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Linear properties of poly-Fuchsian groups

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§ 0. Introduction

Fuchsian groups, that is lattices in $\mathrm{PSL}_2(\mathbb{R})$, are simultaneously amongst the simplest and most complicated objects within the general theory of semisimple lattices. One the one hand, their isomorphism types are completely classified, and they are all commensurable with either Surface groups or finitely generated free groups. Other aspects however, particularly those related to the failure of "rigidity", present more complicated features. As an example, the outer automorphism group $\mathrm{Out}(\Gamma)$ is generally infinite when Γ is a Fuchsian group, whereas for irreducible lattices in other noncompact semisimple Lie groups it is finite [5]. This has nontrivial consequences for the extension theory of Fuchsian groups. Groups obtained by iterated extension of $\mathrm{PSL}_2(\mathbb{R})$-lattices, the poly-Fuchsian groups of the title, are usually not lattices in any semisimple group, and may display other interesting features [1], [15]. By contrast, forming iterated extensions from other semisimple lattices does not give anything essentially new; all such extensions are commensurable with lattices in connected semisimple Lie groups [7].

Except in the trivial case where a poly-Fuchsian group is commensurable with a direct product of Fuchsian groups, it is not known which poly-Fuchsian groups admit faithful linear representations. Nevertheless we are still able to show that the poly-Fuchsian groups have a number of properties in common with finitely generated linear groups, and in particular, with semisimple lattices. We will show that:

(I) a poly-Fuchsian group contains a torsion free subgroup of finite index;

(II) poly-Fuchsian groups are residually finite;

(III) poly-Fuchsian groups satisfy the "Tits alternative" [18]; that is, a solvable subgroup of a poly-Fuchsian groups is polycyclic, whilst a non-solvable subgroup contains a non-abelian free subgroup.

These properties are shared by all finitely generated linear groups; however, poly-Fuchsian groups possess some specifically lattice like properties:

(IV) a torsion free poly-Fuchsian group is a duality group in the sense of [3].

Moreover, poly-Fuchsian groups admit a product decomposition similar to the decomposition of lattices in semisimple Lie groups, up to commensurability, into products of irreducible lattices ([17] p. 86); that is;

(V) each poly-Fuchsian group contains a *characteristic* subgroup of finite index which is isomorphic to a direct product $G_1 \times \ldots \times G_n$ where $G_1, \ldots, G_n$ are *irreducible* poly-Fuchsian groups. Moreover, this decomposition is uniquely determined up to commensurability.

Finally we relate the automorphism group of a poly-Fuchsian group to the automorphism groups of its various composition factors:

(VI) If G is a poly-Fuchsian group with filtration $\mathcal{G} = (G_r)_{0 \leq r \leq n}$, then the group $\mathrm{Aut}(\mathcal{G})$ of filtration-preserving automorphisms is a subgroup of finite index in $\mathrm{Aut}(G)$ and imbeds as a subgroup in $\mathrm{Aut}(Q_1) \times \ldots \times \mathrm{Aut}(Q_n)$, where $Q_r = G_r/G_{r-1}$. In particular, $\mathrm{Aut}(G)$ is commensurable with a subgroup of $\mathrm{Aut}(Q_1) \times \ldots \times \mathrm{Aut}(Q_n)$.

The paper is organised as follows: §1 is a summary of known properties of Fuchsian groups; §2 is a study of groups obtained by extending a finite group by a Fuchsian group; properties (I)-(IV) above are established in §3; §§4–5 are more technical in nature, preparatory to establishing (V); §4 establishes a convenient criterion for a subnormal filtration on a group G to be invariant under all automorphisms of G: in §5 we investigate the direct factors of a reducible poly-Fuchsian group; properties (V) and (VI) are established in §6 and §7 respectively.

In a subsequent paper [13] we shall deal with the more geometrical aspects of poly-Fuchsian groups.

§ 1. Fuchsian groups and their subgroups

By a *Fuchsian group* we mean a discrete subgroup of finite covolume in $\mathrm{PSL}_2(\mathbb{R})$. Such groups may be described geometrically as follows; let $\mathcal{H}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denote the upper halfplane endowed with its natural metric of curvature $\equiv -1$. Fuchsian groups are precisely the discrete groups which act effectively, properly discontinuously, by orientation preserving isometries on $\mathcal{H}_+$, and which possess a fundamental domain with finitely many sides. (Our usage of the term "Fuchsian group" is slightly less general than some; for example, [14], where the possibility of infinite covolume is allowed, at least initially). For nonnegative integers $g, s, m_1, \ldots, m_k$ define

$$\mu(g, s;\, m_1, m_2, \ldots, m_k) = \begin{cases} 2g - 2 + s + \displaystyle\sum_{r=1}^{k} \big(1 - \frac{1}{m_i}\big) & k > 0 \\ 2g - 2 + s & k = 0 \end{cases} .$$

