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Meanders in lattices

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Dedicated to the memory of Professor P. Dubreil

Abstract

Let I be an ideal of a lattice L where $0, 1 \in L$. Let $I^0 := I$ and let the $n-$th meander I^n of I be defined recursively by $I^n = \{w \in L; \forall t \in L\ w \oplus t \in I^{n-1} \;\Rightarrow\; t \in I^{n-1}\}$ where $w \oplus t$ denotes $w \wedge t$ for n odd and $w \oplus t = w \vee t$ for n even. Dually is defined the $n-$th meander F^n of a filter F. In this paper we study the meanders of semiprime and prime ideals and we establish some basic properties of meanders.

I. Introduction

Throughout this note L will denote a lattice having a 0 and a 1.

We assume that the reader is familiar with the standard results and terminology of lattice theory as presented in the book [6] and we refer him to [2] for any unexplained notation.

Let I be an ideal of a lattice L. The set

$$I^1 := \{b \in L; \forall c \in L\ b \wedge c \in I \;\Rightarrow\; c \in I\}$$

will be called the *meander* of I. Given a filter F of L, the *meander* F^1 of F is defined dually by

$$F^1 := \{d \in L; \forall e \in L\ d \vee e \in F \;\Rightarrow\; e \in F\}.$$

Recall (see [9], for instance) that an ideal B of a ring R is said to be *essential* in R if $B \cap C \neq 0$ for any ideal $C \neq 0$ of R. Hence B is essential if and only if it belongs to the meander $(0]^1$ of the ideal $(0]$ in the lattice $Id(R)$ of all the ideals in R.

An element a of a lattice L is said to be *small* (see [8]) if $a \vee c \neq 1$ for any $c \neq 1$. In our terminology is a small if and only if it belongs to the meander $[1)^1$ of the filter $[1)$.

Let I be an ideal of a lattice L and let $K \subset L \setminus I$. An element t is called *K-essential* with respect to I (see [1]) if it belongs to the set

$$\mathrm{Ess}_L^I(K) = \{t \in L \setminus I; \forall k \in K \ t \wedge k \notin I\}.$$

An element t is therefore $K-$essential for $K = L \setminus I$ with respect to I if and only if it belongs to the meander I^1 of the ideal I.

Other examples of meanders can be found in lattices of subalgebras or in lattices of special subalgebras of an algebra.

The following notions and results will form the basis for our proofs so, for the reader's convenience, we list them here in detail.

If $a \leq b$ are elements of a lattice T, we denote the ordered couple (a, b) by b/a, calling it a *quotient* of T. The set of all such quotients will be denoted by $Q(T)$.

If b/a and d/c belong to $Q(T)$ and are such that

$$b \wedge c = a \ \ \& \ \ b \vee c \leq d,$$

we write $b/a \nearrow^w d/c$. The special case where

$$b \wedge c = a \ \ \& \ \ b \vee c = d$$

is denoted by $b/a \nearrow d/c$. The binary relations $\searrow_w$ and $\searrow$ are defined on $Q(T)$ dually.

If either $b/a \nearrow^w d/c$ or $b/a \searrow_w d/c$, we write $b/a \sim_w d/c$.

A quotient b/a is said to be an *allele* of T, provided there exist quotients $b_i/a_i \in Q(T)$ $(i = 0, 1, \cdots, n)$ and $d/c \in Q(T)$ such that

$$b/a = b_0/a_0 \sim_w b_1/a_1 \sim_w \cdots \sim_w b_n/a_n = d/c$$

and such that either $b \leq c$ or $d \leq a$. Let $A(T)$ denote the set of all alleles in T and let $\hat{C}(T)$ be a binary relation defined on T in the following way: The couple (a, b) belongs to $\hat{C}(T)$ if and only if there exist $b_{i+1}/b_i \in A(T)$ $(i = 0, 1, \cdots, n)$ satisfying

$$a \wedge b = b_0 \leq b_1 \leq \cdots \leq b_{n+1} = a \vee b.$$

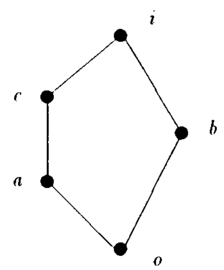

Figure 1

For example, in the pentagon N_5 shown in Figure 1 we see that

$$c/a \searrow_w c/o \nearrow i/b \searrow a/o$$

and so $c/a \in A(N_5)$.

