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Length of ideals in lattices

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Dedicated to the memory of Professor M.-L. Dubreil-Jacotin

Abstract

Let I^k be the $k-$th meander of an ideal I in a lattice L with 0 and 1. Define m to be the smallest nonnegative integer such that $I^m = I^{m+2}$ if such a number exists; in this case we put $l(I) = m+1$; otherwise we set $l(I) = 0$. We show: (i) $l(I) = 1$ for any semiprime ideal I of a lattice satisfying the ascending chain condition (briefly (ACC)); (ii) $l(I) = 1$ for any ideal I of a distributive lattice satisfying the (ACC); (iii) $l(I) \leq 2$ for any ideal I of a modular lattice having no infinite chains; and (iv) given any nonnegative integer n, there exists an ideal I such that $l(I) = n$.

I. Introduction

A study of the relationship between ideals and the corresponding first meanders was undertaken by the author in [4], where, under appropriate conditions, it was shown that the semiprimeness or the primeness of an ideal determinates the same property for its first meander; and vice versa. In this paper, intended as a sequel to [4], we investigate in further detail the $n-$th meanders for $n \geq 2$. As in [4], the chain conditions play a particularly fruitful role.

Throughout this paper, we use the letter L to denote a lattice with 0 and 1.

We refer the reader to [5] and [2] for the basic definitions and results used in this paper. Some concepts from the author's paper [3] may also be useful.

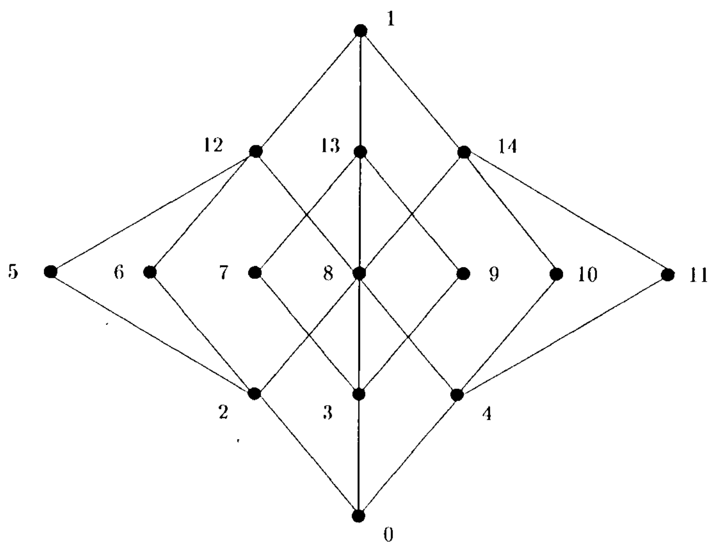

Figure 1

The key notion of the $n\!-\!th$ *meander,* due to the author [4], can be defined as follows: Let $I = I^0$ be an ideal of L. The set

$$I^1 = \{b \in L; \forall c \in L \ b \wedge c \in I \ \Rightarrow \ c \in I\}$$

is called the *first meander* (or simply *meander*) of I. Given a filter F of L, its first meander F^1 is defined dually. Then one can define recursively the *second meander* I^2 of I as the set $(I^1)^1$, the *third meander* I^3 as the set $(I^2)^1$ and so on.

If there exists a nonnegative integer k such that $I^k = I^{k+2}$ and if k_0 is the smallest number with respect to this property, then $k_0 + 1$ is called the *length* of I and we write $l(I) = k_0 + 1$. If there is no such k, we set $l(I) = 0$.

Given a filter F of L, its *length* $l(F)$ is defined dually.

EXAMPLES 1.1: Consider the lattice illustrated in Figure 1. It is the subgroup lattice of a group (cf. [6], the subgroup number 51 in the diagram 32.39) and it is modular.

We list in tabular form the first three meanders of typical ideals in this lattice (see Table 1). The corresponding lengths are given in the last column.

Note that $I^1 = I^3$ for any ideal I of this lattice as it also follows from our Main Theorem.

