Collect. Math. **45**, 3 (1994), 245–261

On certain trigonometric sums in several variables

P. Haukkanen

Department of Mathematical Sciences, University of Tampere,
P.O. Box 607, SF-33101 Tampere, Finland

R. Sivaramakrishnan

Department of Mathematics, University of Calicut, Kozhikode 673 635, India

Received September 22, 1993. Revised October 13, 1994

Abstract

The Hilbert space structure of totally even functions $\pmod r$ which depend on extended Ramanujan sums is described. The function ε_k defined as the quotient of Jordan's J_k-function and Euler's ϕ-function is introduced as a new generalization of the Dedekind ψ-function. Using the basic methods of totally even functions $\pmod r$, we point out that ε_k has also a purpose to serve in obtaining the k-dimensional analogue of an identity due to P. Kesava Menon.

1. Introduction

Let r be an arbitrary but fixed positive integer, and let $\mathbb{Z}/r\mathbb{Z}$ denote the cyclic group of integers $\pmod r$ under addition modulo r. The Euler totient function $\phi(r)$ is defined as the number of integers $a \pmod r$ such that $(a, r) = 1$. For $k \geq 1$, $\mathbb{Z}^k/r\mathbb{Z}^k$ is the group of k-vectors $\{a_i\} \pmod r$ under pointwise addition modulo r. Jordan's totient $J_k(r)$ is the number of k-vectors $\{a_i\} \pmod r$ such that $((a_i), r) = 1$, where (a_i) is the g.c.d. of $a_1, a_2, \ldots, a_k$. Jordan's totient J_k is clearly a generalization of Euler's totient ϕ. For properties of J_k, see Eckford Cohen [1, 2, 4], or Sivaramakrishnan [15].

It is easy to see that

$$\mathbb{Z}^k/r\mathbb{Z}^k = \mathbb{Z}/r\mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r\mathbb{Z} = (\mathbb{Z}/r\mathbb{Z})^k$$

and $\mathbb{Z}^k/r\mathbb{Z}^k$ is thus a direct sum of k (≥ 1) cyclic groups each of order r. For this reason, $\mathbb{Z}^k/r\mathbb{Z}^k$ is a homogeneous finite abelian group of order r^k. Jordan's totient J_k has a bearing on this aspect of $\mathbb{Z}^k/r\mathbb{Z}^k$, see Eckford Cohen [3].

Ramanujan's sum $C(n,r)$ is defined by

$$C(n,r) = \sum_{\substack{a \pmod{r} \\ (a,r)=1}} \exp(2\pi i n a/r), \tag{1.1}$$

where $n \in \mathbb{Z}$. The arithmetical representation of $C(n,r)$ is given [2] by

$$C(n,r) = \sum_{d \mid (n,r)} d\mu(r/d), \tag{1.2}$$

where μ is the Möbius function.

We could naturally anticipate the k-dimensional analogue of Ramanujan's sum $C(n,r)$. This was achieved by Eckford Cohen [2] in the following manner. The extended Ramanujan sum $C^{(k)}(n,r)$ is defined by

$$C^{(k)}(n,r) = \sum_{\substack{\{a_i\} \pmod{r} \\ ((a_i),r)=1}} \exp(2\pi i n(a_1 + \cdots + a_k)/r). \tag{1.3}$$

The evaluation of $C^{(k)}(n,r)$ in terms of $J_k(r)$ is

$$C^{(k)}(n,r) = \frac{J_k(r)\mu(m)}{J_k(m)}, \quad m = \frac{r}{(n,r)}. \tag{1.4}$$

Details are given in [2]. In [5] Eckford Cohen also considers the sum

$$C(n_1,\ldots,n_k;r) = \sum_{\substack{\{a_i\} \pmod{r} \\ ((a_i),r)=1}} \exp(2\pi i(n_1a_1 + \cdots + n_ka_k)/r). \tag{1.5}$$

If $n = (n_1,\ldots,n_k)$, it is shown [5] that

$$C(n_1,\ldots,n_k;r) = C^{(k)}(n,r). \tag{1.6}$$

In applications we need the following evaluation [2]:

$$C^{(k)}(n, r) = \sum_{d \,|\, (n,r)} d^k \mu(r/d). \tag{1.7}$$

The main features of this paper are as detailed below:

(i) An arithmetical function f is said to be an even function of $n \pmod r$ if $f(n, r) = f((n, r), r)$ for all n [2]. Totally even functions (mod r) is a k-vector generalization of even functions (mod r) [5]. We here describe the Hilbert space structure of the class of totally even functions (mod r) in terms of extended Ramanujan sums. See Section 2. This is a generalization of a previous result for even functions (mod r) published in [8].

