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On the Fourier-Laplace representation of analytic functions in tube domains

I.Kh. Musin

Institute of Mathematics with Computer Center, Ufa Scientific Center, Ufa, Chernyshevskii Str. 112, 450000, Russia

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Abstract

Let C denote an acute convex open cone in $\mathbb{R}^n$ with an apex at the origin and let $T(C) = \mathbb{R}^n + iC$ be the corresponding tube in $\mathbb{C}^n$. We define a space of holomorphic functions $f(z)$ of exponential type in $T(C)$ which have boundary values $f_0(x)$, as $y \to 0,\ y \in C$, satisfying some inequality. We obtain Fourier-Laplace integral representation of these functions. As a consequence a weighted version of the edge of the wedge theorem and Fourier-Laplace representation of entire functions of exponential type (with more general growth characteristics than in [2]) are obtained.

1. Introduction

A weighted generalization of theorems of Paley-Wiener [5] and Plancherel-Polya [6], concerning the integral representation of entire functions of exponential type was established in [2]. Also the Fourier-Laplace representation of functions of exponential type in an octant in $\mathbb{C}^n$ and as a consequence a weighted version of the edge of the wedge theorem was obtained there.

In this paper results of [2] are extended to holomorphic functions in tube domains over acute convex open cones in $\mathbb{R}^n$ and entire functions of exponential type with more general growth characteristics. This became possible due to a simple useful statement, namely the lemma below.

Throughout this paper, nonnegative locally integrable functions on $\mathbb{R}^n$ will be called weight functions. We write $f \in L^p_w(\mathbb{R}^n), w$ a weight function, if $f\,w^{1/p} \in L^p(\mathbb{R}^n)$ and $\|f\|_{p,w} = \|w^{1/p} f\|_p$, where $\|\ \|_p$ denotes the norm of L^p.

For $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$ or $\mathbb{R}^n$ we set $\langle \xi, \eta \rangle = \xi_1\eta_1 + \ldots + \xi_n\eta_n$. The euclidean norm in $\mathbb{C}^n$ or $\mathbb{R}^n$ is denoted by $\|\ \|$. $\bar{B}$ will denote the closure of $B \subset \mathbb{C}^n$, respectively, $\mathbb{R}^n$.

If C is a cone in $\mathbb{R}^n$ with an apex at the origin, then the projection of C is $pr\, C = \{y \in C : \|y\| = 1\}$, the dual cone is defined by $C^* = \{\xi \in \mathbb{R}^n : \langle \xi, y \rangle \geq 0 \quad \text{for all} \quad y \in C\}$, and $T(C) = \mathbb{R}^n + iC$ is called a tube domain over cone C. If b is a convex, continuous function on C which is positively homogeneous of order 1, then we define

$$U(b, C) = \bigl\{\xi \in \mathbb{R}^n : -\langle \xi, y \rangle \leq b(y) \quad \text{for all} \quad y \in C\bigr\}\,.$$

It's clear that $U(b, C)$ is closed in $\mathbb{R}^n$. Recall that a cone C is called acute if it doesn't contain any straight line.

For $f \in L^1(\mathbb{R}^n)$

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(t)\, \exp\bigl(-i\langle x, t \rangle\bigr)\, dt\,,\ x \in \mathbb{R}^n$$

is the Fourier transform of f.

The following definition was introduced in [2]:

Definition. Let u and v be weight functions on $\mathbb{R}^n$. We say the pair $(u, v) \in F^p_q,\ p \geq 1,\ q \geq 1,$ if the inequality

$$\|\hat{f}\|_{q,u} \leq c\,\|f\|_{p,v}\ , \quad c = \text{const} > 0 \tag{1}$$

is satisfied for any simple function on $\mathbb{R}^n$.

As it is noted in [2] (1) permits us to define the Fourier transform in $L^p_v(\mathbb{R}^n)$.

A wide class of weight functions, satisfying the F^p_q condition is described in [2, 3].