$I = I^0$	I^1	I^2	I^3	$l(I)$
$(0]$	$[8)$	$(0]$	$[8)$	1
$(1]$	$[0)$	$(1]$	$[0)$	1
$(2]$	$[12)$	$(2]$	$[12)$	1
$(3]$	$[13)$	$(3]$	$[13)$	1
$(5]$	$[12)$	$(2]$	$[12)$	2
$(6]$	$[12)$	$(2]$	$[12)$	2
$(7]$	$[13)$	$(3]$	$[13)$	2
$(8]$	$[1)$	$(8]$	$[1)$	1
$(12]$	$[1)$	$(8]$	$[1)$	2
$(13]$	$[1)$	$(8]$	$[1)$	2

Table 1

2. Second meanders

Recall that an ideal I of a lattice T is called *semiprime* [7] if

$$(x \wedge y \in I \quad \& \quad x \wedge z \in I) \ \Rightarrow \ x \wedge (y \vee z) \in I$$

for any $x, y, z \in T$.

In this section we first prove the following technical lemmas.

Lemma 2.1

If I is a semiprime ideal of a lattice L, then $I \subset I^2$.

Proof. Let $i \in I$. We want to prove that $i \in I^2$, i.e.,

$$\forall x \in L \ i \vee x \in I^1 \ \Rightarrow \ x \in I^1.$$

Were this false, then there would exist x such that

$$i \vee x \in I^1 \quad \& \quad x \notin I^1.$$

Consequently, there exists y for which

$$x \wedge y \in I \quad \& \quad y \notin I.$$

Now, by Lemma 2.1 of [3], $(i \vee (x \wedge y), (i \vee x) \wedge (i \vee y)) \in \hat{C}(L)$. Thus by the definition of the congruence $\hat{C}(L)$ and by [4, Proposition 1.1 (ii)], $(i \vee x) \wedge (i \vee y) \in I$.

Since $i \vee x \in I^1$, the definition of the meander I^1 implies that $i \vee y \in I$. Therefore, $y \in I$, a contradiction. $\square$

It is understood that the remark on the use of the maximal condition made in [4] applies to the proof of Lemma 2.2 and to that of Theorem 3.1.

Lemma 2.2

If I is a semiprime ideal of a lattice L satisfying the (ACC), *then* $I^2 \subset I$.

Proof. We may assume with no loss of generality that $I \neq L$.

Choose $b \in I^2$, i.e.

$$\forall x \in L \; b \vee x \in I^1 \;\Rightarrow\; x \in I^1. \tag{2.1}$$

We claim that $b \in I$. Were this false, we would have $b \notin I^1$ by Lemma 2.1 of [4] and its dual. Consequently, there exists x_1 such that

$$b \wedge x_1 \in I \;\;\&\;\; x_1 \notin I.$$

By the definition of the meander I^1, $x_1 \notin I^1$.

We now construct a chain in L recursively, as follows. Let $n \in \mathbb{N}$. Suppose $x_1, x_2, \cdots, x_n$ are such that

$$x_1 < x_2 < \cdots < x_n; \tag{2.2}$$

$$\forall i = 1, 2, \cdots, n \;\; x_i \notin I^1; \tag{2.3}$$

$$\forall i = 1, 2, \cdots, n \;\; b \wedge x_i \in I \;\;\&\;\; x_i \notin I. \tag{2.4}$$

We shall show that there is x_{n+1} for which

$$x_n < x_{n+1}; \tag{2.5}$$

$$x_{n+1} \notin I^1; \tag{2.6}$$

$$b \wedge x_{n+1} \in I \;\;\&\;\; x_{n+1} \notin I. \tag{2.7}$$

First, one observes that $b \vee x_n \notin I^1$.

Indeed, if $b \vee x_n$ belongs to I^1, then $x_n \in I^1$ by (2.1), yet by (2.3) this is impossible.

Therefore there exists y_n such that

$$(b \vee x_n) \wedge y_n \in I \ \ \& \ \ y_n \notin I. \tag{2.8}$$

Let $x_{n+1} := x_n \vee y_n$. Suppose $x_{n+1} = x_n$. Then $x_n \geq y_n$ and (2.8) yields $y_n = (b \vee x_n) \wedge y_n \in I$, a contradiction. Hence (2.5) is proved.

By (2.4), $b \wedge x_n \in I$, and, by (2.8), $b \wedge y_n \in I$. Since I is semiprime,

$$b \wedge x_{n+1} = b \wedge (x_n \vee y_n) \in I. \tag{2.9}$$

Suppose $x_{n+1} \in I$. Then $x_n \leq x_{n+1}$ would imply $x_n \in I$, contradicting (2.4). Thus, $x_{n+1} \notin I^1$ and this proves (2.7).

Assume $x_{n+1} \in I^1$. Then, by (2.9), $b \in I$, a contradiction. Consequently, (2.6) holds.

Hence, recursively, we could construct an infinite chain $x_1 < x_2 < \cdots$, contrary to hypothesis. This proves Lemma 2.2. □

The following theorems summarize the preceding lemmas.