When $\mu(g, s;\, m_1, m_2, \ldots, m_k) > 0$, we denote by $\mathbb{F}(g; m_1, m_2, \ldots, m_k)$ the group with presentation

$$\mathbb{F}(g, s; m_1, m_2, \ldots, m_k) = \langle C_1, \ldots, C_k, Y_1, \ldots, Y_s, X_1, \ldots, X_{2g} : C_1^{m_1} = \ldots = C_k^{m_2} = \mathcal{W} = 1 \rangle$$

where $\mathcal{W}$ is the word $C_1 \ldots C_k Y_1 \ldots Y_s \prod_{i=1}^{g} X_{2i-1} X_{2i} X_{2i-1}^{-1} X_{2i}^{-1}$.

$\mathbb{F}(g, s; m_1, \ldots, m_k)$ is called the *abstract Fuchsian group of signature* $(g, s; m_1, \ldots, m_k)$. Every Fuchsian group has a presentation of this form; moreover, $\mu\big(\mathbb{F}(g, s; m_1, m_2, \ldots, m_k)\big)$ represents the (suitably normalised) area of a fundamental domain for $\mathbb{F}(g, s; m_1, m_2, \ldots, m_k)$ in $\mathcal{H}_+$.

Although the groups $\mathbb{F}(g, s; m_1, m_2, \ldots, m_k)$ have finite covolume in $\mathrm{PSL}_2(\mathbb{R})$, they need not be cocompact. In fact, the cocompact groups are precisely those for which $s = 0$; that is

Proposition 1.1

The groups

$$\mathbb{F}(g, 0; m_1, m_2, \ldots, m_k) = \langle C_1, \ldots, C_k, X_1, \ldots, X_{2g} : C_1^{m_1} = \ldots = C_k^{m_2} = \mathcal{W} = 1 \rangle$$

are precisely the cocompact subgroups of $PSL_2(\mathbb{R})$, where $\mathcal{W}$ is the commutator word

$$\mathcal{W} = \prod_{i=1}^{g} X_{2i-1} X_{2i} X_{2i-1}^{-1} X_{2i}^{-1} .$$

A subgroup of finite index in a Fuchsian group is also Fuchsian, and the two are related numerically by the Riemann-Hurwitz Theorem:

Theorem 1.2 (Riemann-Hurwitz Theorem)

If G is a Fuchsian group and H is a subgroup of finite index d in G, then H is also a Fuchsian group and $\mu(H) = d\,\mu(G)$. Moreover, G is cocompact if and only if H is cocompact.

We summarise the algebraic properties of Fuchsian groups thus:

(1.3) A Fuchsian group has no nontrivial finite normal subgroup;

(1.4) The centre of a Fuchsian group is trivial;

(1.5) An abelian subgroup of a Fuchsian group is cyclic.

(1.6) An element of finite order in $\mathbb{F}(g, s; m_1, m_2, \ldots, m_k)$ is conjugate to a power of some C_i.

These properties are all "well-known" and can be recovered reasonably easily from the combination of references [2] and [6]. Since Fuchsian groups are finitely generated linear groups, it follows from Selberg's Theorem [4] that

(1.7) $\mathbb{F}(g, s; m_1, m_2, \ldots, m_k)$ has a subgroup of finite index which is torsion free.

Let $\mathcal{F}$ denote the class of Fuchsian groups, and let $\mathcal{F}_0$ denote the subclass of torsion free Fuchsian groups. The groups in $\mathcal{F}_0$ are of two types, according to whether or not the fundamental domain is compact. The torsion free groups with compact fundamental domain are fundamental groups of surfaces of genus ≥ 2 and correspond to the case $s = 0$; they have presentations of the form

$$\mathbb{F}(g, 0; \emptyset) = \left\langle X_1, \ldots, X_{2g} : \prod_{i=1}^{g} [X_{2i-1}, X_{2i}] \right\rangle .$$

Those with noncompact fundamental domain correspond to the cases $s > 0$, and have presentations of the form

$$\mathbb{F}(g, s; \emptyset) = \left\langle Y_1, \ldots, Y_s, X_1, \ldots, X_{2g} : Y_1 \ldots Y_s \prod_{i=1}^{g} [X_{2i-1}, X_{2i}] \right\rangle .$$

In this case, $\mathbb{F}(g, s; \emptyset)$ is free of rank $r = s+2g-1$; r may assume any value ≥ 2. From a purely group theoretic point of view, we may write $\mathcal{F}_0 = \text{FREE} \cup \text{SURFACE}$ where FREE denotes the class of groups of the form $\mathbb{F}(g, s; \emptyset)$ with $s > 0$, and SURFACE denotes the class of groups of the form $\mathbb{F}(g, 0; \emptyset)$ with $g \geq 2$. Within $\mathcal{F}_0$, membership of the subclasses FREE, SURFACE, is determined by the criterion of cohomological dimension; FREE groups have dimension 1 and SURFACE groups have dimension 2. The subgroup structure of $\mathcal{F}_0$-groups is easily described:

(1.8) Let N be a nontrivial normal subgroup of an $\mathcal{F}_0$-group Σ; the following conditions are equivalent:

(i) N is a $\mathcal{F}_0$-group with $\dim(N) = \dim(\Sigma)$;
(ii) N is finitely generated;

(iii) N has finite index in Σ.