With this notation in mind we now can formulate the following proposition.

Proposition 1.1

Let T be a lattice. Then

(i) [2, Theorem VI.1.6] *the binary relation $\hat{C}(T)$ is a congruence of T;*

(ii) [3, Main Theorem] *an ideal I of T is semiprime* (see the definition 2.3) *if and only if there is no $j/i \in A(T)$ with $i \in I$ and $j \notin I$;*

(iii) [4, Corollary 2 (i)] *a principal ideal $I = (i_0]$ of T is semiprime if and only if there is no $j_0/i_0 \in A(T)$ satisfying $j_0 \notin I$.*

2. Meanders and semiprimeness

It is easy to verify the next lemma, which will play an essential role in the following.

Lemma 2.1

(i) *If I is an ideal of a lattice L, then I^1 is a filter of L.*

(ii) *If I is an ideal of L such that $I \neq L$, then $I \cap I^1 = \emptyset$.*

DEFINITION 2.2. Let H be an ideal or a filter of L. Put $H^0 := H$ and let H^n be defined recursively by $H^n = (H^{n-1})^1$. The set H^n will be called the *n–th meander* of H.

The following definition is due to Y. Rav [7].

DEFINITION 2.3. An ideal I of a lattice T is called *semiprime* if the implication

$$(a \wedge b \in I \ \ \& \ \ a \wedge c \in I) \ \Rightarrow \ a \wedge (b \vee c) \in I$$

is true for any $a, b, c \in T$.

A *semiprime filter* of L is defined dually.

DEFINITION 2.4. An ideal I of L is said to be *semiprime of degree n* if its n-th meander I^n is semiprime.

Our first theorem in this section shows that the meanders preserve semiprimeness.

Theorem 2.5

Let I be a semiprime ideal of L. Then the meander I^1 is a semiprime filter of L.

Proof. Assume that I^1 is not semiprime. By the dual of Proposition 1.1 (i) there exists an allele j_0/h_0 such that $j_0 \in I^1$ and $h_0 \notin I^1$. Therefore, there is $x \in L$ such that

$$h_0 \wedge x \in I \ \ \& \ \ x \notin I.$$

From $j_0 \in I^1$ we see that $x \wedge j_0 \notin I$. Now $x \wedge j_0 / x \wedge h_0 \nearrow^w j_0/h_0$. Since $(h_0, j_0) \in \hat{C}(L)$ and since $\hat{C}(L)$ is a congruence of L, we conclude that $(x \wedge h_0, x \wedge j_0) \in \hat{C}(L)$. Consequently, there are elements $a_0, a_1, \cdots, a_n \in L$ satisfying

$$I \ni x \wedge h_0 = a_0 \leq a_1 \leq \cdots \leq a_n = x \wedge j_0 \notin I,$$

a_{j+1}/a_j being an allele of L for every $j = 0, 1, \cdots, n-1$. Hence there exists i such that $a_i \in I$ and $a_{i+1} \notin I$. By Proposition quoted above, the ideal I is not semiprime. □

Theorem 2.5 has the following immediate

Corollary 2.6

Let I be a semiprime ideal of L. Then I is semiprime of degree n for every $n \in \mathbb{N}$.

We briefly review notions that will frequently be used:
DEFINITION 2.7. ([5], p.38). A lattice T satisfies the ascending chain condition, (ACC), if it contains no infinite ascending chain $x_1 < x_2 < \cdots < x_n < \cdots$.

A lattice T *has no infinite chains* if every chain in T is finite.

It is a well-known fact, see [5, 2.26 Lemma], that under the assumption of the Axiom of Choice an ordered set P satisfies the (ACC) if and only if every non-empty subset of P has a maximal element (the *maximal condition,* (MC)).