(ii) The Dedekind ψ-function is given by

$$\psi(r) = r \prod_{p \,|\, r} (1 + p^{-1}), \tag{1.8}$$

and is known to be a totient [11]. Further,

$$\psi(r) = \frac{J_2(r)}{\phi(r)}, \tag{1.9}$$

where J_k is the Jordan totient and ϕ is the Euler totient. In Section 3, we introduce the function ε_k given by

$$\varepsilon_k(r) = \frac{J_k(r)}{\phi(r)} \quad (k \geq 1). \tag{1.10}$$

It appears that ε_k gives the most-suited generalization of ψ.

(iii) Finally, ε_k has also a purpose to serve in obtaining a k-vector extension of the following identity due to P. Kesava Menon [9]

$$\sum_{\substack{a \pmod r \\ (a,r)=1}} (a - 1, r) = \phi(r)\tau(r), \tag{1.11}$$

where $\tau(r)$ denotes the number of divisors of r. This is shown in Theorem 9. See Section 4. The method adopted is on the lines of the generalization of (1.11) given by Nageswara Rao in [13]. This needs the notion of totally even functions (mod r) considered in Section 2.

2. The Hilbert space of totally even functions $(\mathrm{mod}\ r)$

We recall that a complex-valued function f of the integral variable n, written $f(n,r)$, is called [2] an even function of n (mod r) if $f(n,r) = f((n,r),r)$. Ramanujan's sum $C(n,r)$ is an example of an even function (mod r). It is known that the set $B_r(\mathbb{C})$ of even functions (mod r) forms a finite-dimensional normed linear space with

$$\{(r\phi(d))^{-1/2}C(n,d) : d \mid r\} \tag{2.1}$$

forming an orthonormal basis. See [8].

If $f \in B_r(\mathbb{C})$, f can be uniquely expressed as

$$f(n,r) = \sum_{d \mid r} \alpha(d,r)C(n,d), \tag{2.2}$$

where

$$\alpha(d,r) = r^{-1}\sum_{\delta \mid r} f(r/\delta,r)C(r/d,\delta). \tag{2.3}$$

For $d \mid r$, $\alpha(d,r)$ are referred to as the Fourier coefficients of f. For $f, g \in B_r(\mathbb{C})$, their Cauchy product is defined by

$$(f \circ g)(n,r) = \sum_{a+b \equiv n \pmod r} f(a,r)\ g(b,r), \tag{2.4}$$

where the summation is over residues a, b (mod r) such that $a + b \equiv n$ (mod r). Cauchy multiplication is both commutative and associative, and this gives $B_r(\mathbb{C})$ the structure of a finite-dimensional algebra. We do not go into the details.

If $g(n,r)$ is given by

$$g(n,r) = \sum_{d \mid r} \beta(d,r)C(n,d), \tag{2.5}$$

it is known [11, 14] that

$$(f \circ g)(n,r) = r\sum_{d \mid r} \alpha(d,r)\beta(d,r)C(n,d). \tag{2.6}$$

The generalization of $B_r(\mathbb{C})$ to the space of k-vectors (mod r) has been achieved by Eckford Cohen in [5], and in the generalized set-up we confine ourselves to the space of totally even functions (mod r) about which we make a study now.

DEFINITION. A complex-valued function $f(n_1, \ldots, n_k; r)$ is said to be totally even (mod r) if there exists an even function $F(n, r)$ (mod r) such that

$$f(n_1, \ldots, n_k; r) = F((n_1, \ldots, n_k), r). \tag{2.7}$$

DEFINITION. In $\mathbb{Z}^k/r\mathbb{Z}^k$, two k-vectors $\{a_i\}$ and $\{b_i\}$ are called "associates" (or $\{a_i\}$ is an associate of $\{b_i\}$), written $\{a_i\} \sim \{b_i\}$, if and only if

$$((a_1, \ldots, a_k), r) = ((b_1, \ldots, b_k), r). \tag{2.8}$$

It is clear that "associate of" is an equivalence relation on $\mathbb{Z}^k/r\mathbb{Z}^k$ and it partitions $\mathbb{Z}^k/r\mathbb{Z}^k$ into mutually disjoint classes $[1], \ldots, [t], \ldots, [r]$, where a class $[t]$ is uniquely determined by the divisor t of r. Obviously there are $\tau(r)$, the number of divisors of r, associate classes.

It is easy to see that a totally even function (mod r) is completely determined if for each associate class $[t]$, where $t \mid r$, the function value is known in one k-vector of $[t]$.