(1.9) A nontrivial subgroup of infinite index in an $\mathcal{F}_0$-group is a free group.

(1.10) A nontrivial normal subgroup of infinite index in an $\mathcal{F}_0$-group is a free group of infinite rank.

Nonabelian free groups have finitely generated free subgroups of any index, finite or infinite. However,

(1.11) Let H be a subgroup of a Surface group G; then H is itself a Surface group if and only it has finite index in G;

§ 2. Extensions with finite kernel

The following is standard; see, for example, p. 112 of [16];

Proposition 2.1

Let G be a finitely generated group, and let G_0 be a subgroup of finite index in G; then G contains a characteristic subgroup G_1, also of finite index, such that $G_1 \subset G_0 \subset G$.

Proposition 2.2

Let $1 \to \Phi \longrightarrow G \xrightarrow{p} Q \to 1$ be an extension with Φ finite and $Q \in \mathcal{F}_0$; then G contains an $\mathcal{F}_0$-subgroup G' of finite index.

Proof. Since Q is assumed to be a torsion free Fuchsian group, then either Q is a finitely generated free group or a Surface group. If Q is a free group, then the extension

$$1 \to \Phi \longrightarrow G \xrightarrow{p} Q \to 1$$

splits, and we may take $G' = s(Q)$ where $s : Q \longrightarrow G$ is any right inverse homomorphism for p. Thus suppose that Q is a Surface group. For each positive integer n, Q contains a subgroup Q' of index n. Q' is necessarily a Surface group; in fact, if Q is the fundamental group of a surface Σ, Q' can be taken to be the fundamental group of a cyclic covering Σ' of degree n over Σ.

First consider the case where Φ is finite and central in G. In this case, the extension

$$\mathcal{E} = (1 \to \Phi \longrightarrow G \xrightarrow{p} Q \to 1)$$

is completely determined by a cohomology class $c(\mathcal{E}') \in H^2(Q; \Phi) \cong \Phi$. Let Q' be a subgroup of Q with index $n =$ exponent (Φ), and let $\mathcal{E}'$ be the extension

$$\mathcal{E}' = (1 \to \Phi \longrightarrow G' \xrightarrow{p} Q' \to 1)$$

where $G' = p^{-1}(Q')$. It is easy to see that $c(\mathcal{E}') = 0$ so that G' splits as a direct product

$$G' \cong \Phi \times Q'.$$

The result follows since G' has finite index in G.

In the general case, let $c : G \longrightarrow \mathrm{Aut}(\Phi)$ be the homomorphism induced by conjugation, and put $\hat{G} = \mathrm{Ker}(c), \hat{\Phi} = \Phi \cap \hat{G}$ and $\hat{Q} = p(\hat{G})$. Then the extension

$$\hat{\mathcal{E}} = (1 \to \hat{\Phi} \longrightarrow \hat{G} \xrightarrow{p} \hat{Q} \to 1)$$

is in the case considered above. The result follows since $\hat{G}$ has finite index in G. □

As in immediate consequence we obtain:

Corollary 2.3

Let $1 \to \Phi \longrightarrow G \xrightarrow{p} Q \to 1$ be an extension with Φ finite and $Q \in \mathcal{F}$; then G contains an $\mathcal{F}_0$-subgroup of finite index.

§ 3. Linear properties of poly-$\mathcal{F}_+$ groups

Let $\mathcal{C}$ be a class of abstract groups; a group G is a called a *poly-$\mathcal{C}$ group* when it possesses a subnormal filtration $\mathcal{G} = (G_r)_{0 \le r \le n}$ (that is, $G_r \triangleleft G_{r+1}, G_0 = \{1\}$ and $G_n = G$) in which $G_{r+1}/G_r \in \mathcal{C}$ for each r. Poly-$\mathcal{C}$ groups have trivial centre precisely when all $\mathcal{C}$-groups have trivial centre. This is the case when $\mathcal{C} = \mathcal{F}$ is the class of Fuchsian groups, and enables us to describe, in principle at least, the construction of all poly-$\mathcal{F}$ groups; if $\mathcal{G} = (G_r)_{0 \le r \le n}$ is a poly-$\mathcal{F}$ filtration of length n, the extension