While the proofs of Theorems 2.8 and 3.2 given here formally rely on the (ACC), it should be pointed out that the reader may prefer the (MC) to make some steps in the proofs more transparent.

We have the following observation about the semiprime meanders.

Theorem 2.8

Let I be an ideal of a modular lattice satisfying the (ACC). *If the meander I^1 is semiprime, then I is also semiprime.*

Proof. Without loss of generality, we may assume that $L \neq I = (i_0]$. Hence, by Lemma 2.1 (ii), $i_0 \notin I^1$.

Suppose that I is not semiprime. By Proposition 1.1 (iii), there exists j_0 such j_0/i_0 is an allele and $j_0 \notin I$.

We shall consider two cases.

Case I: $j_0 \in I^1$. Then I^1 is not semiprime by the dual of Proposition 1.1 (ii), a contradiction.

Case II: $j_0 \notin I^1$. Put $d_0 := i_0$, $s_0 := j_0$, $x_0 := i_0$ and continue recursively: Suppose that $n \geq 0$ and that $x_0, x_1, \cdots, x_n$ have been defined in such a way that the elements

$$d_q := \bigvee_{i=0}^{q} x_i, s_q := d_q \vee j_0$$

$(q = 0, 1, \cdots, n)$ satisfy

$$\forall q = 0, 1, \cdots, n-1 \;\; s_q/d_q \nearrow s_{q+1}/d_{q+1}; \tag{2.1}$$

$$\forall q = 0, 1, \cdots, n \;\; (d_q, s_q) \in \hat{C}(L); \tag{2.2}$$

$$\forall q = 0, 1, \cdots, n \;\; d_q < s_q; \tag{2.3}$$

$$s_0 < s_1 < \cdots < s_n; \tag{2.4}$$

$$\forall q = 0, 1, \cdots, n \quad d_q \notin I^1. \tag{2.5}$$

By (2.2), $(d_n, s_n) \in \hat{C}(L)$, and by (2.5), $d_n \notin I^1$. It follows, by the dual of Proposition 1.1 (ii), that

$$s_n \notin I^1. \tag{2.6}$$

Hence there exists y_{n+1} such that

$$s_n \wedge y_{n+1} \in I \quad \& \quad y_{n+1} \notin I. \tag{2.7}$$

Let $x_{n+1} := y_{n+1} \vee i_0$, $s_{n+1} := s_n \vee x_{n+1}$ and $d_{n+1} := d_n \vee x_{n+1}$. We need to show that

$$s_n/d_n \nearrow s_{n+1}/d_{n+1}; \tag{2.8}$$

$$(d_{n+1}, s_{n+1}) \in \hat{C}(L); \tag{2.9}$$

$$d_{n+1} < s_{n+1}; \tag{2.10}$$

$$s_n < s_{n+1}; \tag{2.11}$$

$$d_{n+1} \notin I^1. \tag{2.12}$$

Indeed, by (2.7), $s_n \wedge y_{n+1} \leq i_0$. By modularity and since $s_n \geq j_0 \geq i_0$ it follows that

$$s_n \wedge x_{n+1} = s_n \wedge (y_{n+1} \vee i_0) = (s_n \wedge y_{n+1}) \vee i_0 = i_0.$$

We now claim that $x_{n+1} \notin I$. For if this were not the case there would be $y_{n+1} \in I$, which is impossible by (2.7). Therefore

$$s_n \wedge x_{n+1} = i_0 = x_0 \leq d_n \quad \& \quad x_{n+1} \notin I. \tag{2.13}$$

Furthermore $s_n \vee d_{n+1} = s_{n+1}$. By modularity and by (2.13),

$$s_n \wedge d_{n+1} = (d_n \vee j_0) \wedge (d_n \vee x_{n+1}) = d_n \vee (s_n \wedge x_{n+1}) = d_n.$$

Hence, (2.8) is true.

