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Logarithms and imaginary powers of operators on Hilbert spaces

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Abstract

We provide a necessary and sufficient condition for the existence of bounded imaginary powers of certain Hilbert space operators and study the growth of $\|A^{it}\|$ when $|t| \to \infty$.

1. Introduction, notations and definitions

In this paper we study Hilbert space operators A of type θ, which admit bounded imaginary powers. When A^{it} forms a C_0-group, its generator is the operator $i \log A$. In Section 2 we estimate the norm of the resolvent of $i \log A$ and show that the growth bound of the group A^{it} coincides with the (minimal possible) type θ of A. In Section 3 we find an integral representation of Hilbert space C_0-groups, which we apply to A^{it}. The main results in Section 4 include a necessary and sufficient condition for the existence of the imaginary powers A^{it} and an explicit estimate of $\|A^{it}\|$. Our technique is based on the vector-valued Plancherel theorem, true only for Hilbert spaces. The paper, among other things, specifies and complements some results in [1], [2], [4], [10] and [18].

Let now X be a complex Hilbert space and $S_\psi = \{z \in \mathbb{C} : |\mathrm{Arg}(z)| < \psi\}$ be an open sector.

Definition 1.1 A closed, densely defined operator A on X is called an operator of type θ, $0 \le \theta < \pi$, if $\sigma(A) \subset S_\theta$ and

$$\|\lambda(A+\lambda)^{-1}\| \le M_\phi\,, \ \forall\, \lambda \in S_\phi\,, \ \forall\, \phi : 0 \le \phi < \pi - \theta\,. \tag{1.1}$$

Lemma 1.2 (cf. [5])

Let A be an invertible operator of type θ. Then A^{-1} and A^* are also of type θ.

Proof. In a reflexive Banach space any one-to-one operator of type θ has dense range [11, p. 295], therefore A^{-1} is densely defined. Further, for every $\phi < \pi - \theta$ and every $z \in S_\phi$ with $\lambda = 1/z$

$$\left\| \frac{z}{z + A^{-1}} \right\| = \left\| \frac{zA}{zA+1} \right\| = \left\| 1 - \frac{1}{zA+1} \right\| \leq 1 + \left\| \frac{\lambda}{A+\lambda} \right\| \leq 1 + M_\phi$$

since $z \in S_\phi$ if and only if $\lambda \in S_\phi$. Thus A^{-1} is of type θ. The part about A^* is left to the reader. $\square$

Throughout, A denotes an operator as described in the lemma. Note that in this case $D(A) \cap D(A^{-1})$ is dense in X ([16, p. 431]) and A, A^{-1}, A^* are entirely interchangeable in all formulas.

2. Imaginary powers and logarithms of operators

Complex powers of A are usually defined by the formula.

$$A^z x = \frac{\sin \pi z}{\pi} \int_0^{+\infty} \lambda^{z-1} (A+\lambda)^{-1} Ax \, d\lambda$$

for $0 < \operatorname{Re}(z) < 1$, $x \in D(A)$. Integrating by parts here and setting Re $(z) = 0$ we formally get

$$A^{it} = \frac{\sinh \pi t}{\pi t} \int_{\mathbb{R}} \lambda^{it} A(A+\lambda)^{-2} \, d\lambda \quad (t \in \mathbb{R}) \tag{2.1}$$

It is easy to see that the integral is absolutely convergent on the dense subspace $D(A) \cap D(A^{-1})$:

Let $x \in D(A^{-1})$. Then $x = Ay$, $y \in D(A)$ and

$$\frac{A}{(\lambda + A)^2} x = \frac{A^2 y}{(\lambda + A)^2} = \Big(\frac{A}{\lambda + A} \Big)^2 y = \Big(1 - \frac{\lambda}{\lambda + A} \Big)^2 y$$

which is bounded. For $|\lambda| \to \infty$ and $x \in D(A)$, we have

$$x = A^{-1} y, \; y \in D(A^{-1}) \text{ so that } \left\| \frac{Ax}{(\lambda + A)^2} \right\| = \left\| \frac{1}{(\lambda + A)^2} \, y \right\| \leq \frac{M_0 \|y\|}{\lambda^2} \, .$$

Sometimes, the imaginary powers A^{it} extend to bounded operators on the whole space X. In this case they constitute a C_0-group of operators [8], [16]. The importance of A^{it} is demonstrated in [4], [7], [16].

