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CS-barrelled spaces

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Abstract

This note contains a proof of the following result: Let E be a locally convex space. Every absorbing absolutely convex CS-closed subset of E is a neighborhood of zero in E if and only if every sequentially closed linear map from E into an arbitrary Banach space is continuous. Some consequences and remarks are also given.

1. Introduction

Let E= (E, τ) be a Hausdorff locally convex topological vector space (lcs). Recall that a lcs E is *barrelled* if every *barrel* in E (i.e. an absolutely convex absorbing and closed set) is a neighborhood of zero in E. It is known (cf. e.g. [1] or [10]) that E is barrelled iff every closed linear map (i.e. with closed graph) of E into an arbitrary Banach space is continuous.

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The aim of this note is to obtain a similar characterization for locally convex spaces E (called *CS-barrelled*) with "closed linear map" replaced by "sequentially closed linear map".

In [12] Snipes characterized locally convex spaces E (called *C-sequential*, cf. also [4], [5] and [11]) for which every sequentially continuous linear map of E into an arbitrary Banach space is continuous. He showed that (E, τ) is C-sequential iff every convex sequentially open subset of E is open iff τ equals the sequential topology τ_{cs} (for the definition of τ_{cs} see [15], p. 341). It is however unknown if a similar characterization holds for *Mazur* spaces, i.e. lcs for which every sequentially continuous linear functional is continuous; see [11] and [14] for information concerning Mazur spaces. The class of CS-barrelled spaces is strictly included in the class of C-sequential spaces; clearly every metrizable non-barrelled space provides an example of a C-sequential space which is not CS-barrelled.

We have the following obvious implications: CS-barrelled$\Rightarrow$ barrelled,

$$\text{CS-barrelled} \Rightarrow \text{C-sequential} \Rightarrow \text{Mazur}.$$

By $\mathcal{F}(\tau)$ or $\mathcal{F}(E)$ we denote the filter of all neighborhoods of zero in (E, τ). A bounded absolutely convex subsets B of E will be called a *disc*. B is a *Banach disc* if the linear hull E_B of B in E endowed with the Minkowski functional norm topology is a Banach space. Denote $Q = \{(t_n) : t_n \geq 0, \sum_n t_n = 1\}$. By a *convex series* of elements of a subset A of E we will mean a series of the form $\sum_n t_n x_n$, where $x_n \in A$ and $(t_n) \in Q$, $n \in \mathbb{N}$. A subset A of E will be called (cf. [8])

(i) *CS-closed*, if it contains the sum of every convergent convex series of its elements.

(ii) *CS-compact*, if every convex series of its elements converges to an element of A.

Besides the remarkable properties of CS-compact and CS-closed sets, these concepts have been also used to simplify some of fundamental closed graph theorems, cf. e.g. [7], [8].

Let us recall a few interesting properties of these sets; for details and more information we refer to [6], [8], [9].

(1) Every CS-compact set is convex and bounded.

(2) Balanced CS-compact sets are Banach discs. Open or sequentially closed convex sets are CS-closed.

(3) If E is metrizable and A is a CS-closed subset of E, then $\operatorname{Int} \overline{A} = \operatorname{Int} A$, $A \subset E$.

(4) Let $T : E \to F$ be a linear map of E into a lcs F and A a CS-compact subset of F. If T is sequentially closed, i.e. the graph of T is sequentially closed, then $T^{-1}(A)$ is CS-closed in E.

It is well-known, cf. e.g. [1], that if E is locally complete and bornological, then E is ultrabornological (i.e. every absolutely convex set in E which absorbs the Banach discs is a neighborhood of zero). If E is as above, then applying (3) one has that every absorbing absolutely convex CS-closed subset of E (we call such a set a *CS-barrel*) is a neighborhood of zero (since it absorbs any Banach disc of E). This suggests the following definition.

A lcs E will be called CS-barrelled if every CS-barrel in E is a neighborhood of zero in E.

Our main result of this note is the following

Theorem

A lcs E is CS-barrelled iff every sequentially closed linear map of E into an arbitrary Banach space is continuous.