DEFINITION. If f and g are totally even functions (mod r) in the variables $n_1, \ldots, n_k$, then their Cauchy product, written $f \circ g$, is defined by

$$(f \circ g)(n_1, \ldots, n_k; r) = \sum_{\substack{a_i + b_i \equiv n_i \pmod{r} \\ (i=1,\ldots,k)}} f(a_1, \ldots, a_k; r)\, g(b_1, \ldots, b_k; r), \tag{2.9}$$

where a_i, b_i range over residues (mod r) such that $a_i + b_i \equiv n_i \pmod{r}$ for $i = 1, 2, \ldots, k$.

We write $T_{k,r}(\mathbb{C})$ to denote the set of totally even functions (mod r) in the variables $n_1, n_2, \ldots, n_k$. It is known [5] that $f \in T_{k,r}(\mathbb{C})$ has the representation

$$f(n_1, \ldots, n_k; r) = \sum_{d \mid r} \alpha(d, r) C(n_1, \ldots, n_k; d), \tag{2.10}$$

where

$$\alpha(d, r) = r^{-k} \sum_{\delta \mid r} F(r/\delta, r) C^{(k)}(r/d, \delta) \tag{2.11}$$

with $F(n, r)$ being the even function (mod r) associated with $f \in T_{k,r}(\mathbb{C})$, or

$$\alpha(d, r) = r^{-k} [J_k(d)]^{-1} \sum_{\{a_i\} \pmod{r}} f(a_1, \ldots, a_k; d) C(a_1, \ldots, a_k; d). \tag{2.12}$$

In (2.11) or (2.12), $\alpha(d, r)$, for $d \mid r$, are called the Fourier coefficients of f.

If $f, g \in T_{k,r}(\mathbb{C})$ have Fourier coefficients $\alpha(d, r)$ and $\beta(d, r)$, respectively, then their Cauchy product (2.9) has the representation

$$(f \circ g)(n_1, \ldots, n_k; r) = r^k \sum_{d \mid r} \alpha(d, r)\beta(d, r)C(n_1, \ldots, n_k; d). \tag{2.13}$$

It is easy to verify that $T_{k,r}(\mathbb{C})$ has the structure of a finite-dimensional algebra using Cauchy multiplication. Just as $B_r(\mathbb{C})$ is a normed linear space, we can make $T_{k,r}(\mathbb{C})$ a normed linear space. This is shown by obtaining an inner product for pairs of elements in $T_{k,r}(\mathbb{C})$.

Theorem 1

For $f, g \in T_{k,r}(\mathbb{C})$, let F and G be the respective associated even functions (mod r). Further, let $\bar{f}$ (resp. $\bar{F}$) denote the complex conjugate of f (resp. F). Then

$$\langle f, g \rangle = \sum_{d \mid r} J_k(r/d)\ F(d, r)\ \overline{G(d, r)} \tag{2.14}$$

defines an inner product in $T_{k,r}(\mathbb{C})$, and

$$\langle f, g \rangle = r^k \sum_{d \mid r} \alpha(d, r)\overline{\beta(d, r)}J_k(d), \tag{2.15}$$

where α and β are respectively the Fourier coefficients of f and g.

Proof. It is easy to check that for $f_1, f_2, g \in T_{k,r}(\mathbb{C})$ and $z \in \mathbb{C}$
(i) $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$,
(ii) $\langle z\, f_1, g \rangle = z\, \langle f_1, g \rangle$,
(iii) $\langle g, f_1 \rangle = \overline{\langle f_1, g \rangle}$,
(iv) $\langle f_1, f_1 \rangle\ >\ 0$ if $f_1 \neq 0$.

Now, by (2.11), the right side of (2.15) is equal to

$$r^{-k} \sum_{d \mid r} \Big\{ \sum_{e_1 \mid r} F(r/e_1, r) C^{(k)}(r/d, e_1) \sum_{e_2 \mid r} \overline{G(r/e_2, r)} C^{(k)}(r/d, e_2) \Big\} J_k(d)$$

or

$$r^{-k} \sum_{e_1 \mid r} \sum_{e_2 \mid r} F(r/e_1, r)\overline{G(r/e_2, r)}\ \sum_{d \mid r} C^{(k)}(r/d, e_1) C^{(k)}(r/d, e_2) J_k(d). \tag{2.16}$$

Since

$$C^{(k)}(r/d, e_1)J_k(d) = J_k(e_1)C^{(k)}(r/e_1, d),$$

the inner sum in (2.16) can be written as

$$J_k(e_1)\sum_{d\,|\,r} C^{(k)}(r/e_1, d)C^{(k)}(r/d, e_2).$$

Further, by the orthogonal property [11] of $C^{(k)}(n, r)$,

$$\sum_{d\,|\,r} C^{(k)}(r/e_1, d)C^{(k)}(r/d, e_2) = \begin{cases} r^k & \text{if } e_1 = e_2, \\ 0 & \text{if } e_1 \neq e_2. \end{cases}$$