The operator logarithm is defined by the formula (cf. Nollau [13])

$$\big(\log A\big)\ x = \int_0^\infty (A+\lambda)^{-1}(Ax-x)\,\frac{d\lambda}{1+\lambda}\,, \quad \big(x \in D(A)\cap D(A^{-1})\big)\,.$$

In the same way as in [13, Satz 3], one can show that the linear operator $\log A$ is closable and $D(A)\cap D(A^{-1})$ is a core for it. We shall keep the same notation for its closure. Nollau proved that

$$\log Ax = \lim_{\alpha\to 0+} \frac{A^\alpha x - x}{\alpha}$$

for $x \in D(A)\cap D(A^{-1})$ [13, Satz 4]. For every such x, the vector function $A^z x$, with $A^{-z}x = (A^{-1})^z x$, is holomorphic in the strip $|\mathrm{Re}(z)| < 1$, which yields

$$i\log Ax = \lim_{t\to 0+} \frac{A^{it}x - x}{t} \tag{2.2}$$

since $(d/dz)\,A^z x = (d/d\,\alpha)\,A^\alpha x = (d/dit)\,A^{it}x$.

Proposition 2.1

If the imaginary powers $A^{it}(t\in\mathbb{R})$ form a C_0-group, then its generator is the operator $i\log A$ defined above.

Proof. According to (2.2), the generator, say, B of A^{it} coincides with $i\log A$ on the dense set $D(A)\cap D(A^{-1})$. This set is invariant for A^{it} and therefore a core for B [6, Theorem 1.9]. Since both operators are closed, they coincide. □

Conversely, we have also the following.

Proposition 2.2

If $i\log A$ is a generator of a C_0-group, then this group is the extension of A^{it} from $D(A)\cap D(A^{-1})$ to X.

The proof is given in Section 5.

We remind some definitions. Given a C_0-semigroup $e^{-tB},\ t\geq 0$, then

$$\omega(B) = \lim_{t\to\infty} \big(\log\|e^{-tB}\|/t\big)$$

is the exponential type (or growth bound) of that semigroup. If e^{tB}, $t \in \mathbb{R}$, is a group, then its exponential type is the number

$$\omega_g(B) = \max\left\{\omega(B),\ \omega(-B)\right\} = \lim_{|t|\to\infty} \sup\left(\log\|e^{-tB}\| \,/\, |t|\right).$$

When A^{is} is a C_0-group, an interesting question is how the growth bound $\omega_g(i\log A)$ of that group is related to the type θ of the operator A. We shall prove here that $\omega_g \leq \theta$.

First, we need a result that can be found in [15] or [12, p. 96].

Lemma 2.3

For any Hilbert space semigroup $e^{-tB},\ t \geq 0$, we have

$$\omega(B) = \inf\left\{\lambda \in \mathbb{R} : \lambda + i\mathbb{R} \subset \rho(-B) \text{ and } \|(\lambda + i\mu + b)^{-1}\| \text{ is bounded } \forall\, \mu \in \mathbb{R}\right\}.$$

We combine this now with the following lemma, which is needed also for the proof of the main theorem in Section 4.

Lemma 2.4

Suppose that A is an operator of type $\theta,\ 0 \leq \theta < \pi$. Then for every $\psi,\ \theta < \psi < \pi$, the following estimate holds

$$\left\|\frac{1}{\psi \pm i\log A + \mu}\right\| \leq \frac{K_\psi}{\mathrm{Re}(\mu)}\ , \quad \left(\mathrm{Re}(\mu) > 0\right) \tag{2.3}$$

where $K_\psi = \pi M_{\pi-\psi}$, ($M$ as in (1.1)), depends only on ψ. The operators $\psi \pm i\log A$ are of type $\pi/2$ and the spectrum of $i\log A$ lies in the strip $|\mathrm{Re}(z)| \leq \theta$.

Note that the lemma is true for general Banach spaces. Its proof, for convenience, is given in Section 5.

Corollary 2.5

Given the operator A of type θ, suppose that A^{it} forms a C_0-group. Then $\omega_g \leq \theta$, where ω_g is the growth bound of that group. More precisely, $\omega_g = \inf\left\{\theta : A \text{ is of type } \theta\right\}$.

Proof. The inequality (2.3) implies, according to Lemma 2.3, that for every $\varepsilon : 0 < \varepsilon < \pi - \theta$ we have $\omega(\pm i \log A) \leq \theta + \varepsilon$ (take $\psi = \theta + \varepsilon/2$, then $\|(\lambda \pm i \log A)^{-1}\|$ is uniformly bounded for $\mathrm{Re}(\lambda) \geq \theta + \varepsilon$).

