}, a_j) \sum_{a_j \le a < a_{j+1}} r(n-a+1, a_{j+1})\,.$$

Observing that jA_j is increasing, one has from (27):

$$\sum_{a_j \le a < a_{j+1}} r(n-a+1, a_{j+1}) \le \sum_{a=0}^{a_{j+1}-1} r(n-a+1, a_{j+1}) = R(n+1)$$
$$\le \frac{(n+1)^{k-j-1} \exp(kA_k/(n+1))}{(k-j-1)!a_{j+2}a_{j+3}\dots a_k}\,. \tag{45}$$

Further, from (26) and (45), one gets:

$$S_{h-2} \le \sum_{j=1}^{h-2} \frac{a_{j+1}^{j-1} \exp(jA_j/a_{j+1})(n+1)^{k-j-1} \exp(kA_k/(n+1))}{(j-1)!a_2\dots a_j(k-j-1)!a_{j+2}\dots a_k}\,. \tag{46}$$

Noting that from (29) and (2), one has

$$jA_j/a_{j+1} \le e_j a_j/a_{j+1} \le e$$

$$kA_k/(n+1) \le kA_k/n \le kea_k/n \le ek\,,$$

$$(n+1)^{k-j-1} \le n^{k-j-1}(1+1/n)^k \le n^{k-j-1}(1+1/n)^n \le en^{k-j-1}\,,$$

and

$$\frac{(k-1)!}{(j-1)!(k-j-1)!} = (k-1)\binom{k-2}{j-1} \le k2^k\,,$$

(46) becomes

$$S_{h-2} \le \frac{e^{e+ke+1}k2^k}{(k-1)!a_2a_3\dots a_k}\sum_{j=1}^{h-2} a_{j+1}^j n^{k-j-1}\,. \tag{47}$$

Now, from (31), one has:

$$a_{j+1}^j \le \frac{n^j}{[k(k-1)\dots(j+2)]^j}\,. \tag{48}$$

The denominator of the right hand side of (48) is equal to $k!/2$ for $j=1$, to k^{k-2} for $j=k-2$, and for $2 \le j \le k-3$, it is the product of $j(k-j+1) \ge k$ factors all greater than $j+2$ and it is a multiple of $k, k-1, \dots, j+2$, so that it is greater than $k!$. Then it follows from (48) that

$$a_{j+1}^j \le \frac{2n^j}{k!}, 2 \le j \le h-2,\ k \text{ large enough} \tag{49}$$

and (47) and (26) yield:

$$S_{h-2} \le \frac{n^{k-1}}{(k-1)!a_2a_3\dots a_k}\frac{2e^{e+ke+1}k^2 2^k}{k!} \le \frac{(32)^k}{k!}p(n) \tag{50}$$

for k large enough.

In conclusion, (42), (44) and (50) give

$$M(n) \le \frac{(32)^k}{k!}p(n) + 3p(\lfloor n/2 \rfloor)$$

which together with (32) completes the proof of (6) and of Theorem 2. □

5. An example due to D. Hickerson proving theorem 3

Let us define the sequence ℓ_r by $\ell_0 = 0$, and for $r \geq 0$, by

$$\ell_{r+1} = \ell_r + 2^{\ell_r} f(\ell_r)\,. \tag{51}$$

Let

$$m_r = \ell_{r+1} - \ell_r \tag{52}$$

and

$$n_r = 2\ell_r m_r\,. \tag{53}$$

Let us define the set $\mathcal{A} = \{a_1, a_2, \ldots\}$ by:

$$a_{\ell_r+i} = n_r + i, r \geq 0, 1 \leq i \leq m_r\,. \tag{54}$$

We claim that $1 = a_1 < a_2 < \ldots$. First, $a_1 = a_{\ell_0+1} = n_0 + 1 = 1$. To see that $a_1 < a_2 < \ldots$ it suffices to show that $a_{\ell_r} < a_{\ell_r+1}$ for $r \geq 1$. But

$$\begin{aligned} a_{\ell_r} &= a_{\ell_{r-1}} + m_{r-1} = n_{r-1} + m_{r-1} = (2\ell_{r-1}+1)m_{r-1} \\ &\leq (2\ell_{r-1}+1)\ell_r < 2\ell_r^2 \end{aligned} \tag{55}$$

while

$$a_{\ell_r+1} = n_r + 1 > 2\ell_r m_r \geq 2\ell_r f(\ell_r) 2^{\ell_r} > 2\ell_r^2\,. \tag{56}$$

Note that, from (55) and (56), it follows that $n_r \geq 2^{\ell_r}$, which implies $(\log n_r)^2 \geq \ell_r^2 \log 2 \geq a_{\ell_r}(\log 2)/2$, and so,

$$a_{\ell_r} \leq \frac{2}{\log 2}(\log n_r)^2 \leq 3(\log n_r)^2\,. \tag{57}$$

We must prove

$$a_k = O(kf(k)2^k)\,. \tag{58}$$

Given k, let $\ell_r < k \leq \ell_{r+1}$. Note that

$$m_r = \ell_{r+1} - \ell_r < f(\ell_r)2^{\ell_r} + 1 = O(2^k f(k))\,.$$

Hence

$$\begin{aligned} a_k &= n_r + k - \ell_r \leq n_r + m_r = (2\ell_r+1)m_r \\ &= O(\ell_r m_r) = O(k2^k f(k))\,, \end{aligned}$$

as required.