$$1 \to G_{n-1} \xrightarrow{i} G_n \xrightarrow{\pi} G_n/G_{n-1} \to 1$$

is determined, up to congruence, by an operator homomorphism $h_{n-1} : G_n/G_{n-1} \longrightarrow \mathrm{Out}(G_{n-1})$; we regard G_n as a fibre product

$$G_n = \mathrm{Aut}(G_{n-1}) \underset{\lambda, h_{n-1}}{\times} G_n/G_{n-1},$$

where $\lambda : \mathrm{Aut}(G_{n-1}) \longrightarrow \mathrm{Out}(G_{n-1})$ is the canonical mapping. Inductively, the study of poly-$\mathcal{F}$ groups of length n may be reduced to that of the outer automorphism groups of pol-$\mathcal{F}$ groups of length $(n-1)$, starting from outer automorphism groups of groups in $\mathcal{F}$.

It is only slightly more difficult to work with the class of poly-$\mathcal{F}_+$ groups, where $\mathcal{F}_+$ denotes the augmented class $\mathcal{F}_+ = \mathcal{F} \cup \{\text{finite groups}\}$. This class includes the "non-orientable Fuchsian groups", that is, discrete subgroups of finite covolume in $\mathrm{PGL}_2(\mathbb{R})$, and in particular, the fundamental groups of non-orientable surfaces of genus ≥ 2.

Selberg showed that a finitely generated linear group contains a torsion free subgroup of finite index [4]. Here we establish a similar property for poly-$\mathcal{F}_+$ groups.

Proposition 3.1

A poly-$\mathcal{F}_+$ group contains a poly-$\mathcal{F}_0$ subgroup of finite index; this subgroup may be assumed to be characteristic.

Proof. Let $(G_r)_{0\leq r\leq n}$ be a poly-$\mathcal{F}$ filtration on G. We prove it by induction on n. For $n = 1$, this is (2.1) above; therefore, suppose proved for $n-1$. By induction, choose a characteristic poly-$\mathcal{F}_0$ subgroup H of finite index in G_{n-1}. H is then normal in G, and we have an exact sequence

$$1 \to G_{n-1}/H \longrightarrow G/H \longrightarrow G/G_{n-1} \to 1$$

in which G_{n-1}/H is finite, and $G/G_{n-1} \in \mathcal{F}$. By (2.3) above, choose an $\mathcal{F}_0$-subgroup Q_1 of finite index in G/H. Put $\hat{G} = p^{-1}(Q_1)$ where $p : G \longrightarrow G/H$ is the canonical map. Then $\hat{G}/H \cong Q_1 \in \mathcal{F}_0$. Hence $\hat{G}$ is a poly-$\mathcal{F}_0$ group of length n. □

The next two propositions are easy to verify.

Proposition 3.2

Let G be a poly-$\mathcal{F}_0$ group of length n, and let H be a subgroup of finite index in G; then H is also a poly-$\mathcal{F}_0$ group of length n.

Proposition 3.3

All poly-$\mathcal{F}_0$ groups are torsion free.

Corollary 3.4

A poly-$\mathcal{F}_+$ group contains a torsion free subgroup of finite index.

We have shown elsewhere that a torsion free poly-$\mathcal{F}_+$ group need not be poly-$\mathcal{F}_0$ [12].

Both Surface groups and finitely generated free groups are duality groups in the sense of [3]. Since the class of duality groups is closed under extension, it follows that a poly-$\mathcal{F}_0$ group is a duality group; however, since duality groups are also closed under torsion free extension by finite groups, we see that

Corollary 3.5

A torsion free poly-$\mathcal{F}_+$ group satisfies homological/cohomological duality in the sense of [3].

A group G is *residually finite* when, for each nontrivial element $g \in G$, there exists an epimorphism onto a finite group $\varphi : G \longrightarrow \Phi$ such that $\varphi(g) \neq 1$. It is straightforward to see that if H is a subgroup of finite index in a finitely generated group G then H is residually finite $\Longleftrightarrow$ G is residually finite.