Thus, (2.16), or the right side of (2.15), is equal to

$$\sum_{e_1\,|\,r} F(r/e_1, r)\overline{G(r/e_1, r)}J_k(e_1).$$

Therefore (2.15) is a consequence of (2.14). This completes the proof of Theorem 1. □

Remark. By (2.13) and (2.15), for $f, g \in T_{k,r}(\mathbb{C})$,

$$\langle f, g\rangle = (f \circ \overline{g})(0, \ldots, 0; r) \tag{2.17}$$

as $C(0, \ldots, 0; d) = J_k(d)$.

Theorem 2

$T_{k,r}(\mathbb{C})$ forms a Hilbert space under the inner product defined in (2.14).

Proof. As in the case of the Hilbert space of even functions (mod r) [8], we make use of a measure-theoretic approach to get at the structure of $T_{k,r}(\mathbb{C})$.

Let X denote the set of associate classes $[t]$ considered in (2.8). If B denotes the power set of X, then (X, B) is a measurable space. We define a measure m on B by writing

$$m([t]) = J_k(r/t) \text{ for all } t \mid r.$$

This yields a measure space (X, B, m). For a complex-valued function f on X, the integral of f over X is given by

$$\int_X f dm = \sum_{t\,|\,r} f([t])J_k(r/t).$$

The vector space $L^2(X, B, m)$ of all measurable functions $f : X \to \mathbb{C}$ such that $|f|^2$ is integrable over X forms a Hilbert space under the inner product

$$\langle f, g\rangle = \int_X f\bar{g}\; dm.$$

It is clear that $L^2(X, B, m)$ consists of all complex-valued functions on X. Since every totally even function $f \pmod r$ is completely determined by its values on X, the vector-spaces $L^2(X, B, m)$ and $T_{k,r}(\mathbb{C})$ are identical. We thus arrive at Theorem 2. □

Analogous to the case $B_r(\mathbb{C})$ of even functions $\pmod r$ [8], an orthonormal basis for $T_{k,r}(\mathbb{C})$ could be given in terms of $C(n_1, \ldots, n_k; r)$.

Theorem 3

The set

$$\{C(n_1, \ldots, n_k; d)(r^k J_k(d))^{-1/2} : d \mid r\} \tag{2.18}$$

forms an orthonormal basis of the vector space $T_{k,r}(\mathbb{C})$ *under the inner product (2.14).*

Proof. It is easy to see that the dimension of the vector space of complex-valued functions on X is $\tau(r)$, the number of divisors of r. That is, the dimension of $T_{k,r}(\mathbb{C})$ is $\tau(r)$. Further, it can be proved that the set (2.18) is orthonormal (cf. [8]). We thus obtain Theorem 3. □

Remark. The finite Fourier series representation of $f \in T_{k,r}(\mathbb{C})$ with respect to the orthonormal basis (2.18) leads to (2.10) and the application of the inner product expressions (2.14) and (2.15) gives the Fourier coefficient expressions (2.11) and (2.12), respectively.

3. An analogue of the Dedekind ψ-function

In (1.10), we gave the expression for ε_k which reduces to the Dedekind ψ-function when $k = 2$. We note that ε_k is the quotient of two totients for $k \geq 2$ and is itself a totient only for $k = 2$.

It is known [3] that Jordan's totient $J_k(r)$ gives the number of elements of order r in a homogeneous finite abelian group $G_r^{(k)} = C_r^{(1)} \oplus C_r^{(2)} \oplus \cdots \oplus C_r^{(k)}$, where $C_r^{(i)}$ is a cyclic group of order r $(i = 1, 2, \ldots, k)$, see [12] also. In fact, $G_r^{(k)}$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^k$, and $\{a_i\} \in (\mathbb{Z}/r\mathbb{Z})^k$ is of order r if and only if the l.c.m.$[o(a_1), o(a_2), \ldots, o(a_k)] = r$, where $o(a_i)$, $i = 1, 2, \ldots, k$, is the order of a_i in $\mathbb{Z}/r\mathbb{Z}$. Since $o(a_i) = r/(r, a_i)$, we see that

$$\big[o(a_1), o(a_2), \ldots, o(a_k)\big] = r \iff ((a_i), r) = 1.$$

This proves the group-theoretic interpretation of $J_k(r)$. Analogously, we have

Theorem 4

$\varepsilon_k(r)$ counts the number of cyclic subgroups of order r in $G_r^{(k)}$.