2. Results

Let C be an open convex acute cone in $\mathbb{R}^n$ with apex at the origin ([8], p. 73). Let $a(z)$ be a convex continuous function on $T(\bar{C})$ which is positively homogeneous of order 1. By $P_a(C)$ we denote the space of holomorphic functions on $T(C)$ satisfying inequality

$$|f(z)| \le c_\varepsilon \exp\big(a(z) + \varepsilon\|z\|\big)\,,\ c_\varepsilon > 0$$

for any $\varepsilon > 0$.

Lemma

Let $g \in P_a(C)$ and for any $\xi \in \mathbb{R}^n$ $\overline{\lim\limits_{z\to\xi, z\in T(C)}}$ $|g(z)| \le M$, then

$$|g(x+iy)| \le M\ \exp\big(a(iy)\big)\,,\ x+iy \in T(C)\,. \qquad (2)$$

Proof. Note that for some $\sigma > 0$ $a(z) \le \sigma\|z\|\,,\ z \in T(C)$. Let us fix $\eta = (\eta_1, \dots, \eta_n) \in pr\, C$ and define a linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ whose matrix has the form

$$\begin{pmatrix} \eta_1 & \eta_2 & \eta_n \\ a_{21} & a_{22} & a_{2n} \\ & \dots\dots\dots\dots\dots\dots & \\ a_{n1} & a_{n2} & a_{nn} \end{pmatrix},$$

where the elements $a_{kj} \in \mathbb{R}$ are chosen so that A is unitary. Then $g(x+i\eta s) = g\big(A(u_1 + is,\, u_2, \dots, u_n)\big)$, where $u = (u_1, u_2, \dots, u_n) = A^{-1}x$, $s > 0$. Fix $u^1 = (u_2, \dots, u_n)$. Then function $\varphi(u_1 + is) = g\big(A(u_1 + is,\, u^1)\big)$ is analytic in the upper half-plane $G = \{w \in \mathbb{C} : Im\, w > 0\}$. Using the unitary of the operator A we obtain the estimate on G

$$\big|\varphi(u_1+is)\big| \le c_\varepsilon\ \exp\big((\sigma+\varepsilon)\|u^1\| + (\sigma+\varepsilon)|u_1| + (\sigma+\varepsilon)s\big)$$

and by the Phragmen-Lindelöf principle ([7], p. 120) we get

$$\big|\varphi(u_1+is)\big| \le M\ \exp(\sigma s)\,,\ u_1 + is \in G\,. \qquad (3)$$

Using convexity of function $a(z)$ we have for $x + i\eta s \in T(C)$ and any $\varepsilon > 0$

$$\big|g(x+i\eta s)\big| \le c_\varepsilon\ \exp\big(a(x) + \varepsilon\|x\| + (a(i\eta)+\varepsilon)s\big)\,.$$

Therefore,

$$\overline{\lim_{s\to+\infty}}\ \frac{\log|\varphi(is)|}{s} \le a(i\eta)\,.$$

Applying the Phragmen-Lindelöf principle again ([4], p. 119) and (3) we obtain

$$\log \left|\varphi(u_1+is)\right| \le \log M + a(i\eta)s\,,\ u_1+is \in G\,,$$

that is,

$$\left|g(x+i\eta s)\right| \le M \exp\left(a(iy)\right),\ x = A\left(u_1, u^1\right),\ s>0\,.$$

The right side of this last inequality does not depend on u^1. This means that the inequality holds for any $x \in \mathbb{R}^n$. Since $\eta \in pr\,C$ was arbitrary, (2) is proved. □

Let $b(y) = a(iy)\,,\ y \in C\,,\ B_r = \{\xi \in \mathbb{R}^n : \|\xi\| \le r\}\,.$

Theorem 1

Let $(u,v) \in F^p_q$ and suppose that the following conditions hold:

a) The inequality

$$\int_{\|y\|\le 1} v(x+\varepsilon y)\,dy \le c_1\,v(x) + c_2\,,\ c_1>0\,,\ c_2>0$$

is satisfied on $\mathbb{R}^n$ for sufficiently small $\varepsilon > 0$.

b) The weight u is even and

$$u(x) \ge c_\varepsilon\ \exp\left(-\varepsilon\|x\|\right),\ x \in \mathbb{R}^n,\ c_\varepsilon > 0 \tag{4}$$

holds for any $\varepsilon > 0$.