Therefore $\omega_g \leq \theta$. Pruss and Sohr [16, Theorem 2] proved that if A^{it} is a C_0-group of growth order ω, then A is of type at most ω. This completes the last part of the statement. □

We note that this corollary can also be derived from [10].

3. Representation of groups of operators on a Hilbert space

For our main result we need an integral representation of A^{it}, which is a particular case of the following general theorem.

Theorem 3.1

Let iB be the generator of the C_0-group e^{itB}, $t \in \mathbb{R}$, on X with growth $\|e^{itB}\| \leq M\, e^{a|t|}$, $a \geq 0$. Then $\sigma(B) \subseteq \{z : |\mathrm{Im}(z)| \leq a\}$ and for every $c > a$ we have the representation

$$e^{-itB} = e^{c|t|} \frac{1}{\pi} \int_{\mathbb{R}} e^{its} \frac{c}{c^2 + (B+s)^2}\, ds \quad (\forall\, t \in \mathbb{R}) \tag{3.1}$$

which is absolutely convergent in the weak operator topology.

Proof. For every $c > a$ we can write:

$$\big(iB + c + is\big)^{-1} = \int_0^{+\infty} e^{-t(iB+c+is)} dt = \int_0^{+\infty} e^{-ist} e^{-ct} e^{-itB}\, dt\,. \tag{3.2}$$

By the Fourier (or Laplace) inversion we have

$$e^{-itB} e^{-ct} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist} \big(iB + c + is\big)^{-1} ds \quad (t > 0)\,, \tag{3.3}$$

$$0 = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist} \big(iB + c + is\big)^{-1} ds \quad (t < 0)$$

(the convergence we shall specify later). In the same way

$$(-iB + c - is)^{-1} = \int_0^{+\infty} e^{-t(-iB+c-is)} dt = \int_0^{+\infty} e^{ist} e^{-ct} e^{itB}\, dt \tag{3.4}$$

which implies

$$e^{itB}e^{-ct} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ist}\big(-iB+c-is\big)^{-1}\,ds \quad (t>0) \tag{3.5}$$

$$0 = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ist}\big(-iB+c-is\big)^{-1}\,ds \quad (t<0)\,.$$

The second integral can be rewritten as

$$0 = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ist}\big(-iB+c-is\big)^{-1}\,dt \quad (t>0)\,.$$

Combining this with (3.3) we get

$$\begin{aligned} e^{-itB}\,e^{-ct} &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ist}\big[(c+iB+is)^{-1}+(c-iB-is)^{-1}\big]\,ds \\ &= \frac{c}{\pi}\int_{\mathbb{R}} e^{ist}\big(c^2+(B+s)^2\big)^{-1}\,ds\,, \quad \text{or} \end{aligned}$$

$$e^{-itB} = e^{ct}\frac{1}{\pi}\int_{\mathbb{R}} e^{ist}\frac{c}{c^2+(B+s)^2}\,ds \quad (t>0) \tag{3.6}$$

In the same way, starting with the resolvents $(iB+c-is)^{-1}$ and $(-iB+c+is)^{-1}$, we come to the representation

$$e^{iuB} = e^{cu}\frac{1}{\pi}\int_{\mathbb{R}} e^{ius}\frac{c}{c^2+(B-s)^2}\,ds \quad (u>0)\,.$$

Substituting here s by $-s$ and u by $-t$ we get

$$e^{-itB} = e^{-ct}\frac{1}{\pi}\int_{\mathbb{R}} e^{its}\frac{c}{c^2+(B+s)^2}\,ds \quad (t<0)$$

which in combination with (3.6) gives the representation (3.1).

Now convergence. Because of the estimate $\|e^{-ist}\,e^{-ct}\,e^{-itB}\| \le M\,e^{-(c-a)t}$ the integral in (3.2) is absolutely convergent in the uniform topology. The same is true for the integral in (3.4). At the same time, we have for any Banach space.

Lemma 3.2

Suppose $-T$ is the generator of the C_0-semigroup e^{-sT} with growth $\|e^{-tT}\| \le M\,e^{at}$, $(t \ge 0)$. Then for every $c>a$

$$e^{-tT}x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{zt}(T+z)^{-1}\,x dz \quad (t \ge 0)$$

the integral being absolutely convergent for $\forall\, x \in D(T^2)$.