In [2], Eckford Cohen has proved among other things the following identity (see (41) of Section 5)

$$\frac{J_k((n,r))}{(n,r)^k} = \frac{J_{k+1}(r)}{r^{k+1}} \sum_{d \mid r} \frac{\mu(d)}{J_{k+1}(d)} C(n,d), \tag{3.1}$$

where μ is the Möbius function. Taking $n = 0$ in (3.1) and noting that $C(0,r) = \phi(r)$, we obtain

$$\frac{r\,\varepsilon_k(r)}{\varepsilon_{k+1}(r)} = \sum_{d \mid r} \frac{\mu(d)}{\varepsilon_{k+1}(d)} \quad (k \geq 1). \tag{3.2}$$

Next, for $k \geq 2$, let D_k denote the set of positive integers which are k-free in the sense that the highest power of a prime occurring as a factor of an integer t (> 1) in D_k is $\leq (k-1)$. Let L_k denote the set of positive integers which are k-full, that is, every integer $s \in L_k$ has the property that all its prime factors occur to a power $\geq k$. By convention, we take $D_k \cap L_k = \{1\}$.

Theorem 5

ε_k has the arithmetical representation

$$\varepsilon_k(r) = \sum_{d\delta = r^{k-1}} d\, q_k(\delta) \quad (k \geq 2), \tag{3.3}$$

where q_k is the characteristic function of D_k.

Proof. We write $F_k(r) = \sum_{d\delta=r^{k-1}} d\, q_k(\delta)$, the right side of (3.3). It is easy to see that F_k is multiplicative, that is,

$$F_k(r)F_k(s) = F_k(rs) \quad \text{whenever} \quad (r,s)=1.$$

Therefore it suffices to verify the truth of (3.3) when $r = p^m$, p a prime, $m \geq 1$. We have

$$F_k(p^m) = p^{m(k-1)}q_k(1) + p^{m(k-1)-1}q_k(p) + \cdots + p^{m(k-1)-(k-1)}q_k(p^{k-1}),$$

since $q_k(p^n) = 0$ for $n \geq k$. Then

$$F_k(p^m) = p^{m(k-1)}(1 + p^{-1} + p^{-2} + \cdots + p^{-(k-1)}) = \varepsilon_k(p^m).$$

This completes the proof of Theorem 5. □

Remark. The case $k = 2$ of (3.3) reads

$$\psi(r) = \sum_{d\delta=r} d\, q_2(\delta), \tag{3.4}$$

where q_2 is the characteristic function of the set of positive square-free integers. Equation (3.4) is due to Eckford Cohen [6].

The following theorem gives inequalities connected with J_k and ε_k.

Theorem 6

There exists a positive constant C such that, for $k \geq 2$,

$$\frac{1}{\zeta(k)}\, r^k < J_k(r) < r^k \quad (r > 1), \tag{3.5}$$

$$\frac{1}{\zeta(k)}\, r^{k-1} < \varepsilon_k(r) < C^{-1} r^{k-1} \log\log r \quad (r > 3), \tag{3.6}$$

where ζ is the Riemann ζ-function.

Proof. Jordan's totient J_k has the arithmetical representation [1]

$$J_k(r) = r^k \prod_{p \mid r}(1 - p^{-k}) > r^k \prod_{p}(1 - p^{-k}) = \frac{1}{\zeta(k)}\, r^k,$$

as $\zeta(k) = \prod_p (1 - p^{-k})^{-1}$. This proves (3.5). It is known [10, Chapter 6] that there exists a positive constant C such that

$$\frac{Cr}{\log\log r} < \phi(r) < r \quad (r > 3).$$

As $\varepsilon_k = J_k/\phi$, (3.6) follows from (3.5). This completes the proof of Theorem 6. □

Corollary

The series $\sum_{r=1}^{\infty} \frac{1}{\varepsilon_k(r)}$ *diverges for* $k \leq 2$, *and converges for* $k \geq 3$.