Suppose that $f \in P_a(C)$ has boundary values

$$f_0(x) = \lim_{y\to 0,\, y\in C} f(x+iy)$$

a.e. in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |f_0(x)|^p\left(1+v(x)\right)dx < \infty\,.$$

Then the following representation holds

$$f(z) = \int_{\mathbb{R}^n} \exp\left(-i\langle z,t\rangle\right) g(t)\,dt\,,\ z \in T(C)\,,$$

where $g \in L^q_u(\mathbb{R}^n)\,,\ \mathrm{supp}\ g \subseteq -U(b,C)$.

Proof. We follow the proof of Theorem 4 in [2], hence for this reason some details are omitted. Let V be the volume of unit ball B_1 in $\mathbb{R}^n$. Set

$$F_\varepsilon(z) = \begin{cases} V^{-1} \displaystyle\int_B f(z+\varepsilon\,\xi)\ d\xi\,, & z \in T(C) \\ V^{-1} \displaystyle\int_B f(z+\varepsilon\,\xi)\ d\xi\,, & z = x \in \mathbb{R}^n\,, \end{cases}$$

where $\varepsilon > 0$. Obviously, $F_\varepsilon \in P_a(C)$. Since $f_0 \in L^p(\mathbb{R}^n)$, Hölder's inequality implies that $\mathbb{R}^n\,|F_\varepsilon(x)| \le M_\varepsilon$ for some $M_\varepsilon > 0$. By the Lebesgue dominated convergence theorem it follows that $F_\varepsilon(z)$ is continuous at every point of $\mathbb{R}^n$. From the lemma we see that

$$\big|F_\varepsilon(x+iy)\big| \le M_\varepsilon\ \exp\big(a(iy)\big)\,,\ x+iy \in T(C)\,.$$

Now let $\varphi \in C_0^\infty$, such that, supp $\varphi \subseteq \operatorname{int}(-C^*)$ and $\int_{\mathbb{R}^n}\varphi(t)\,dt = 1$. Then

$$\psi(z) = \int_{\mathbb{R}^n} \exp\big(-i\langle z,t\rangle\big)\,\varphi(t)\ dt$$

is an entire function satisfying the inequalities

$$\big|\psi(z)\big| \le c_m\big(1+\|z\|\big)^{-m},\ z \in \mathbb{C}^n\,, \quad \text{for all} \quad m \ge 0 \tag{5}$$

Setting

$$g_\varepsilon(t) = (2\pi)^{-n}\int_{\mathbb{R}^n} \exp\big(i\langle x,t\rangle\big)\,F_\varepsilon(x)\ \psi(\varepsilon x)\,dx\,,\ t \in \mathbb{R}^n\,,$$

It is obvious that, $g_\varepsilon \in C^\infty(\mathbb{R}^n)$ and g_ε is bounded on $\mathbb{R}^n$. Arguing as in [1], [2] we get

$$g_\varepsilon(t) = (2\pi)^{-n}\int_{\mathbb{R}^n} \exp\big(i\langle x+iy,\ t\rangle\big)\ F_\varepsilon(x+iy)\,\psi(\varepsilon\,x+i\varepsilon y)\,dx \tag{6}$$

where $y \in C$. And applying the estimate (5) we obtain

$$g_\varepsilon(t) \le A_\varepsilon \exp\Big(\inf_{y\in C}\big(-\langle y,t\rangle + a(iy)\big)\Big)\,,\ A_\varepsilon > 0\,.$$