(This is Corollary 7.5 in [14, Chapter 1]). Therefore, the integrals above, in (3.3) and further, are convergent for all $x \in D(B^2)$. As we shall see now, in Hilbert spaces the integral (3.1) is absolutely weakly convergent everywhere. First, formula (3.2) says that for every $x \in X$ the function $(c+i(B+s))^{-1}x$ is the Fourier transform of the function defined to be $e^{-t(c+iB)}x$ for $t \geq 0$, and zero for $t \leq 0$. By the vector valued Plancherel theorem (which holds in Hilbert spaces – see [17, p. 139]) we have the estimate

$$\int_{\mathbb{R}} \left\| \big(c+i(B+s)\big)^{-1} x \right\|^2 ds = 2\pi \int_0^\infty \left\| e^{-ct} e^{-itB} x \right\|^2 dt \tag{3.7}$$

$$\leq \; 2\pi \int_0^\infty \big(M\, e^{-ct}\, e^{a|t|} \|x\|\big)^2 \, dt = 2M^2\pi \|x\|^2 \int_0^\infty e^{-2(c-a)t}\, dt = \frac{\pi M^2}{c-a} |x\|^2 \, .$$

In the same way $\int_0^\infty \left\| \big(c+i(B^*+s)\big)^{-1} x \|^2 \, ds \leq \frac{\pi M^2}{c-a} \|x\| 2 \right.$, since B^* generates the C_0-group e^{itB^*} and $\|e^{itB}\| = \|(e^{itB})^*\| = \|e^{-itB^*}\| \; (\forall\, t \in \mathbb{R})$. For every $x, y \in X$ we can write

$$\begin{aligned}
\int_{\mathbb{R}} \left| \Big\langle \frac{c}{c^2+(B+s)^2}\, x, y \Big\rangle \right| ds &= c \int_{\mathbb{R}} \left| \Big\langle \frac{1}{c-i(B+s)} \cdot \frac{1}{c+i(B+1)}\, x, y \Big\rangle \right| ds \\
&= c \int_{\mathbb{R}} \left| \Big\langle \frac{1}{c+i(B+s)}\, x \, , \; \frac{1}{c+i(B^*+s)}\, y \Big\rangle \right| ds \\
&\leq c \int_{\mathbb{R}} \left\| \frac{1}{c+i(B+s)}\, x \right\| \left\| \frac{1}{c+i(B^*+s)}\, y \right\| ds \\
&\leq c \left[\int_{\mathbb{R}} \left\| \frac{1}{c+i(B+s)}\, x \right\|^2 ds \right]^{1/2} \left[\int_{\mathbb{R}} \left\| \frac{1}{c+i(B^*+s)}\, y \right\|^2 ds \right]^{1/2} \\
&\leq \frac{c\,\pi M^2}{c-a} \, \|x\| \, \|y\| \, .
\end{aligned}$$

That is:

$$\int_{\mathbb{R}} \left| \Big\langle \frac{c}{c^2+(B+s)^2}\, x, y \Big\rangle \right| ds \leq \frac{c\,\pi M^2}{c-a} \, \|x\| \, \|y\| \quad \big(\forall x, \, y \in X\big) \tag{3.8}$$

and the proof is completed. □

Remark . For bounded groups of operators, the absolute convergence of the integral in (3.8), based on Plancherel's theorem, was shown by Van Casteren [17]. He obtained this way a very nice characterization of operators, similar to self-adjoint. The representation (3.1) seems to be new.

Suppose now that $\omega(iB) = \limsup_{|t|\to\infty} \big(\log\|e^{-itB}\| \,/\, |t|\big)$ is the growth order of the group. We see that for every $b > \omega(iB)$ there exists a constant L_b, depending on b, such that

$$\sup_{c\geq b}\int_{\mathbb{R}} \left|\left\langle \frac{c}{c^2+(B+s)^2}\, x, y\right\rangle\right| ds \leq L_b\|x\|\,\|y\| \quad \big(\forall\, x,\ y\in X\big) \tag{3.9}$$

(simply take $a = \big(b+\omega(iB)\big)/2$ in (3.8)).

The representation (3.1) suggests the following.

Theorem 3.3

Let B be a closed, densely defined linear operator on X. Let $\omega \geq 0$ be a number such that $\sigma(B) \subseteq \{z : |\mathrm{Im}(z)| \leq \omega\}$ and for every $b > \omega$ there exists a constant L_b for which the inequality (3.9) holds. Then iB generates (via (3.1)) a C_0-group e^{itB} of growth order ω.

We omit the proof, which is similar to that of [17, Theorem 7.9]. Apply now the representation (3.1) to A^{it}. In this case, $B = \log A$ and in view of theorems 3.1 and 3.3 we have the following.