Theorem 2.3

Let L be a lattice satisfying the (ACC)*. Then $I^2 = I$ for every semiprime ideal I of L.*

Theorem 2.4

Let L be a distributive lattice satisfying the (ACC)*. Then $I^2 = I$ for every ideal I of L.*

In other words, the length of every ideal in L is equal to 1.

We conclude this section with one example.

Examples 2.5: Both Theorem 2.3 and Theorem 2.4 are false without the assumption that L satisfies the (ACC). This shows the distributive lattice $L = (\mathbb{N}_0, |)$, where $\mathbb{N}_0$ denotes the set of all the nonnegative integers and "$|$" is the usual divisibility relation on $\mathbb{N}_0$. The set

$$I = \{m \in \mathbb{N}; \exists e \in \mathbb{N}_0 \ \ m = 2^e\}$$

is an ideal of L for which $I^1 = \{0\}$ and $I^2 = \mathbb{N} \neq I$.

3. Meanders in modular lattices

We first prove the following theorem which plays a crucial role in the proof of our Main Theorem.

Theorem 3.1

Let I be an ideal of a modular lattice L satisfying the (ACC)*. Then $I^2 \subset I$.*

Proof. Suppose by way of contradiction that there exists w such that

$$w \in I^2 \quad \& \quad w \notin I. \tag{3.1}$$

Note that $w \notin I^1$. To prove this assertion it suffices to observe that $w \in I^1$ would imply $w \in I^1 \cap I^2$. By the dual of Lemma 2.1 in [4], $I^1 = L$, and so $I \cap I^1 = I \neq \emptyset$. According to the same Lemma, $I = L$ which is absurd since $w \notin I$.

Set $I = (i_0]$ and let $w_0 := w \vee i_0$, $z_0 := i_0$. Then $w_0 \notin I^1$. Were this false, we would have

$$w_0 = w \vee i_0 \in I^1 \quad \& \quad w \in I^2.$$

By the definition of I^2, $i_0 \in I \cap I^1$. From the same Lemma as above we conclude that $I = L$. This contradicts (3.1).

We assume by induction that n is a nonnegative integer and that there are elements $z_0, z_1, \cdots, z_n$ such that the elements z_i and $w_j := w \vee \bigvee_{i=0}^{j} z_i$ satisfy

$$z_0 < z_1 < \cdots < z_n; \tag{3.2}$$

$$w_j/zj \searrow w_0/z_0 \tag{3.3}$$

for every $j = 1, 2, \cdots, n$;

$$w_j \notin I^1 \tag{3.4}$$

for every $j = 0, 1, \cdots, n$.

We shall prove that there exists z_{n+1} such that

$$z_n < z_{n+1}; \tag{3.5}$$

$$w_{n+1}/z_{n+1} \searrow w_0/z_0; \tag{3.6}$$

$$w_{n+1} \notin I^1. \tag{3.7}$$

In fact, by (3.4) $w_n \notin I^1$. Hence there exists y_{n+1} for which

$$w_n \wedge y_{n+1} \in I \quad \& \quad y_{n+1} \notin I. \tag{3.8}$$

Set $z_{n+1} := z_n \vee y_{n+1}$ and $w_{n+1} := w_n \vee z_{n+1}$.

Clearly z_{n+1} cannot equal to z_n; otherwise we would have $z_n \geq y_{n+1}$ and (3.8) would imply

$$I \ni w_n \wedge y_{n+1} = (w \vee z_0 \vee \cdots \vee z_n) \wedge y_{n+1} = y_{n+1}.$$

This contradicts (3.8) and so (3.5) holds.

Further, by modularity and since $z_n \leq w_n$ it follows that

$$w_n \wedge z_{n+1} = w_n \wedge (z_n \vee y_{n+1}) = z_n \vee (w_n \wedge y_{n+1}).$$

From (3.8) we see that $w_n \wedge y_{n+1} \leq i_0 = z_0 \leq z_n$. Hence $w_n \wedge z_{n+1} = z_n$ and $w_{n+1}/z_{n+1} \searrow w_n/z_n$. Applying (3.3), we have $w_{n+1}/z_{n+1} \searrow w_0/z_0$. This establishes (3.6).