From this estimate it follows that $g_\varepsilon(t) = 0$ for $t \notin -U(b,C)$. Further, as in [2] it can be shown that there exists a sequence $\{g_{\varepsilon_k}\} \subset L^q_u(\mathbb{R}^n)$ such that $\{g_{\varepsilon_k}\}$ converges weakly to some $g \in L^q_u(\mathbb{R}^n)$ in $L^q_u(\mathbb{R}^n)$, as $\varepsilon_k \to 0$. Note that $g_\varepsilon(t)\exp\big(\langle y,t\rangle\big) \in L^1(\mathbb{R}^n)$, if $y \in C$. Indeed we know that $g_\varepsilon(t) = 0$ for $t \notin -U(b,C)$. Further,

$-U(b,C) \subseteq -C^* + B_r\,,$ where $r = \max_{y \in pr\, C} b(y)$. But for $t \in -C^* + B_r\,,\ y \in C$ the following inequality holds ([8], p. 172):

$$\langle y, t\rangle \le -\Delta(y)\big(\|t\| - r\big)\,\theta\big(\|t\| - r\big) + r\|y\|\,, \tag{7}$$

where $\theta(\mu) = 1\,,\ \mu > 0\,,\ \theta(\mu) = 0\,,\ \mu < 0\,,\ \Delta(y) = \inf_{\xi \in pr\, C} \langle \xi, y\rangle \ge c_y\|y\|\,, c_y > 0\,,$ so

$$\int_{-U(b,C)} |g_\varepsilon(t)| \exp\big(\langle y,t\rangle\big)dt \le \int_{(-C^*+B_r)\setminus B_r} |g_\varepsilon(t)| \exp\big(r\|y\| - \Delta(y)\big(\|t\| - r\big)\big)dt$$
$$+ \int_{B_r} |g_\varepsilon(t)| \exp\big(r\|y\|\big)\, dt < \infty\,.$$

If $X(t)$ is the characteristic function of the set $-U(b,C)$, then, taking into account (4) and (7), we can show that function $X(t)\exp\big(-i\langle z,t\rangle\big)$ belongs to dual for $L^q_u(\mathbb{R}^n)$, $z \in T(C)$.

From (6) and the Fourier inversion formula

$$F_\varepsilon(z)\,\psi(\varepsilon z) = \int_{-U(b,C)} \exp\big(-i\langle z,t\rangle\big)\, g_\varepsilon(t)\, dt\,,\ z \in T(C)\,.$$

Note that $F_\varepsilon(z) \to f(z)$, and $\psi(\varepsilon z) \to 1$, as $\varepsilon \to 0$. Replacing ε by ε_k and letting $\varepsilon_k \to 0$, we get

$$f(z) = \int_{-U(b,C)} \exp\big(-i\langle z,t\rangle\big)\, g(t)\, dt\,,\ z \in T(C)$$

where $g \in L^q_u(\mathbb{R}^n)$, supp $g \subseteq -U(b,C)$. This proves the result. □

Theorem 2

Let a_1, a_2 be nonnegative convex continuous functions on $T(\bar{C})$, respectively, $T(-\bar{C})$, which are positively homogeneous of order 1. Let $f_1 \in P_{a_1}(C)$, $f_2 \in P_{a_2}(-C)$ and suppose the limits

$$\lim_{y\to 0, y \in C} f_1(x+iy) = \tilde{f}_1(x)\,,$$
$$\lim_{y\to 0, y \in -C} f_2(x+iy) = \tilde{f}_2(x)$$

exist a.e. in $\mathbb{R}^n$. Let $\tilde{f}_1 = \tilde{f}_2$ a.e. in $\mathbb{R}^n$, and $\tilde{f}_1(x)$, $\tilde{f}_2(x)$, u and v satisfy the conditions of Theorem 1.