Corollary 3.4

The imaginary powers A^{it} exist as a C_0-group if and only if

$$\int_{\mathbb{R}} \left|\left\langle \frac{c}{c^2+(\log A+s)^2}\, x, y\right\rangle\right| ds \leq C_\theta\|x\|\,\|y\| \quad \big(\forall\, c>\theta,\ x,\ y\in X\big)$$

in which case

$$A^{it} = \frac{e^{c|t|}}{\pi}\int_{\mathbb{R}} e^{ist}\frac{c}{c^2+(\log A+s)^2}\, ds, \quad \big(\forall\, c>\theta\big) \tag{3.10}$$

4. Main Results

Setting $\lambda = e^u$ in the representation (2.1) we get

$$A^{it} = \frac{\sinh \pi t}{\pi t}\int_{\mathbb{R}} e^{itu} A e^u (A+e^u)^{-2}\, du \quad (t\in\mathbb{R})\,.$$

Under the condition

$$\int_{\mathbb{R}} \Big| \Big\langle \frac{Ae^u}{(A+e^u)^2}\, x, y \Big\rangle \Big| \, du \le L\|x\|\,\|y\| \qquad \big(\forall\, x, y \in X;\ L\text{– a constant}\big) \tag{4.1}$$

the purely imaginary powers extend to a C_0-group (e.g. [1]) with

$$\|A^{it}\| \le L \frac{\sinh \pi t}{\pi t} \le L\, e^{\pi |t|} \tag{4.2}$$

Therefore, (4.1) appears in a natural manner. The estimate (4.2) however, is very rough. Since A is of type θ, the growth order of A^{it} is at most θ (Corollary 2.5). We shall obtain now a better estimate of $\|A^{it}\|$ in terms of the original constants.

Theorem 4.1

Condition (4.1) implies that $A^{it} (t \in \mathbb{R})$ constitutes a C_0-group of operators with

$$\|A^{it}\| \le K(\theta, \varepsilon)\, e^{(\theta+\varepsilon)|t|} \big(\forall\, \varepsilon,\ 0 < \varepsilon < \pi - \theta\big) \tag{4.3}$$

where $K(\theta,\varepsilon) = \frac{\theta+\varepsilon}{\pi}\big\{1 + \frac{2}{\varepsilon}(2\pi - \theta - \varepsilon) M_{\pi-\theta-\varepsilon}\big\}^2\ L^2 \le \big(1 + \frac{4\pi}{\varepsilon}\, M_{\pi-\theta-\varepsilon}\big)^2 L^2$ *and* $M_{\pi-\theta-\varepsilon}$ *is the constant appearing in (1.1).*

Proof. Take $\varepsilon : 0 < \varepsilon < \pi - \theta$ and set $c = \theta + \varepsilon, \psi = \theta + \varepsilon/2$, $\mu = \varepsilon/2$, so that $c = \psi + \mu$. For every $x \in X$ we have

$$\big\|\big(c + is + i \log A\big)^{-1} x\big\| = \Big\| \frac{2\pi + is + i \log A}{c + is + i \log A} \big(2\pi + is + i \log A\big)^{-1} x \Big\|$$

$$\le \Big\| 1 + \frac{2\pi - c}{c + is + i \log A} \Big\| \; \big\|\big(2\pi + is + i \log A\big)^{-1} x\big\|$$

and since $\big\|\frac{2\pi - c}{c+is+i\log A}\big\| = \big\|\frac{2\pi-\theta-\varepsilon}{\psi+\mu+is+i\log A}\big\| \le \frac{2}{\varepsilon}\big(2\pi - \theta - \varepsilon\big)\, M_{\pi-\theta-\varepsilon}$ according to (2.3), we get

$$\big\|(c + is + i\log A)^{-1}\, x\big\| \le C(\theta,\varepsilon) \big\|(2\pi + is + i \log A)^{-1}\, x\big\|$$

where $C(\theta, \varepsilon) = 1 + \frac{2}{\varepsilon}(2\pi - \theta - \varepsilon)\, M_{\pi-\theta-\varepsilon}$.