2. Proof of Theorem

The proof of Theorem follows from the following two propositions.

Proposition 1

Let E be a CS-barrelled space and F a lcs containing an absorbing absolutely convex CS-compact set A. If $T : E \to F$ is a sequentially closed linear map, then T is continuous as a map from E into F_A.

Proof. Since A is CS-compact , then $T^{-1}(A)$ is a CS-barrel in E; hence $T^{-1}(A) \in \mathcal{F}(E)$ and $T : E \to F_A$ is continuous. □

In [9] Mahowald proved that E is barrelled if every closed linear map of E into an arbitrary Banach space is continuous. We have also the following

Proposition 2

If every sequentially closed linear map of a lcs E into an arbitrary Banach space is continuous, then E is CS-barrelled.

Proof. Assume that U is a CS-barrel in E. Set $U_n = 2^{-n}U$, $n \in \mathbb{N}$, and $P = \bigcap_n U_n$. Let $q : E \to E/P$ be the canonical map. Put $W_n = q(U_n)$, $n \in \mathbb{N}$. Since $\bigcap_n 2^{-n}q(U) = 0$, then $q(U)$ generates a normed topology β on E/P. Let H be the completion of $(E/P, \beta)$ and q again denotes the canonical map of E into H.

Observe that the graph of q is sequentially closed in $E \times H$: Let (x_n) be a sequence in E such that $x_n \to 0$ and $q(x_n) \to y$ in H. We have to show that $y = 0$. For every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $q(x_s - x_p) \in W_k$ for all $s, p \geq n(k)$. Put $y_k = x_{n(k+1)} - x_{n(k+2)}$, $k \in \mathbb{N}$. Then $y_k \in U_{k+1} + P \subset U_k$, so there exist $u_k \in U$ such that $y_k = 2^{-k}u_k$, $k \in \mathbb{N}$. Since U is CS-closed and $x_{n(k+1)} = \sum_{m=k}^{\infty} y_m = \sum_{m=k}^{\infty} 2^{-m}u_m = 2^{-k+1}\sum_{m=1}^{\infty} 2^{-m}u_{m+k-1}$, then $x_{n(k+1)} \in 2^{-k+1}U$. Therefore $q(x_{n(k+1)}) \in W_{k-1}$, $k \in \mathbb{N}$. Consequently, $q(x_{n(k)}) \to 0$ in $(E/P, \beta)$. Hence $y = 0$.

By assumption q is continuous. Hence $q^{-1}(\overline{q(U)}) \in \mathcal{F}(E)$. On the other hand

$$q^{-1}(\overline{q(U)}) = q^{-1}(\overline{q(U)} \cap E/P) \subset q^{-1}(q(U) + q(U)) \subset 3U.$$

Hence $U \in \mathcal{F}(E)$ and E is CS-barrelled. □

3. Remarks and more properties

Using our Theorem one obtains the following

Corollary 1

Inductive limits of CS-barrelled spaces are CS-barrelled. Hausdorff quotients of CS-barrelled spaces are CS-barrelled. In particular inductive limits of metrizable barrelled spaces are CS-barrelled.

Note that there exist CS-barrelled spaces which are not the inductive limit of a family of metrizable barrelled spaces, [3].

Since every CS-barrelled space is barrelled and for barrelled spaces the Banach-Steinhaus theorem holds, our Theorem applies also to get

Corollary 2

A lcs E is CS-barrelled iff every pointwise bounded family of sequentially closed linear maps of E into an arbitrary Banach space is equicontinuous.

Recall that there exist metrizable and barrelled spaces which are not Baire, cf. e.g. [1]; therefore CS-barrelled spaces which are not Baire exist. Clearly Baire lcs are barrelled; there exist however Baire lcs which are not CS-barrelled, see Proposition 3. It is known also that a lcs which contains a dense barrelled (or Baire) subspace is barrelled (Baire); for CS-barrelled (or C-sequential or Mazur) spaces a similar result fails (comp. also [1], Proposition 6.2.16):

Proposition 3

Let $(E_s)_{s\in S}$ be a family of metrizable and complete lcs, where card $S > \aleph_0$ and E its topological product. Let $E_0 = \{(x_s) \in E : \{s : x_s \neq 0\}$ is countable$\}$. Then E_0 endowed with the relative topology is Baire and CS-barrelled but no subspace L of E such that $E_0 \subset L \subset E$ and $0 <$dim$(L/E_0) < \infty$ is a Mazur space.