Let $1 \to K \longrightarrow G \longrightarrow Q \to 1$ be an exact sequence of groups in which K is a finitely generated residually finite group, and $Q \in \mathcal{F}_0$. Let g be a nontrivial element of G; there exists a subgroup K_1 of finite index in K such that $g \notin K_1$. If K is also finitely generated, K_1 may be assumed to be characteristic in K, by (2.1), so that G/K_1 occurs in an extension

$$1 \to K/K_1 \longrightarrow G/K_1 \longrightarrow Q \to 1\,.$$

Observe that $\pi(g) \neq 1$ where $\pi : G \longrightarrow G/K_1$ is the canonical epimorphism; since $Q \in \mathcal{F}_0$, we may, by (2.2), choose an $\mathcal{F}_0$ subgroup H of finite index in G/K_1. H is residually finite, as it admits a faithful finite dimensional real linear representation [19], so we can ensure that $\pi(g) \notin H; G' = \pi^{-1}(H)$ is a subgroup of finite index in G such that $g \notin G'$. Since G is finitely generated, we may choose a normal subgroup G'' of finite index in G such that $G'' \subset G' \subset G$. Then $\psi(g) \neq 1$ where ψ is the canonical epimorphism of G onto G/G''. We have established:

Theorem 3.6

Let $1 \to K \longrightarrow G \longrightarrow Q \to 1$ be an exact sequence in which K is a finitely generated residually finite group and $Q \in \mathcal{F}_0$: then G is residually finite.

Following (3.6), an induction on the length of a filtration shows that:

Theorem 3.7

A poly-$\mathcal{F}_0$ group is residually finite.

From (2.5) and (3.1), we obtain, as an immediate Corollary;

Corollary 3.8

A poly-$\mathcal{F}_+$ group is residually finite.

A class $\mathcal{C}$ of groups is said satisfy the "Tits' alternative" when, given a group Γ in $\mathcal{C}$ and a subgroup Δ of Γ, then either Δ is polycyclic or Δ contains a non-abelian free group. In [18] Tits showed that the class of finitely generated linear groups satisfies de Tits' alternative. We will show that the class of poly-$\mathcal{F}_+$ groups satisfy the Tits' alternative.

Proposition 3.9

Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an extension where Q is a nonabelian subgroup of an $\mathcal{F}_0$-group. Then G contains nonabelian free group.

Proof. Let $Q' \subset Q$ be a nonabelian free group and put $G' = p^{-1}(Q')$. Then the extension $1 \to K \to G' \xrightarrow{p} Q' \to 1$ splits, so that G', and hence also G, contains a subgroup isomorphic to Q'. $\Box$

Proposition 3.10

Let Γ be a poly-$\mathcal{F}_0$ group and let Δ be a nontrivial subgroup of Γ. If Δ is solvable, then Δ is poly-$\{$infinite cyclic$\}$.

Proof. If $\Gamma \in \mathcal{F}_0$ and Δ is a nontrivial solvable subgroup of Γ, then Δ is infinite cyclic. The result follows by induction on the length of a poly–$\mathcal{F}_0$ filtration on Γ. $\Box$

Corollary 3.11

The class of poly-$\mathcal{F}_+$ groups satisfies the Tits' alternative.

Proof. Since each poly-$\mathcal{F}_+$ group contains a poly-$\mathcal{F}_0$ subgroup of finite index, it clearly suffices to establish that the class of poly-$\mathcal{F}_0$ groups satisfies the Tits' alternative. However, this follows easily by induction from (3.9) and (3.10), starting from the observation that $\mathcal{F}_0$ itself satisfies the Tits' alternative. $\Box$

§ 4. Strong and characteristic filtrations

Let $\mathcal{G} = (G_r)_{0 \le r \le n}$ be a *poly-$\mathcal{C}$* filtration on a group G; we say that $\mathcal{G}$ is a *strong poly-$\mathcal{C}$ filtration* when, in addition, $G_r \triangleleft G$ for each r, and that $\mathcal{G}$ is *characteristic* when G_{r-1} is a characteristic subgroup of G_r for each r. The following are easy to verify:

(4.1) A characteristic poly-$\mathcal{C}$ filtration is strong.

(4.2) Let H be a subgroup of finite index in a strongly poly-$\mathcal{F}_0$ group G; then H is also a strongly poly-$\mathcal{F}_0$ group.

We observed, (1.10), that a nontrivial normal subgroup of infinite index in an $\mathcal{F}_0$-group is a free group of infinite rank. By induction on the length of a filtration, we obtain a sort of Noetherian property:

(4.3) Let $H \in \mathcal{F}_0$ and let $H_0 \subset H_1 \subset \ldots \subset H_n = H$ be a sequence of finitely generated subgroups such that $H_r \triangleleft H_{r+1}$ for each r; then there exists $m, 1 \leq m \leq n$, such that H_r has finite index in H for $m \leq r$, and $H_r = \{1\}$ for $r < m$.