We now come to the expression for $\zeta(k)$ in terms of ε_k. In [6], Eckford Cohen has shown that

$$\sum_{\substack{r=1 \\ r \in L_2}}^{\infty} \frac{1}{\psi(r)} = \zeta(2), \tag{3.7}$$

where the summation on the left is over the positive integers r whose prime factors occur to a power ≥ 2, that is, r is square-full. We give a generalization of (3.7) in

Theorem 7

For $k \geq 2$,

$$\sum_{\substack{r=1 \\ r \in L_k}}^{\infty} \frac{r^{k-2}}{\varepsilon_k(r)} = \zeta(k), \tag{3.8}$$

where r *runs through* k*-full positive integers.*

Proof. If a_k denotes the characteristic function of L_k, we have

$$\sum_{\substack{r=1 \\ r\in L_k}}^{\infty} \frac{r^{k-2}}{\varepsilon_k(r)} = \sum_{r=1}^{\infty} \frac{a_k(r)\, r^{k-2}}{\varepsilon_k(r)}.$$

By (3.6) we note that for $r > 3, k \geq 2$

$$\frac{a_k(r)\, r^{k-2}}{\varepsilon_k(r)} \leq \frac{a_k(r)\, \zeta(k)}{r}.$$

As $a_k(r) = 0$ for $r \notin L_k$, $\sum\limits_{r=1}^{\infty} \frac{a_k(r)}{r}$ is comparable with $\sum\limits_{m=1}^{\infty} \frac{1}{m^k}$, which is convergent for $k \geq 2$. Thus the series on the left side of (3.8) converges for $k \geq 2$. We shall denote its sum by S_k. As ε_k is multiplicative, we have

$$\begin{aligned}
S_k &= \prod_p \sum_{m=0}^{\infty} \frac{a_k(p^m)\, p^{m(k-2)}}{\varepsilon_k(p^m)} \\
&= \prod_p \left\{ 1 + \sum_{m=k}^{\infty} \frac{p^{m(k-2)} \phi(p^m)}{J_k(p^m)} \right\} \\
&= \prod_p \left\{ 1 + \sum_{m=k}^{\infty} \frac{p-1}{p^{m-k+1}(p^k-1)} \right\} \\
&= \prod_p \left\{ 1 + \frac{p-1}{p(p^k-1)} \sum_{m=0}^{\infty} p^{-m} \right\} \\
&= \prod_p \left\{ 1 + \frac{1-p^{-1}}{p^k-1} (1-p^{-1})^{-1} \right\} \\
&= \prod_p (1-p^{-k})^{-1},
\end{aligned}$$

which is the Euler product form for $\zeta(k)$. This proves Theorem 7. □

DEFINITION. Liouville's function λ is defined by $\lambda(r) = (-1)^{\Omega(r)}$, where $\Omega(r)$ denotes the total number of prime factors of r, each being counted according to its multiplicity. Further, $\Omega(1) = 0$.

Theorem 8

For $k \geq 2$,

$$\sum_{r=1}^{\infty} \frac{\lambda(r)}{\varepsilon_{k+1}(r)} = \frac{\zeta(k+1)\zeta(2k)}{\zeta(2k+1)\zeta(k)}. \tag{3.9}$$

Proof. As λ and ε_{k+1} are multiplicative and as $\sum_{r=1}^{\infty} \frac{1}{\varepsilon_{k+1}(r)}$ converges for $k \geq 2$, we obtain the Euler product form of the left side of (3.9) as

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{\lambda(r)}{\varepsilon_{k+1}(r)} &= \prod_{p} \left\{ 1 + \sum_{m=1}^{\infty} (-1)^m \, p^{-mk} \, \frac{1-p^{-1}}{1-p^{-(k+1)}} \right\} \\
&= \prod_{p} \left\{ 1 - \frac{1-p^{-1}}{1-p^{-(k+1)}} \, p^{-k} \sum_{m=0}^{\infty} (-1)^m p^{-mk} \right\} \\
&= \prod_{p} \left\{ 1 - \frac{(1-p^{-1}) \, p^{-k}}{(1-p^{-(k+1)})(1+p^{-k})} \right\} \\
&= \prod_{p} \frac{1-p^{-(2k+1)}}{(1-p^{-(k+1)})(1+p^{-k})}
\end{aligned}$$

or

$$\sum_{r=1}^{\infty} \frac{\lambda(r)}{\varepsilon_{k+1}(r)} = \prod_{p} \frac{(1-p^{-(2k+1)})(1-p^{-k})}{(1-p^{-(k+1)})(1-p^{-2k})}.$$

The right side simplifies into the right side of (3.9), by virtue of the Euler product form of the ζ-function. This proves Theorem 8. □

More identities of the type (3.9) could be derived. We mention without proof the following identities valid for $k \geq 2$.