Proof. Using our Corollary 1 and Proposition 2 of [2] one deduces that E_0 is CS-barrelled. By Theorem 4.11 of [13] E_0 is Baire; clearly E_0 is dense and sequentially closed in E. To complete the proof it is enough to show that for $x_0 \in E\backslash E_0$, the space $L = E_0 + \text{lin}\{x_0\}$ is not a Mazur space. In fact, the functional $x + \alpha x_0 \to \alpha$, $x \in E_0$, is sequentially continuous but not continuous. □

Hence the space $\mathbb{R}^{\mathbb{R}}$ provides a simple example of a CS-barrelled Baire space containing a dense subspace which is Baire but not CS-barrelled. The space $(l^1, \sigma(l^1, l^\infty))$ is an example of a Mazur space which is not C-sequential (because of the Schur property of l^1); for another examples, see [11].

Nevertheless one obtains the following: If E is a lcs containing a sequentially dense subspace which is CS-barrelled (or C-sequential or Mazur), then E is a space of the same type, respectively.

Similarly as for barrelled spaces one has the following

Proposition 4

Let E be a CS-barrelled space and F its closed countable-codimensional subspace. Then F is CS-barrelled and every algebraic complement of F in E is a topological complement and carries the finest locally convex topology.

Proof. Since E is barrelled, F is barrelled and any algebraic complement of F in E is a topological complement and carries the finest locally convex topology, cf. e.g. [1], Proposition 4.5.22. Now Corollary 1 applies to complete the proof. □

Another permanence properties of CS-barrelled spaces, their applications to continuous function spaces $C(X)$ will be given in a separate paper.

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References

1. P. Peréz-Carreras and J. Bonet, *Barrelled locally convex spaces,* North-Holland Math. Studies, North-Holland, Amsterdam, 1978.
2. S. Dierolf, The barrelled space associated with a bornological space need not be bornological, *Bull. London Math. Soc.* **12** (1980), 60–62.
3. S. Dierolf and J. Kąkol, On s-barrelled spaces, (to appear).
4. R.M. Dudley, On sequential convergence, *Trans. Amer. Math. Soc.* **112** (1964), 483–507.
5. R.M. Dudley, Convergence of sequences of distributions, *Proc. Amer. Math. Soc.* **27** (1971), 531–534.
6. D. H. Fremlin and M. Talagrand, On CS-closed sets, *Mathematica* **26** (1979), 30–32.
7. R.B. Holmes, *Geometric Functional Analysis and its Applications,* Graduate texts in Math. **24** Springer Berlin, 1975.
8. G.J. O. Jameson, Convex series, *Proc. Camb. Phil. Soc.* **72** (1972), 37–47.
9. H. König and G. Wittstock, Superconvex sets and σ-convex sets, and the embedding of convex and superconvex spaces, *Note di Mat.* (Memorial Volume G. Köthe) (to appear).
10. M. Mahowald, Barrelled spaces and the closed graph theorem, *J. London Math. Soc.* **36** (1961), 108–110.
11. M. Oberguggenberger, Über Folgenkonvergenz in lokalkonvexen Räumen, *Math. Nachr.* **129** (1986), 219–234.
12. R.F. Snipes, C-sequential and S-bornological Topological Vector Spaces, *Math. Ann.* **202** (1973), 273–283.
13. A. Todd and S. Saxon, A property of locally convex Baire spaces, *Math. Ann.* **206** (1973), 23–29.
14. B. Tsirulnikov, Inheritance and preservation of barrelledness properties in locally convex spaces, Thesis 1980, Tel-Aviv.
15. J.H. Webb, Sequential convergence in locally convex spaces, *Proc. Cambr. Phil. Soc.* **64** (1968), 341–364.