Let $(1 \to K \longrightarrow G \xrightarrow{p} Q \to 1)$ be an exact sequence of groups in which $Q \in \mathcal{F}_0$ and K admits a subnormal filtration $(K_r)_{0 \leq r \leq n}$ in which each K_r/K_{r-1} is finitely generated with $\mathrm{rank}(K_r/K_{r-1}) < \mathrm{rank}(Q)$. Let $\alpha \in \mathrm{Aut}(G)$; put $Q_r = p\alpha(K_r)$ for $0 \leq r \leq n$, and $Q_{n+1} = Q$; then the hypotheses of (4.3) are satisfied; that is, $Q \in \mathcal{F}_0; (Q_r)_{1 \leq r \leq n+1}$ is a sequence of finitely generated subgroups of $Q = Q_{n+1}$ with $Q_0 \subset Q_1 \subset \ldots \subset Q_n \subset Q_{n+1}$ and $Q_{r-1} \triangleleft Q_r$ for each r. From (4.3), it follows that either

(i) Q_n is trivial or
(ii) there exists $m (1 \leq m \leq n+1)$ such that $Q_r = \{1\}$ for $r < m$, and Q_r has finite index in Q for $m \leq r$.

If (ii) holds, then Q_m has finite index (j, say) in Q, and the Riemann-Hurwitz formula gives

$$(j-1)(\mathrm{rank}(Q) - 2) = \mathrm{rank}(Q) = \mathrm{rank}(Q_m)\,;$$

in particular:

$$\mathrm{rank}(Q) \leq \mathrm{rank}(Q_m)\,.$$

Since $p\alpha(K_{m-1}) = \{1\}$, $p\alpha$ induces an epimorphism $(p\alpha)_* : K_m/K_{m-1} \longrightarrow Q_m$, whence

$$\mathrm{rank}(Q_m) \leq \mathrm{rank}(K_m/K_{m-1})\,.$$

However, by hypothesis, $\mathrm{rank}(K_m/K_{m-1}) < \mathrm{rank}(Q)$; that is, we have

$$\mathrm{rank}(Q) \leq \mathrm{rank}(Q_m) \leq \mathrm{rank}(K_m/K_{m-1}) < \mathrm{rank}(Q)$$

which is a contradiction. Thus our supposition (ii) is false, and we must have $p\alpha(K) = Q_n = \{1\}$. Thus $\alpha(K) \subset K$. Repeating the argument with α^{-1} instead of α gives $\alpha^{-1}(K) \subset K$ or $K \subset \alpha(K)$. Hence $\alpha(K) = K$, and K is a characteristic subgroup of G.

By a *stable* poly-$\mathcal{F}_0$ filtration $\mathcal{G} = (G_r)_{0 \leq r \leq n}$ on a group $G = G_n$ we shall mean one for which $\mathrm{rank}(G_r/G_{r-1}) < \mathrm{rank}(G_{r+1}/G_r)$ for all $r \in \{1, \ldots, n-1\}$; a poly-$\mathcal{F}_0$ group is called stable when it admits a stable poly-$\mathcal{F}_0$ filtration. The above argument establishes:

(4.4) A stable poly-$\mathcal{F}_0$ filtration $\mathcal{G} = (G_r)_{0 \leq r \leq n}$ is characteristic.

Proposition 4.5

A poly-$\mathcal{F}_0$ group contains a subgroup of finite index which is stably (and hence characteristically) poly-$\mathcal{F}_0$.

Proof. By (4.4), it clearly suffices to prove the stability statement. The proof goes by induction on the length n of a poly-$\mathcal{F}_0$ filtration. The case $n = 1$ is trivial, so suppose proved for $n - 1$ and let $(G_r)_{0 \le r \le n}$ be a poly-$\mathcal{F}_0$ filtration of length n. Inductively, we may choose a subgroup H_{n-1} of finite index in G_{n-1} admitting a poly-$\mathcal{F}_0$ filtration $(H_r)_{0 \le r \le n-1}$ satisfying the condition

$$\operatorname{rank}(H_r/H_{r-1}) < \operatorname{rank}(H_{r+1}/H_r) \quad \text{for all} \quad r \in \{1, \ldots, n-2\}\,.$$

A finitely generated group has only a finite number of subgroups of a given finite index; thus the set $\{\alpha(H_{n-1}) : \alpha \in \operatorname{Aut}(G_{n-1})\}$ is finite. Put $S(H_{n-1}) = \{\alpha \in \operatorname{Aut}(G_{n-1}) : \alpha(H_{n-1}) = H_{n-1}\}$. Then $S(H_{n-1})$ is a subgroup of finite index in $\operatorname{Aut}(G_{n-1})$. Let $c : G_n \longrightarrow \operatorname{Aut}(G_{n-1})$ denote the conjugation map, and put $\tilde{H}_n = c^{-1}(S(H_{n-1}))$. Observe that H_{n-1} is normal $\tilde{H}_n$, and we have an extension

$$1 \longrightarrow \big(G_{n-1} \cap \tilde{H}_n\big)/H_{n-1} \longrightarrow \tilde{H}_n/H_{n-1} \longrightarrow \tilde{H}_n/\big(G_{n-1} \cap \tilde{H}_n\big) \longrightarrow 1$$