Proof of (7). The above example and the proof of (7) below are due to D. Hickerson and we thank him very much for allowing us to include them in this paper. Let $r \geq 1$, and consider the partitions of $n = 2n_r$ with parts in $\mathcal{A}$. Let $C = p(n, a_{\ell_r})$ be the number of such partitions with all parts $\leq a_{\ell_r}$, and D be the number with at least one part $\geq a_{\ell_r+1}$. Thus $p(n) = C + D$. Also, every partition counted by D contains a part $\geq a_{\ell_r+1} = n_r + 1$, so the sum of the other parts is $\leq n_r - 1$. Hence, such a partition cannot represent n_r ; therefore, $\tilde{p}(n) \leq C$, so

$$\tilde{p}(n)/p(n) \leq C/(C+D) \leq C/D\,. \tag{59}$$

We now estimate C and D. By applying (26) with $j = \ell_r$, and $A_j \leq j a_j$,

$$C = p(n, a_{\ell_r}) \leq \frac{(2n_r)^{\ell_r-1}\exp(\ell_r^2 a_{\ell_r}/(2n_r))}{(\ell_r-1)!a_2a_3\dots a_{\ell_r}}\,.$$

For $1 \leq i \leq m_r$, the number of partitions counted by D which contain $a_{\ell_r+i} = n_r + i$ is $p(n_r - i, a_{\ell_r})$. Hence, by (26),

$$\begin{aligned} D &\geq \sum_{i=1}^{m_r} p(n_r - i, a_{\ell_r}) \geq \sum_{i=1}^{m_r} \frac{(n_r - i)^{\ell_r-1}}{(\ell_r-1)!a_2a_3\dots a_{\ell_r}} \\ &\geq \frac{m_r(n_r - m_r)^{\ell_r-1}}{(\ell_r-1)!a_2a_3\dots a_{\ell_r}}\,, \end{aligned}$$

so

$$\frac{C}{D} \leq \frac{2^{\ell_r-1}\exp(\ell_r^2 a_{\ell_r}/(2n_r))}{m_r(1 - m_r/n_r)^{\ell_r-1}}\,.$$

Note that

$$m_r = \ell_{r+1} - \ell_r \geq f(\ell_r)2^{\ell_r},$$

$$(1 - m_r/n_r)^{\ell_r-1} = (1 - 1/(2\ell_r))^{\ell_r-1} \geq 1 - \frac{\ell_r - 1}{2\ell_r} > \frac{1}{2}$$

and from (57)

$$\frac{\ell_r^2 a_{\ell_r}}{2n_r} \leq \frac{(a_{\ell_r})^3}{2n_r} \leq \frac{27}{2}\frac{(\log n_r)^6}{n_r}$$

Hence,

$$\frac{C}{D} \leq \frac{\exp(27(\log n_r)^6/(2n_r))}{f(\ell_r)}\,,$$

which, with (59), proves (7). □

Proof of (8). Here we shall choose $n = n_r$. Observe that $k = k(n)$ defined by (2) is equal to ℓ_r, and that, from (57), a_k is very much smaller than n:

$$k = \ell_r \le a_k \le 3(\log n)^2 \,, \tag{60}$$

and $a_{k+1} = n + 1$. This implies in particular that, for $m = \lfloor n/2 \rfloor$,

$$p(m) \le (1 + o(1))p(n)2^{-k+1} \tag{61}$$

since, $p(m) = p(m, a_k)$ and $p(n) = p(n, a_k)$, and in (26) $kA_k/n \le k^2 a_k/n$ tends to 0 because of (60).