Finally, we claim that $w_{n+1} \notin I^1$. For if this were not the case then I^1 would contain $w \vee z_0 \vee z_1 \vee \cdots \vee z_n \vee z_{n+1}$. Putting together (3.2) and (3.5), we obtain $w \vee z_{n+1} \in I^1$. Since $w \in I^2$, we get from the definition of I^2 that $z_{n+1} \in I^1$. By (3.6), $w_0 \wedge z_{n+1} = z_0 \in I$. Therefore, by the definition of I^1, $w_0 = w \vee i_0 \in I$. Now I is an ideal, and so we would have $w \in I$, contradicting (3.1). Hence (3.7) is true.

In summary, we obtain an infinite sequence $z_1 < z_2 < \cdots$, contradicting the (ACC). Thus, our proof has been completed. □

Remark 3.2. The ideal I of the lattice mentioned in Remark 2.10.B of [4] is such that $I \subsetneqq L \setminus \{1\} = I^2$. Hence the hypothesis of the (ACC) in Theorem 3.1 is essential.

Definition 3.3. (cf. [1]): Let I be an ideal of a lattice T. An element $a \in T$ is said to be an $I-$*atom* if

(i) $a \notin I$

and if

(ii) for all $m \in T, \quad m < a \;\Rightarrow\; m \in I.$

Dually is defined an $F-$*atom* for a given filter F of T.

To determine the first meander of an ideal is not always easy to practise. However, for the case where L has no infinite chains we have the following useful result:

Theorem 3.4

Let I be an ideal of a lattice L which has no infinite chains and let i_1 be the join of all the $I-$atoms. Then $I^1 = [i_1)$.

Proof. If $I = L$, then the join is equal to 0 and the assertion holds.

Suppose $I \neq L$ and set $H := [i_1)$.

First, we show that for any $h \in H$ and any $x \in L$ such that $h \wedge x \in I$ we necessarily have $x \in I$.

Indeed, if $x \notin I$, then there would exist an $I-$atom x_0 such that $x_0 \leq x$. Since x_0 is an $I-$atom, $x_0 \notin I$. On the other hand, we have $x_0 \leq i_1 \leq h$ and so $x_0 = h \wedge x_0 \leq h \wedge x \in I$ which gives $x_0 \in I$, a contradiction. Consequently, $H \subset I^1$.

Conversely, let $y \in I^1$. We want to show that $y \in H$, i.e. that $a \leq y$ for every $I-$atom a. If $a \wedge y < a$, then $a \wedge y \in I$, since a is an $I-$atom. Notice that $y \in I^1$ implies $a \in I$, a contradiction. Hence $a = a \wedge y \leq y$. $\square$

The following lemma (whose trivial proof is omitted) can be proved.

Lemma 3.5

Let $I = (i_0]$ be a principal ideal of a lattice T and let a be an $I-$atom of T. Then the element $a_0 := a \wedge i_0$ is the greatest element which is contained in a.

With the above definition of an $I-$atom in hand we are prepared to prove the main theorem of this paper.

Theorem 3.6 (Main theorem)

Let L be a modular lattice which has no infinite chains. Then $I^1 = I^3$ for every ideal I of L.

In other words, there is no ideal in L having its length greater than 2.

Proof. By the dual of Theorem 3.1, $I^3 \subset I^1$.

Set $I^1 = [i_1)$. To finish the proof, it remains to show that $I^1 \subset I^3$, or more precisely, that $i_1 \in I^3$.

Were this false, then there would exist $t \in L$ such that

$$i_1 \wedge t \in I^2 \ \ \& \ \ t \notin I^2.$$

Since $t \notin I^2$, there exists an I^2-atom t_1 such that $t_1 \leq t$. Clearly, $i_1 \wedge t_1 \in I^2$, and, by Theorem 3.1, also $i_1 \wedge t_1 \in I$. Then, in view of $i_1 \in I^1$, $t_1 \in I$. Hence, in summary

$$i_1 \wedge t_1 \in I^2 \ \ \& \ \ t_1 \notin I^2 \ \ \& \ \ t_1 \in I. \tag{3.9}$$

Set $I^2 = (i_2]$ and let $t_0 := i_2 \wedge t_1$.