in which $\big(G_{n-1} \cap \tilde{H}_n\big)/H_{n-1}$ is finite, and $\tilde{H}_n/(G_{n-1} \cap \tilde{H}_n) \in \mathcal{F}_0$. In particular, $\tilde{H}_n/(G_{n-1} \cap \tilde{H}_n)$ contains subgroups of arbitrary finite index. It follows easily from the Riemann Hurwitz Theorem that we may choose a subgroup Q of finite index in $\tilde{H}_n/(G_{n-1} \cap \tilde{H}_n)$ such that

$$\operatorname{rank}(H_{n-1}/H_{n-2}) < \operatorname{rank}(Q)\,.$$

Let $\varphi : \tilde{H}_n \longrightarrow \tilde{H}_n/H_{n-1}$ denote the identification mapping; then $H_n = \varphi^{-1}(Q)$ is a subgroup of finite index in G_n, and the poly-$\mathcal{F}_0$ filtration $(H_r)_{0 \le r \le n-1}$ satisfies the condition

$$\operatorname{rank}(H_r/H_{r-1}) < \operatorname{rank}(H_{r+1}/H_r)$$

for all $r \in \{1, \ldots, n-1\}$. This completes the proof. $\square$

§ 5. Products of strongly poly-$\mathcal{F}_0$ groups

First we recall some facts about subdirect products. Let $G_1, \ldots, G_k$ be groups, and let $\pi_1 : \prod_{j=1}^{k} G_j \longrightarrow G_i$ be the projection onto the i^{th} factor; a (normal) subgroup H of $\prod_{i=1}^{k} G_i$ is called a *(normal) subdirect product* when $\pi_i(H) = G_i$ for each i; a subgroup H is a *normal subdirect product* when, in addition, H is a normal subgroup of $\prod_{i=1}^{k} G_i$. The following is easily proved; see, for example Proposition (1.2) of [9].

Proposition 5.1

If H is a normal subdirect product in $G = G_1 \times \ldots \times G_k$, then

$$\left[G_1, G_1\right] \times \ldots \times \left[G_k, G_k\right] \subset H\,.$$

We write $G = G_1 \circ G_2$ when the group G is the internal direct product of its normal subgroups G_1, G_2, A group G is said to have property $\Re$ when every nontrivial normal subgroup of G is nonabelian. The proofs of the next two propositions are straightforward:

Proposition 5.2

Let $1 \longrightarrow H_1 \longrightarrow G \longrightarrow H_2 \longrightarrow 1$ be an exact sequence; if both H_1 and H_2 have property $\Re$ then so also does G.

Proposition 5.3

Let H_1, H_2 be groups; then

$$H_1 \times H_2 \quad \textit{has property} \quad \Re \Longleftrightarrow \quad \textit{both} \quad H_1 \quad \textit{and} \quad H_2 \quad \textit{have property} \quad \Re\,.$$

Let G be a Fuchsian group, and let N be a nontrivial normal subgroup. If $[G; N]$ is finite then N is also a Fuchsian group; if $[G; N]$ is infinite then N contains a subgroup of finite index which is a free group of infinite rank; either way, N is nonabelian. It follows by induction that

Proposition 5.4

Each poly-$\mathcal{F}$ group has property $\Re$.

Let $G = G_1 \circ G_2$ be the (internal) direct product of normal subgroups G_1, G_2 which both have property $\Re$, and let H be a torsion free normal subgroup of G, with the property that every abelian subgroup of H is cyclic. Then H is a normal subdirect product of $H_1 \circ H_2$, where H_i is the image of H under the projection $\pi_i : G_1 \circ G_2 \longrightarrow G_i$.

From (2.2) of [9] we infer

$$(*) \qquad \left[H_1, H_1\right] \circ \left[H_2, H_2\right] \subset H\ .$$

For $i = 1, 2, H_i$ is a normal subgroup of G_i; if H_i is nontrivial, then since G_i has property $\Re, [H_i, H_i] \neq \{1\}$, and since H is torsion free, $[H_i, H_i]$ contains an infinite cyclic group. Suppose that both projections H_1, H_2 are nontrivial; then, from (*), H

contains a free abelian subgroup of rank 2, which contradicts our assumption that every abelian subgroup of H is cyclic. At least one projection $H_i = \pi_i(H)$ must therefore be trivial, from which we see that:

Proposition 5.5

Let H be a torsion free normal subgroup of $G_1 \circ G_2$, where G_1, G_2 both have the property $\Re$. If H has the property that every abelian subgroup is cyclic, then either

$$H \subset G_1 \quad \text{or} \quad H \subset G_2 \,.$$

Proposition 5.6

Let $G = K_1 \circ K_2$ be the (internal) direct product of nontrivial normal subgroups K_1, K_2; if G is a strongly poly-$\mathcal{F}_0$ group, then K_1, K_2 are also strongly poly-$\mathcal{F}_0$ groups.