$$\sum_{r=1}^{\infty} \frac{1}{\varepsilon_{k+1}(r)} = \zeta(k)\zeta(k+1) \prod_{p} (1 - 2p^{-(k+1)} + p^{-(2k+1)}), \tag{3.10}$$

$$\sum_{r=1}^{\infty} \frac{\mu(r)}{\varepsilon_{k+1}(r)} = \frac{\zeta(k+1)}{\zeta(k)}, \tag{3.11}$$

$$\sum_{r=1}^{\infty} \frac{1}{\phi(r)\varepsilon_k(r)} = \sum_{r=1}^{\infty} \frac{1}{J_k(r)} = \frac{\zeta(k)\zeta(2k)\zeta(3k)}{\zeta(6k)}, \tag{3.12}$$

$$\sum_{r=1}^{\infty} \frac{\mu^2(r)}{\phi(r)\varepsilon_k(r)} = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{J_k(r)} = \zeta(k). \tag{3.13}$$

4. An identity involving ε_k

In (1.11), $(a-1,r)$ for $(a,r)=1$ occurs as the number of solutions of the congruence

$$(a-1)x \equiv 0 \pmod{r}$$

as pointed out in [9]. To seek a generalization in terms of k-vectors, we look at the number of solutions of the congruence

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k \equiv b \pmod{r}. \tag{4.1}$$

A necessary and sufficient condition for (4.1) to have a solution $\{x_1, x_2, \ldots, x_k\}$ is that $(a_1, a_2, \ldots, a_k, r) \mid b$. If this condition is satisfied, the number of incongruent (mod r) solutions is $r^{k-1}(a_1, a_2, \ldots, a_k, r)$, see P.J. McCarthy [11, Chapter 3]. Therefore the number of solutions of

$$(a_1-1)x_1 + (a_2-1)x_2 + \cdots + (a_k-1)x_k \equiv 0 \pmod{r}$$

is $r^{k-1}(a_1-1, a_2-1, \ldots, a_k-1, r)$. This motivates the generalization of (1.11) we are attempting. Our aim is to evaluate $\sum(a_1-1, a_2-1, \ldots, a_k-1, r)$, where the summation is over $\{a_i\}$ (mod r) with $((a_i), r) = 1$. The method is to make use of totally even functions (mod r). We need two particular totally even functions (mod r) (see (2.7)). They are

$$\eta(n_1, \ldots, n_k; r) = ((n_1, \ldots, n_k), r) \tag{4.2}$$

and

$$\rho(n_1, \ldots, n_k; r) = \begin{cases} 1 & \text{if } ((n_1, \ldots, n_k), r) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The lemmas given below furnish the representation of η and ρ in terms of $C(n_1, \ldots, n_k; r)$ on the basis of (2.10).

Lemma 1

η given by (4.2) has the representation

$$\eta(n_1, \ldots, n_k; r) = \sum_{d \mid r} \alpha(d,r) C(n_1, \ldots, n_k; d), \tag{4.4}$$

where

$$\alpha(d,r) = r^{-k} \sum_{\delta \mid r/d} \phi(r/\delta)\delta^k.$$

Proof. Using (2.10), (2.11) and (1.7), we note that the Fourier coefficients of η are given by

$$\begin{aligned}\alpha(d,r) &= r^{-k}\sum_{\delta\,|\,r}(r/\delta)C^{(k)}(r/d,\delta)\\ &= r^{-k}\sum_{\delta\,|\,r}(r/\delta)\sum_{\substack{t\,|\,r/d\\ t\,|\,\delta}} t^k\mu(\delta/t)\\ &= r^{-k}\sum_{\delta\,|\,r}\delta\sum_{\substack{t\,|\,r/d\\ t\,|\,r/\delta}} t^k\mu(r/(\delta t))\\ &= r^{-k}\sum_{t\,|\,r/d} t^k\sum_{\delta\,|\,r/t}\delta\mu(r/(\delta t))\\ &= r^{-k}\sum_{t\,|\,r/d} t^k\phi(r/t).\end{aligned}$$

This proves Lemma 1. □

Lemma 2

ρ given by (4.3) has the representation

$$\rho(n_1,\dots,n_k;r) = r^{-k}\sum_{\delta\,|\,r} C^{(k)}(r/d,r)C(n_1,\dots,n_k;d). \tag{4.5}$$

Proof. Using (2.10) and (2.11), we obtain the Fourier coefficients $\beta(d,r)$ of ρ as

$$\beta(d,r) = r^{-k}\sum_{\delta\,|\,r}\rho(r/\delta;r)C^{(k)}(r/d,\delta),$$

where $\rho(r/\delta;r) = 0$ for $\delta \neq r$, and $= 1$ for $\delta = r$. Therefore

$$\beta(d,r) = r^{-k}C^{(k)}(r/d,r)$$

and this gives (4.5), proving Lemma 2. □

Theorem 9

If J_k is Jordan's totient and ε_k as defined in (1.10), then

$$\sum_{\substack{\{a_i\}\ (\mathrm{mod}\ r)\\ ((a_i),r)=1}} (a_1-1,a_2-1,\dots,a_k-1,r) = J_k(r)\sum_{d\,|\,r}\frac{1}{\varepsilon_k(d)}. \tag{4.6}$$