The same holds with $(\log A)^*$ in the place of $\log A$, since

$$\Big\| \frac{2\pi - c}{c + is + i(\log A)^*} \Big\| = \Big\| \Big(\frac{2\pi - c}{c - is - i \log A} \Big)^* \Big\| = \Big\| \frac{2\pi - c}{c - is - i \log A} \Big\| \le C(\theta, \varepsilon)\,.$$

In view of (3.10), with the above choice of c and $\forall\, x,\; y \in X$,

$$|\langle A^{it}x, y\rangle| \le \frac{c\, e^{c|t|}}{\pi} \int_{\mathbb{R}} \left| \left\langle \frac{1}{c^2 + (\log A + s)^2}\, x, y \right\rangle \right| ds$$

$$= \frac{c\, e^{c|t|}}{\pi} \int_{\mathbb{R}} \left| \left\langle \frac{1}{c + is + i\log A}\, x,\; \frac{1}{c + is + i(\log A)^*}\, y \right\rangle \right| ds$$

$$\le \frac{c\, e^{c|t|}}{\pi} \int_{\mathbb{R}} \left\| \frac{1}{c + is + i\log A}\, x \right\| \left\| \frac{1}{c + is + i(\log A)^*}\, y \right\| ds$$

$$\le \frac{c\, e^{c|t|}}{\pi} \int_{\mathbb{R}} \left\| \frac{1}{c + is + i\log A}\, x \right\| \left\| \frac{1}{c + is + i(\log A)^*}\, y \right\| ds$$

$$\le C^2(\theta, \varepsilon) \frac{c\, e^{c|t|}}{\pi} \left\{ \int_{\mathbb{R}} \left\| \frac{1}{2\pi + is + i\log A}\, x \right\|^2 ds \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left\| \frac{1}{2\pi + is + i(\log A)^*}\, y \right\|^2 ds \right\}^{1/2}$$

$$\le C^2(\theta, \varepsilon) \frac{c\, e^{c|t|}}{\pi}\, L^2 \|x\|\, \|y\|$$

For the last inequality we use (3.7) with $c = 2\pi,\; a = \pi,\; B = \log A, (\log A^*)$ and also (4.2). This brings to (4.3). The proof is completed. □

In a sense, the converse is also true.

Proposition 4.2

Suppose that the imaginary powers of A constitute a C_0-group with growth $\|A^{it}\| \le M\, e^{a|t|}$ for some $a < \pi$. Then (4.1) holds with $L = 2\pi M^2/(\pi - a)$.

Proof. For every $x \in X$ we have ([3, Corollary 3.5])

$$\frac{A^{1/2} e^{r/2}}{A + e^r}\, x = A^{1/2} e^{r/2} (A + e^r)^{-1}\, x = \int_{\mathbb{R}} \frac{A^{irs} x}{2\cosh(\pi s)}\, e^{-irs}\, ds$$

By Plancherel's theorem

$$\int_{\mathbb{R}} \left\| \frac{A^{1/2} e^{r/2}}{A + e^r}\, x \right\|^2 dr = 2\pi \int_{\mathbb{R}} \left\| \frac{A^{irs} x}{2\cosh(\pi s)} \right\|^2 ds \le 2\pi M^2 \|x\|^2 \int_{\mathbb{R}} \frac{e^{as}}{2\cosh(\pi s)}\, ds$$

$$\le 2\pi M^2 \|x\|^2 \int_0^\infty e^{-(\pi - a)s} ds = \frac{2\pi M^2 \|x\|^2}{\pi - a}$$

The same is true if we replace A by A^*. For every $x, y \in X$

$$\int_{\mathbb{R}} \left| \left\langle \frac{A\, e^r}{(A+e^r)^2} x, y \right\rangle \right| dr = \int_{\mathbb{R}} \left| \left\langle \frac{A^{1/2} e^{r/2}}{A+e^r}\, x,\ \frac{(A^*)^{1/2} e^{r/2}}{A^*+e^r}\, y \right\rangle \right| dr$$

$$\leq \int_{\mathbb{R}} \left\| \frac{A^{1/2} e^{r/2}}{A+e^r}\, x \right\| \left\| \frac{(A^*)^{1/2} e^{r/2}}{A^*+e^r}\, y \right\| dr$$

$$\leq \left\{ \int_{\mathbb{R}} \left\| \frac{A^{1/2} e^{r/2}}{A+e^r}\, x \right\|^2 dr \right\}^{1/2} \left\{ \int_{\mathbb{R}} \left\| \frac{(A^*)^{1/2} e^{r/2}}{A^*+e^r}\, y \right\|^2 dr \right\}^{1/2} \leq \frac{2\pi M^2}{\pi - a} \|x\|\, \|y\| \,.$$

This way, (4.1) is a necessary and sufficient condition for the existence of boundary imaginary powers. □