Then $f_1(z)$ and $f_2(z)$ are analytically continuable to entire function $f(z)$ and

$$f(z) = \int_K \exp\big(-i\langle z,t\rangle\big)\, g(t)\, dt\,,\ z \in \mathbb{C}^n\,,$$

where $K = \big(-U(b_1,C)\big) \cap \big(-U(b_2,C)\big)$, $g \in L^q_u(\mathbb{R}^n)$ and supp $g \subseteq K$.

Proof. By Theorem 1 the functions $f_j(z)$, $j = 1, 2$, have representation

$$f_j(z) = \int_{\mathbb{R}^n} \exp\big(-i\langle z, t\rangle\big)\, g_j(t)\, dt\,,\ z \in \mathbb{R}^n + i(-1)^{j+1}\, C\,,$$

where $g_j(t)$, $j = 1, 2$, satisfy the conditions of Theorem 1. Since $\tilde{f}_1 = \tilde{f}_2$ a.e. in $\mathbb{R}^n$, then as in Lemma 6 of [1], it may be shown that $g_1(t) = g_2(t)$ a.e. in $\mathbb{R}^n$. Set $g(t) = g_1(t) = g_2(t)$. Then supp $g \subseteq \big(-U(b_1, C)\big) \cap \big(-U(b_2, C)\big)$. Let $R = \max_{y \in \mathrm{pr}\, C} \big(b_1(y),\, b_2(y)\big)$. Then $K \subseteq (-C^* + B_r) \cap (C^* + B_r)$. Since C^* is an acute convex cone ([8], p. 74) the set $(-C^* + B_r) \cap (C^* + B_r)$ is bounded in $\mathbb{R}^n$. Hence, $f(z)$ is entire and $f(z) = f_1(z)$, $z \in T(C)$, $f(z) = f_2(z)$, $z \in T(-C)$. Besides that, $|f(z)| \le C \exp(H_K(Imz))$, $z \in \mathbb{C}^n$, where $H_K(y) = \max\limits_{t \in K} \langle y, t\rangle$ is the support function of convex compact K. □

Theorem 3

Let $a(z)$ be a nonnegative convex continuous function on $\mathbb{C}^n$ which is positively homogeneous of order 1, and u and v as in Theorem 1. Suppose the entire function $f(z)$ satisfies

$$|f(z)| \le C_\varepsilon \exp\big(a(z) + \varepsilon\|z\|\big)\,,\ C_\varepsilon > 0\,,\ z \in \mathbb{C}^n$$

and

$$\int_{\mathbb{R}^n} |f(x)|^p \big(1 + v(x)\big)\, dx < \infty\,.$$

Then

$$f(z) = \int_K \exp\big(-i\langle z, t\rangle\big)\, g(t)\, dt\,,\ z \in \mathbb{C}^n\,,$$

where $K = \{t \in \mathbb{R}^n : -\langle t, y\rangle \le a(iy)\,,\ \ \text{for all}\ \ y \in \mathbb{R}^n\}$, $g \in L^q_u(\mathbb{R}^n)$, *supp* $g \le K$.

Proof. Let C_j, $j = 1, 2, \ldots, m$, be acute convex open cones in $\mathbb{R}^n$, such that $\bigcup\limits_{j=1} \bar{C}_j = \mathbb{R}^n$. By Theorem 1

$$f(z) = \int_{\mathbb{R}^n} \exp\big(-i\langle z, t\rangle\big)\, g_j(t)\, dt\,,\ z \in \mathbb{R}^n + iC_j\,,$$

$g_j \in L^q_u(\mathbb{R}^n)$, supp $g_j \subseteq -U(b, C_j)$, $j = 1, 2, \ldots, m$. As in Lemma 6 of [1] we have

$$\int_{\mathbb{R}^n} g_j(t)\, \varphi(t)\, dt = (2\pi)^{-n} \int_{\mathbb{R}^n} f(x)\, \hat{\varphi}(-x)\, dx\,,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. From this equality it follows that functions $g_j(t)$, $j = 1, 2, \ldots, m,$ as elements of $L^q_u(\mathbb{R}^n)$ coincide. Now our statement follows. □

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