Since $t_1 \notin I^2$, there exists an element z such that

$$t_1 \vee z \in I^1 \ \ \& \ \ z \notin I^1.$$

Let z_1 be an I^1-atom for which $z \leq z_1$. As an immediate consequence of the fact that I^1 is a filter we have

$$t_1 \vee z_1 \in I^1 \ \ \& \ \ z_1 \notin I^1. \tag{3.10}$$

From $z_1 \notin I^1$ we see that there exists an element w for which

$$z_1 \wedge w \in I \ \ \& \ \ w \notin I.$$

Let w_1 be an $I-$atom such that $w_1 \leq w$. Since I is an ideal, we obtain

$$z_1 \wedge w_1 \in I \ \ \& \ \ w_1 \notin I. \tag{3.11}$$

By Theorem 3.4, $w_1 \leq i_1$ so that $w_1 \leq i_1 \vee t_0$. Now, $t_0 \leq t_1$ and L is modular. Therefore,

$$\begin{aligned} \xi :&= \big[z_1 \wedge (w_1 \vee t_1)\big] \wedge (i_1 \vee t_0) = z_1 \wedge \big\{w_1 \vee [t_1 \wedge (i_1 \vee t_0)]\big\} \\ &= z_1 \wedge \big[w_1 \vee t_0 \vee (t_1 \wedge i_1)\big]. \end{aligned}$$

By (3.9), $i_1 \wedge t_1 \in I^2$ and, at the same time $i_1 \wedge t_1 \leq t_1$ where t_1 is an I^2-atom. Using Lemma 3.5, we get $i_1 \wedge t_1 \leq t_0$. This implies that $\xi = z_1 \wedge (w_1 \vee t_0)$. By the dual of Theorem 3.4 (with I in the Theorem equal to the filter I^1), $i_2 \leq z_1$, and, a fortiori, $t_0 \leq z_1$. Since L is modular, ξ can be rewritten in the form $\xi = (z_1 \wedge w_1) \vee t_0$. Moreover, by (3.11), $z_1 \wedge w_1 \in I$ and we know that $I^2 \subset I$. Hence $t_0 \leq i_2 \leq i_0 \in I$ and so $t_0 \in I$. Thus $\xi \in I$. Now i_1 belongs to the filter I^1 and $i_1 \leq i_1 \vee t_0$. Therefore,

$$\xi = [z_1 \wedge (w_1 \vee t_1)] \wedge (i_1 \vee t_0) \in I \ \ \& \ \ i_1 \vee t_0 \in I^1.$$

Thus, by the definition of I^1, we get

$$z_1 \wedge (w_1 \vee t_1) \in I. \tag{3.12}$$

Set $\eta := t_1 \vee [z_1 \wedge (w_1 \vee t_1)]$. In view of (3.9) and (3.12) it is easily seen that $\eta \in I$. By modularity, $\eta = (t_1 \vee z_1) \wedge (w_1 \vee t_1)$. Here z_1 is an I^1-atom and, from (3.10) we have $z_1 \leq t_1 \vee z_1 \in I^1$. Let $z_0 := i_1 \vee z_1$. By the dual of Lemma 3.5, $t_1 \vee z_1 \geq z_0$. Therefore $\eta \geq z_0 \wedge w_1$. It follows immediately from Theorem 3.4 that

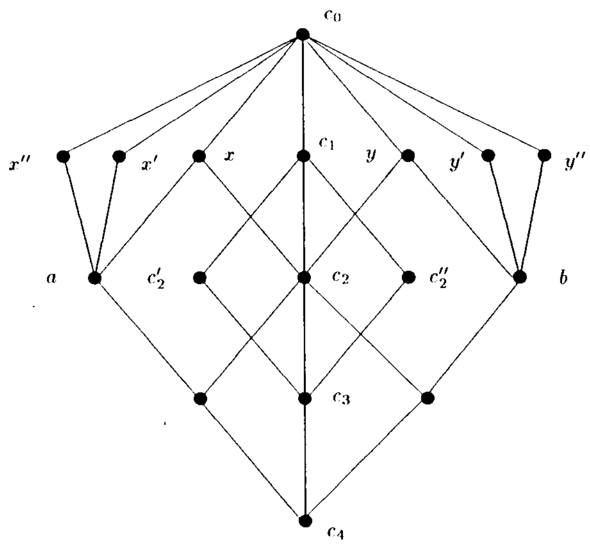

Figure 2

$w_1 \leq i_1 \leq z_0$; hence $\eta \geq w_1$. Since $\eta \in I$ and I is an ideal, we get $w_1 \in I$. This contradicts (3.11) and completes the proof. $\square$

4. Examples of ideals with a given length

In this section we give an explicit construction which makes clear the fact that for every nonnegative integer n there exists an ideal I such that $l(I) = n$.

We start with two lattices which will pave the way.

Let L_1 be the lattice shown in Figure 2 and let $I = (c_3]$. From Theorem 3.4 it follows that $I^1 = [c_1)$, $I^2 = (c_4]$, $I^3 = [c_2)$, $I^4 = I^2$. Thus $l(I) = 3$ and $l(I^1) = 2$.