5. Appendix

Proof of Lemma 2.4. The following representation is convergent in the uniform topology [13, Satz 7, p. 170]:

$$(\log A - z)^{-1} = \int_0^\infty (A+t)^{-1} \frac{dt}{\pi^2 + (\log t - z)^2}\,, \quad |\mathrm{Im}(z)| > \pi\,.$$

We replace here z by $-i\lambda$ and set $t = e^u$ to get

$$\frac{1}{i \log A - \lambda} = \int_{-\infty}^{+\infty} \frac{e^u}{A + e^u} \frac{du}{\pi^2 - (\lambda - iu)^2}\,, \quad |\mathrm{Re}(\lambda)| > \pi\,. \tag{5.1}$$

Now take $\phi : |\phi| < \pi - \theta$. The operator $A\, e^{i\phi}$ is of type $\theta + |\phi|$ and we have $\log(A\, e^{i\phi}) = \log A + i\phi$ ([13]). Substituting $Ae^{i\phi}$ for A we get

$$\frac{1}{i \log A - \phi - \lambda} = \int_{\mathbb{R}} \frac{e^u}{A\, e^{i\phi} + e^u} \frac{du}{\pi^2 - (\lambda - iu)^2}\,, \quad |Re(\lambda)| > \pi\,.$$

Setting $\lambda = \alpha + i\beta$ we have the estimate

$$\Big\| \int_{\mathbb{R}} \frac{e^u}{A\, e^{i\phi} + e^u} \, \frac{1}{\pi^2 - (\lambda - iu)^2} \, du \Big\| \le M_\phi \int_{\mathbb{R}} \Big| \frac{1}{\pi^2 - (\alpha + i\beta - iu)^2} \Big| \, du$$

$$= M_\phi \int_{\mathbb{R}} \Big| \frac{1}{\pi^2 - (\alpha - is)^2} \Big| \, ds \quad (s = u - \beta)\,, \quad \text{and since}$$

$$\Big| \frac{1}{\pi^2 - (\alpha - is)^2} \Big| = \frac{1}{\big|\pi - (\alpha - is)\big| \; \big|\pi + (\alpha - is)\big|}$$

$$= \frac{1}{\big|is - (\alpha - \pi)\big| \; \big|is - (\alpha + \pi)\big|} \le \frac{1}{\sqrt{s^2 + \big(|\alpha| - \pi\big)^2} \; \sqrt{s^2 + \big(|\alpha| - \pi\big)^2}}$$

$$= \frac{1}{s^2 + \big(|\alpha| - \pi\big)^2} \quad \text{(here use} \quad |\alpha \pm \pi| \ge |\alpha| - \pi)\,, \quad \text{we finally get}$$

$$\Big\| \frac{1}{i \log A - \phi - \lambda} \Big\| \le M_\phi \int_{\mathbb{R}} \frac{ds}{s^2 + \big(|\alpha| - \pi\big)^2} = \frac{M_\phi \pi}{|\alpha| - \pi}\,, \quad \text{i.e.}$$

$$\Big\| \frac{1}{i \log A - \phi - \lambda} \Big\| \le \frac{M_\phi \pi}{|\mathrm{Re}\lambda| - \pi}\,, \quad \big(|Re\,\lambda| > \pi\big)\,.$$

Since this is true for $\pm\,\phi$, it can be rewritten as

$$\Big\| \frac{1}{\psi \pm i \log A + \mu} \Big\| \le \frac{\pi M_{\pi - \psi}}{Re(\mu)}\,, \quad \big(\mathrm{Re}(\mu) > 0\big)$$

$\forall\, \psi : \theta < \psi < \pi$ with $\pm\psi = \pi - \psi,\ \mu = \pm(\lambda - \pi)$, which shows that the operators $\psi \pm i \log A$ are of type $\pi/2$ and also implies (2.3). □

Proof of Proposition 2.2. We start with the representation (5.1) above. In view of the formula [9; 4.9. (1)]

$$\frac{1}{\pi^2 + (u + i\lambda)^2} = \frac{-1}{\pi} \int_0^\infty \sinh(\pi\, t)\, e^{-\lambda t}\, e^{iut}\, dt\,, \quad \lambda > \pi\,, \tag{5.2}$$

we can write for every $\lambda > \pi$:

$$\frac{i}{\lambda - i\log A} = \int_{\mathbb{R}} \frac{e^u}{A+e^u} \left\{ \int_0^\infty \frac{\sinh(\pi t)}{\pi}\, e^{-\lambda t}\, e^{iut}\, dt \right\} du$$

$$= \frac{1}{i} \int_{\mathbb{R}} \frac{e^u}{A+e^u} \frac{d}{du} \left\{ \int_0^\infty \frac{\sinh(\pi t)}{\pi t}\, e^{-\lambda t}\, e^{iut}\, dt \right\} du$$

or, after integration by parts, in weak sense,

$$\frac{1}{\lambda - i\log A} = \int_{\mathbb{R}} \frac{A\, e^u}{(A+e^u)^2} \left\{ \int_0^\infty \frac{\sinh(\pi t)}{\pi t}\, e^{-\lambda t}\, e^{iut} \right\} du$$