Note that $s_0 = j_0, d_0 = i_0$ and that the relation $\nearrow$ is transitive (cf. [2], Lemma VI.1.1, p. 200). Together with (2.1) and (2.8) this yields

$$j_0/i_0 \nearrow s_{n+1}/d_{n+1}. \tag{2.14}$$

Since $(i_0, j_0) \in \hat{C}(L)$, we conclude from the same Lemma that $(d_{n+1}, s_{n+1}) \in \hat{C}(L)$, and so the assertion (2.9) must hold. Since $i_0 \neq j_0$, (2.14) implies that $d_{n+1} \neq s_{n+1}$; hence also (2.10) is true.

By (2.3) and (2.13), $d_n \wedge x_{n+1} \leq s_n \wedge x_{n+1} = i_0$. However, by (2.13) $i_0 \leq d_n$ and, moreover, $i_0 \leq x_{n+1}$, so that $i_0 = d_n \wedge x_{n+1}$. Thus

$$d_{n+1}/d_n \searrow x_{n+1}/i_0. \tag{2.15}$$

By (2.13), $x_{n+1} \notin I$ and it follows that $i_0 < x_{n+1}$. Applying (2.15), we have

$$d_n < d_{n+1}. \tag{2.16}$$

Now, (2.8) is equivalent to $s_{n+1}/s_n \searrow d_{n+1}/d_n$. From (2.16) we, therefore, infer that $s_n < s_{n+1}$; hence (2.11) holds.

Further, by (2.14), $d_{n+1} \wedge j_0 = i_0 \in I$. But so we see that $d_{n+1} \in I^1$ is impossible, since it would imply $j_0 \in I$. This proves (2.12).

The assumption that I is not semiprime implies therefore the existence of an infinite ascending sequence $s_0 < s_1 < \cdots$. This contradicts the hypothesis on L, and establishes our assertion. □

From the above theorem one can deduce the following corollary.

Corollary 2.9

Let L be a modular lattice which has no infinite chains and let $n \in \mathbb{N}$. Then an ideal I of L is semiprime of degree n if and only if I is semiprime.

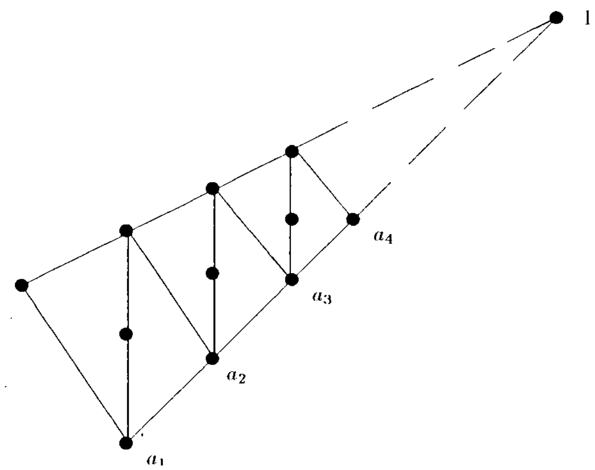

Figure 2

Remark 2.10. A. Consider the lattice N_5 of Figure 1 and choose $I = (a]$. Then $I^1 = [i)$ is a semiprime filter but I is not semiprime. Hence the hypothesis of modularity is essential in Theorem 2.8.

B. Let L be the lattice shown in Figure 2 and let $I = \{a_i; i \in \mathbb{N}\}$. Then $I^1 = \{1\}$ is a semiprime filter but I is not semiprime. Therefore, the hypothesis (ACC) cannot drop out in Theorem 2.8.

3. Meanders and primeness

The following results were motivated by Theorem 2.5 and Theorem 2.8.

Theorem 3.1

Let I be a prime ideal of a lattice L. Then I^1 is a prime filter of L.