Proof. Let $\{s_i\}$ be a k-vector with $((s_i), r) = 1$. In the notation of (2.9), taking $f = \eta$ and $g = \rho$, one obtains

$$\sum_{\substack{\{a_i\} \pmod r \\ ((a_i),r)=1}} (s_1 - a_1, s_2 - a_2, \ldots, s_k - a_k, r) = (\eta \circ \rho)(s_1, \ldots, s_k; r). \tag{4.7}$$

Denoting the left side of (4.7) by S, we have using (2.13) and Lemmas 1 and 2,

$$S = r^{-k} \sum_{d \mid r} \left\{ \sum_{\delta \mid r/d} \phi(r/\delta)\delta^k \right\} C^{(k)}(r/d, r) C(s_1, \ldots, s_k; d).$$

By virtue of the equations (1.4) and (1.6), since $((s_i), r) = 1$, $C(s_1, \ldots, s_k; r) = \mu(r)$. Therefore, from (1.4) again, we get

$$\begin{aligned} S &= r^{-k} \sum_{d \mid r} \left\{ \sum_{\delta \mid r/d} \phi(r/\delta)\delta^k \right\} J_k(r)\mu^2(d)/J_k(d) \\ &= r^{-k} J_k(r) \sum_{\delta \mid r} \phi(r/\delta)\delta^k \sum_{d \mid r/\delta} \mu^2(d)/J_k(d). \end{aligned}$$

But

$$\sum_{d \mid r} \mu^2(d)/J_k(d) = r^k/J_k(r),$$

(see [13, Lemma 2]). Therefore

$$\begin{aligned} S &= r^{-k} J_k(r) \sum_{\delta \mid r} \phi(r/\delta)\delta^k (r/\delta)^k / J_k(r/\delta) \\ &= J_k(r) \sum_{\delta \mid r} \frac{1}{\varepsilon_k(r/\delta)} \quad \text{as} \quad \varepsilon_k = J_k/\phi. \end{aligned}$$

Thus, when $\{s_i\} = \{1, 1, \ldots, 1\}$, we arrive at (4.6). This completes the proof of Theorem 9. □

Remarks. (i) When $k = 1$, $\varepsilon_k(r) = 1$ for all r, and so (1.11) is a special case of (4.6).

(ii) In [7], Haukkanen and McCarthy have studied (1.11) in greater detail.

Acknowledgement. The authors are grateful to the referee for valuable comments.

References

1. E. Cohen, Generalizations of the Euler ϕ-function, *Scripta Math.* **23** (1957), 157–161.
2. E. Cohen, Trigonometric sums in elementary number theory, *Amer. Math. Monthly* **66** (1959), 105–117.
3. E. Cohen, Partitions in homogeneous, finite abelian groups, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 1290–1291.
4. E. Cohen, A trigonometric sum, *Math. Student* **28** (1960), 29–32.
5. E. Cohen, A class of arithmetical functions in several variables with applications to congruences, *Trans. Amer. Math. Soc.* **96** (1960), 355–381.
6. E. Cohen, A property of Dedekind's ψ-function, *Proc. Amer. Math. Soc.* **12** (1961), 996.
7. P. Haukkanen and P.J. McCarthy, Sums of values of even functions, *Portugal. Math.* **48** (1991), 53–66.
8. P. Haukkanen and R. Sivaramakrishnan, Cauchy multiplication and periodic functions $(mod\, r)$, *Collect. Math.* **42** (1991), 33–44.
9. P. Kesava Menon, On the sum $\Sigma\,(a-1,n)\,[(a,n)=1]$, *J. Indian Math. Soc. (N.S.)* **29** (1965), 155–163.
10. W.J. LeVeque, *Topics in Number Theory (Vol. I)*, Addison-Wesley Publishing Co., Reading, Mass. 1956.
11. P.J. McCarthy, *Introduction to Arithmetical Functions*, Universitext, Springer-Verlag, New York 1986.
12. G.A. Miller, On the totatives of different orders, *Amer. Math. Monthly* **11** (1904), 129–130.
13. K. Nageswara Rao, On certain arithmetical sums, *Springer-Verlag Lecture Notes in Math.* **251** (1972), 181–192.
14. R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, No. 126, Marcel Dekker, Inc., New York 1989.
15. R. Sivaramakrishnan, The many facets of Euler's totient (II): Generalizations and analogues, *Nieuw Arch. Wisk. (4)* **8** (1990), 169–187.