(note that the intermediate term is zero by the Riemann-Lebesgue lemma, since $e^u(A+e^u)^{-1}$ is bounded). Changing the order of integration, we get for every $x \in D(A) \cap D(A^{-1})$

$$\big(\lambda - i\log A\big)^{-1} x = \int_0^\infty e^{-\lambda t} \left\{ \frac{\sinh(\pi t)}{\pi t} \int_{\mathbb{R}} e^{iut} \frac{A\, e^u}{(A+e^u)^2}\, x\, du \right\} dt$$

$$= \int_0^\infty e^{-\lambda t}\, A^{it}\, x\, dt\,. \tag{5.3}$$

Suppose now that $i\log A$ generates a C_0-group of operators, say $T(t)$, of exponential type ω. Then we have the standard Laplace representation

$$\big(\lambda - i\log A\big)^{-1} x = \int_0^\infty e^{-\lambda t}\, T(t)\, x\, dt \quad \big(\forall\, \lambda > \omega,\ \forall\, x \in X\big)\,.$$

Comparing this to (5.3) we find that $T(t)x = A^{it}x$ for all $t \geq 0$ and all $x \in D(A) \cap D(A^{-1})$. In a similar manner the equality is proved for all $t < 0$. We see that $T(t)$ extends A^{it} from $D(A) \cap A(A^{-1})$ to X. $\square$

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References

1. K. Boyadzhiev and R. deLaubenfels, Semigroups and resolvents of bounded variation, imaginary powers and H^∞-calculus, *Semigroup Forum* **45** (1992), 372–384.
2. K. Boyadzhiev and R. deLaubenfels, Spectral theorem for unbounded groups of operators on Hilbert space, *Proc. Amer. Math. Soc.* **120** (1994), 127–136.
3. I. Cioranescu and L. Zsido, Analytic generators for one-parameter groups, *Tohoku Math. Journal* **28** (1976), 327–362.
4. M. Cowling, I. Doust, A. McIntosh and A. Yagi, Banach space operators with a bounded H^∞-functional calculus, (Preprint).
5. R. deLaubenfels, Inverses of generators, *Proc. Amer. Math. Soc.* **104** (1988), 443–448.
6. E.B. Davies, *One-parameter Semigroups,* Academic Press 1980.
7. G. Dore and A. Venni, On the closedness of the sum of two closed operators, *Math. Z.* **196** (1987), 189–201.
8. G. Dore and A. Venni, Some results about complex powers of closed operators, *J. Math. Anal. Appl.* **149** (1990), 124–136.
9. A. Erdélyi (Ed), *Tables of Integral Transforms,* vol. I, McGraw-Hill, New York, 1954.
10. A. McIntosh, Operators which have an H^∞-functional calculus. Miniconference on operator theory and partial differential equations, *Proc. Centre Math. Analysis, A.N.U. Canberra* **14** (1986), 210–231.
11. H. Komatsu, Fractional powers of operators, *Pacific J. Math.* **19** (1966), 285–346.
12. R. Nagel (ed), *One-Parameter Semigroups of Positive Operators,* Lecture Notes in Mathematics 1184, Springer Verlag, 1986.
13. V. Nollau, Über den Logarithmus abgeschlossener Operatoren in Banaschen Raümen, *Acta Sci. Math. (Szeged)* **30** (1969), 161–174.
14. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer Verlag, 1983.
15. J. Pruss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* **284** (1984), 847–857.
16. J. Pruss and H. Sohr, On operators with bounded imaginary powers in Banach spaces, *Mat. Z.* **203** (1990), 429–452.
17. J.A. Van Casteren, *Generators of Strongly Continuous Semigroups,* Pitman, 1985.
18. A. Yagi, Coincidance entre des espaces d'interpolation et des domaines de puissances fractionaire d'operateurs, *C.R. Acad. Sci. Paris (Ser. I)* **299** (1984), 173–176.