Proof. Let $a, b \in L$ be such that $a \vee b \in I^1$, and suppose by way of contradiction that

$$a \in L \setminus I^1 \;\;\&\;\; b \in L \setminus I^1.$$

Then there are z, w such that

$$a \wedge z \in I \;\;\&\;\; z \notin I; \;\; b \wedge w \in I \;\;\&\;\; w \notin I. \tag{3.1}$$

Since I is prime, $a \in I$ and $b \in I$. Consequently, $a \vee b \in I \cap I^1$. By Lemma 2.1 (ii), we have $I = L$. But this violates (3.1). □

The next result provides a partial converse to Theorem 3.1:

Theorem 3.2

Let I be an ideal of a lattice L satisfying the (ACC). *Then I is prime, provided I^1 is prime.*

Proof. Suppose there are elements $a, b \in L$ such that

$$a \wedge b \in I \;\;\&\;\; a \notin I \;\;\&\;\; b \notin I.$$

We then reason to a contradiction as follows.

It is not difficult to show that $a \vee b \notin I^1$. Indeed, otherwise the primeness of I^1 implies that either $a \in I^1$ or $b \in I^1$. Say $a \in I^1$. Then, by the definition of a meander, $b \in I$, a contradiction.

Let $s_1 := a \vee b$ and assume inductively that $n \in \mathbb{N}$ and that there exist elements $s_1, s_2, \cdots, s_n$ such that

$$a \vee b = s_1 < s_2 < \cdots < s_n \tag{3.2}$$

and

$$s_i \notin I^1 \tag{3.3}$$

for every $i = 1, 2, \cdots, n$.

We shall produce an element s_{n+1} for which

$$s_n < s_{n+1} \tag{3.4}$$

and

$$s_{n+1} \notin I^1. \tag{3.5}$$

First, by (3.3) there exists $q \in L$ such that

$$s_n \wedge q \in I \;\;\&\;\; q \notin I. \tag{3.6}$$

We claim that

$$q \notin I^1. \tag{3.7}$$

Were this false, we would have $s_n \in I$ by (3.6). Now, by (3.2), $a \vee b \leq s_n$ and, therefore, $a \vee b \in I$. But then $a \in I$, a contradiction.

Next we claim that $s_{n+1} := q \vee s_n \notin I^1$.

Suppose it is false. By (3.3), $s_n \notin I^1$, and since I^1 is a prime filter it follows that $q \in I^1$. This contradicts (3.7). Thus we get (3.5).

Moreover $s_{n+1} = s_n$ is impossible, since it would imply $s_n \geq q$, and, by (3.6), I would contain the element $s_n \wedge q = q$, contrary to (3.6). Therefore, $s_{n+1} > s_n$, proving (3.4).

The preceding arguments can be summarized in the statement that L contains an infinite ascending chain $s_1 < s_2 < \cdots$. This contradicts the (ACC) and proves the theorem. □

Remark 3.3. A. The ideal I defined in Remark 2.10.B is not prime but the filter I^1 is prime. Hence the (ACC) hypothesis cannot be dropped in Theorem 3.2.

B. We leave a more detailed study of the $n-$th meanders where $n \geq 2$ for a later note. Suffice it to say that they enjoy some very interesting properties.

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References

1. L. Beran, Lattice socles and radicals described by a Galois connection, *Acta Univ. Carolinae* **12** (1971), 55–63.
2. L. Beran, *Orthomodular Lattices (Algebraic Approach),* Reidel, Dordrecht, 1985.
3. L. Beran, On semiprime ideals in lattices, *Journ. Pure Appl. Algebra* **64** (1990), 223–227.
4. L. Beran, Remarks on special ideals in lattices, *Comment. Math. Univ. Carolinae* **35** (4) (1994), 607–615.
5. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order,* Cambridge University Press, Cambridge, 1990.
6. G. Grätzer, *General Lattice Theory,* Academic Press, New York, N.Y.; Birkhäuser Verlag, Basel; Akademie Verlag, Berlin, 1978.
7. Y. Rav, Semiprime ideals in general lattices, *Journ. Pure Appl. Algebra* **56** (1989), 105–118.
8. B. Stenström, Radicals and socles of lattices, *Arch. Math.* **20** (1969), 258–261.
9. H. Zand, A note on prime essential rings, *Bull. Austral. Math. Soc.* **49** (1994), 55–57.