To simplify the following diagrams, we shall use a more concise way to denote the groups of elements such as a, x, x', x'', c_0 or b, y, y', y'', c_0 or $c_3, c_2, c_2', c_3'', c_1$ as it is depicted in Figure 3. With this convention Diagram in Figure 3 represents the same lattice as Figure 2.

Now let L_2 be the lattice of Figure 4. Then L_2 contains L_1 as a sublattice and the ideal $I = (c_3]$ is such that $I^1 = [c_1)$, $I^2 = (c_4]$, $I^3 = [c_2)$, $I_4 = (c_5]$, $I_5 = [c_4)$, $I_6 = I_4$. Hence $l(I) = 5$ and $l(I^1) = 4$.

It is possible to proceed recursively along these lines, to obtain a more general result:

Indeed, if L_n has been defined, let L_{n+1} be constructed in the way indicated in Figure 5.

A glance at Figure 5 shows that $L_1 \subset L_2 \subset \cdots \subset L_n \subset L_{n+1}$ and that for the ideal $I = (c_3]$ of L_{n+1} we have $l(I) = 2n + 3$ and $l(I^1) = 2n + 2$.

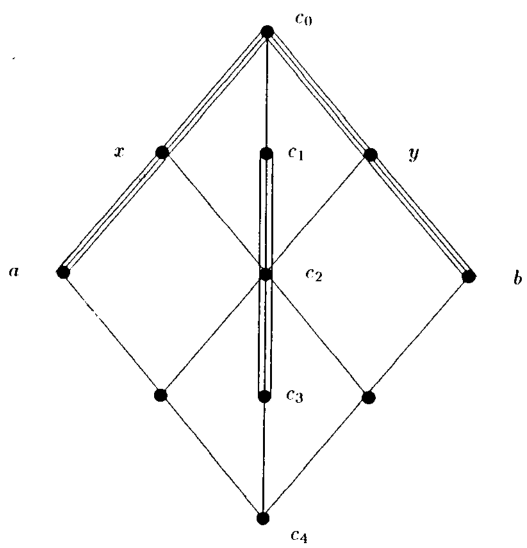

Figure 3

We end this section with the following consequence of the preceding examples: Let $S_\infty := \bigcup\{L_n; n \in \mathbb{N}\}$ and let L^* be the lattice we get from the lattice S_∞ by adjoining a zero 0, and a unit 1 to S_∞.

Then the ideal $I = (c_3]$ of L^* satisfies $l(I) = 0$.

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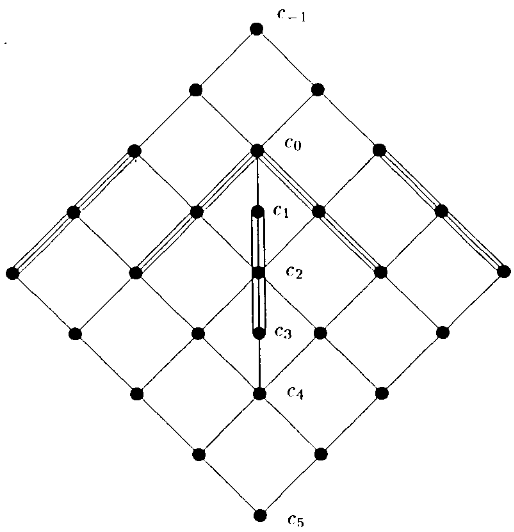

Figure 4

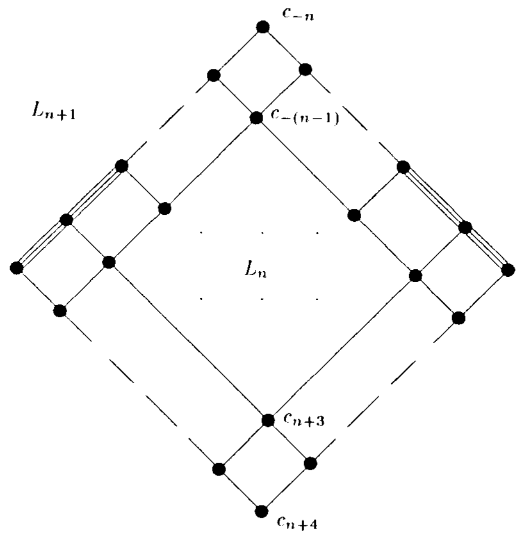

Figure 5

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