The proof of (8) is very close to the proof of Theorem 2 in section 4. From (40) and (60), one gets $h = k$, so (44) holds with $p(m)$ instead of $3p(m)$ on the right hand side, and we have:

$$M(n) \le S_{k-2} + p(m) \,. \tag{62}$$

(45) is still valid, and from (60),

$$kA_k/n \le k^2 a_k/n \le a_k^3/n \le 27(\log n)^6/n$$

hence,

$$\exp(kA_k/n) = (1 + o(1)).$$

Now, in S_{k-2}, from (4), one has:

$$p(a_{j+1}, a_j) \le p(a_k, a_j) \le p(y, a_j)$$

with $y = \lfloor (\log n)^7 \rfloor$, and from (26)

$$p(a_{j+1}, a_j) \le \frac{y^{j-1} \exp(jA_j/y)}{(j-1)! a_2 a_3 \ldots a_j} \,.$$

As above,

$$jA_j/y \le j^2 a_j/y \le k^2 a_k/y = O(1/\log n) \,,$$

and therefore,

$$\begin{aligned} S_{k-2} &\le (1 + o(1)) \sum_{j=1}^{k-2} \frac{y^{j-1}(n+1)^{k-j-1} a_{j+1}}{(j-1)!(k-j-1)! a_2 a_3 \ldots a_k} \\ &\le \frac{(1+o(1)) a_k (k-1)}{(k-1)! a_2 a_3 \ldots a_k} \sum_{j=1}^{k-2} \binom{k-2}{j-1} y^{j-1} (n+1)^{k-j-1} \\ &\le \frac{(1+o(1)) 9(\log n)^4}{(k-1)! a_2 a_3 \ldots a_k} (n+y+1)^{k-2} \\ &\le \frac{n^{k-1}}{(k-1)! a_2 a_3 \ldots a_k} (1+o(1)) \frac{9(\log n)^4}{n} (1 + (1+y)/n)^{k-2} \,. \end{aligned}$$

But, by (26),

$$p(n) = p(n, a_k) \geq \frac{n^{k-1}}{(k-1)!a_2 a_3 \dots a_k},$$

$$\begin{aligned}(1+y/n)^{k-2} &\leq \exp((k-2)\log(1+(1+y)/n) \leq \exp(k(y+1)/n) \\ &\leq \exp(O(\log n)^9/n) = 1 + o(1)\,,\end{aligned}$$

so that $S_{k-2}/p(n)$ tends to 0, and by (61), (62) and (3), the proof of (8) is completed. □

Table of binary partitions

n	$p(n)$	$M(n)$	n	$p(n)$	$M(n)$
1	1	0	51	786	54
2	2	1	52	900	168
3	2	0	53	900	88
4	4	2	54	1014	202
5	4	1	55	1014	80
6	6	3	56	1154	220
7	6	0	57	1154	122
8	10	4	58	1294	262
9	10	2	59	1294	134
10	14	6	60	1460	300
11	14	2	61	1460	187
12	20	8	62	1626	353
13	20	5	63	1626	0
14	26	11	64	1828	202
15	26	0	65	1828	36
16	36	10	66	2030	238
17	36	4	67	2030	20
18	46	14	68	2268	258
19	46	4	69	2268	66
20	60	18	70	2506	304
21	60	10	71	2506	24
22	74	24	72	2790	308
23	74	6	73	2790	78
24	94	26	74	3074	362
25	94	14	75	3074	68
26	114	34	76	3404	398
27	114	16	77	3404	136
28	140	42	78	3734	466
29	140	27	79	3734	52
30	166	53	80	4124	442
31	166	0	81	4124	124
32	202	36	82	4514	514
33	202	10	83	4514	112
34	238	46	84	4964	562
35	238	8	85	4964	202
36	284	54	86	5414	652
37	284	22	87	5414	160
38	330	68	88	5938	684
39	330	12	89	5938	266
40	390	72	90	6462	790
41	390	30	91	6462	272
42	450	90	92	7060	870
43	450	32	93	7060	402
44	524	106	94	7658	1000
45	524	56	95	7658	166
46	598	130	96	8350	858
47	598	26	97	8350	286
48	692	120	98	9042	978
49	692	52	99	9042	270
50	786	146	100	9828	1056

References

1. P.T. Bateman and P. Erdös, Monotonicity of partition functions, *Mathematika* **3** (1956), 1–14.
2. N.G. de Bruijn, On Mahler's partition problem, Nederl, Akad. Wetensch. Proc. 51, 659–669 = *Indagationes Math.* **10** (1948), 210–220
3. R.F. Churchhouse, Binary partitions, Computers in number theory, edited by A.O.L. Atkin and B.J. Birch, Acad. Press, 1971, 397-400.
4. J. Dixmier et J.L. Nicolas, Partitions without small parts, Colloquia Mathematica Societatis János Bolyai 51, Number Theory, Budapest, Hungary, 1987, 9-33.
5. P. Erdös and M. Szalay, On some problems of J. Dénes and P. Turán, Studies in Pure Mathematics to the memory of P. Turán, Editor P. Erdös, Budapest 1983, 187-212.
6. H. Gupta, C.E. Gwyther and J.C.P. Miller, *Table of partitions,* Cambridge Univ. Press, 1962.
7. K. Mahler, On a special functional equation, *J. London Math. Soc.* **15** (1940), 115–123.
8. G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres,* Publication de l'Université de Nancy 1, $n^\circ 13$, Nancy, 1990.