Proof. The proof goes by induction on the length n of a strong poly-$\mathcal{F}_0$ filtration on G. The case $n = 1$ is empty, since a Fuchsian group is not isomorphic to a nontrivial direct product. Suppose that $n \geq 2$, and that the statement is proved for strong filtrations of length $\leq n-1$; let $(G_r)_{0 \leq r \leq n}$ be a strong poly-$\mathcal{F}_0$ filtration on a group $G = G_n = K_1 \circ K_2$. By (5.4), G has property $\Re$, so that, by (5.2), K_1, K_2 also have property $\Re$. Since $G_1 \triangleleft G$, it follows from (5.5) that either $G_1 \subset K_1$ or $G_1 \subset K_2$. Without loss of generality, we may suppose that $G_1 \subset K_1$. Then

$$G/G_1 \cong (K_1/G_1) \times K_2$$

and G/G_1 admits a strong poly-$\mathcal{F}_0$ filtration of length $n-1$. If $G_1 = K_1$, then $K_2 \cong G/G_1$ is strongly poly-$\mathcal{F}_0$ whilst $K_1 \in \mathcal{F}_0$. If $G_1 \neq K_1$, then K_1/G_1 and K_2 are both nontrivial, so that, by induction, both K_1/G_1 and K_2 are strongly poly-$\mathcal{F}_0$. However, K_1 is an extension

$$1 \longrightarrow G_1 \longrightarrow K_1 \longrightarrow K_1/G_1 \longrightarrow 1$$

in which $G_1 \in \mathcal{F}_0$ and K_1/G_1 is strongly poly-$\mathcal{F}_0$; it follows that K_1 is also strongly poly-$\mathcal{F}_0$. In either case, both K_1, K_2 are strongly poly-$\mathcal{F}_0$ groups. This completes the induction, and also the proof. $\square$

§ 6. Commensurability and decomposition into irreducibles

Recall that two abstract groups G_1, G_2 are *commensurable,* written $G_1 \sim G_2$, when there exists a group H, and injections $\ell_r : H \longrightarrow G_r (r = 1, 2)$, such that $\ell_r(H)$ has finite index in G_r; without loss of generality, we may suppose a pair H, K of commensurable groups have intersection $H \cap K$ of finite index in each of H, K. An infinite group G is *reducible* when it is commensurable to a direct product $G \sim H_1 \times H_2$ where H_1, H_2 are infinite groups; otherwise, G is *irreducible*. It is straightforward to see that:

Proposition 6.1

A finitely generated infinite group G is irreducible if and only if it contains no subgroup of finite index which is isomorphic to a direct product of infinite groups.

It is a consequence of the Borel Density Theorem that a lattice in a connected linear semisimple Lie group admits a decomposition, up to commensurability, into a product of irreducible semisimple lattices ([17], p. 86). Moreover, this decomposition is essentially unique. The irreducible factors correspond either to arithmetic lattices in $\mathbb{Q}$-simple algebraic groups, or to nonarithmetic lattices in $\mathbb{R}$-simple Lie groups. In this section, we will show analogously that, up to commensurability, a poly-$\mathcal{F}$ group admits a decomposition as a product of irreducible poly-$\mathcal{F}$ groups.

It is technically convenient to work within a wider context. Let $\mathcal{L}$ denote the class of finitely generated infinite groups of finite cohomological dimension which have the property that every subgroup of finite index has trivial centre. The class of poly-$\mathcal{F}_0$ groups is a subclass of $\mathcal{L}$. Let $\mathcal{L}_0$ denote the subclass of $\mathcal{L}$ consisting of irreducible groups; we show that an $\mathcal{L}_0$-product structure on a group is unique up to commensurability.

First recall that if $\mathcal{C}$ is a class of groups, then by a *$\mathcal{C}$-product structure* on a group G we mean a finite sequence $\mathcal{P} = (G_\lambda)_{\lambda \in \Lambda}$ where each $G_\lambda \in \mathcal{C}$ is a nontrivial normal subgroup of G such that G is the internal direct product

$$G = \prod_{\lambda \in \Lambda} G_\lambda = G_{\lambda_1} \circ \ldots \circ G_{\lambda_n}, \quad \text{where} \quad \Lambda = \{\lambda_1, \ldots, \lambda_n\}.$$

Proposition 6.2

An $\mathcal{L}_0$-product structure on a group is unique up to commensurability; that is, if H and K are commensurable groups having $\mathcal{L}_0$-product structures $H = H_1 \circ \ldots \circ H_m$ and $K = K_1 \circ \ldots \circ K_n$ respectively, then

(i) $m = n$ *and*

(ii) for some unique bijection $\tau : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}, H_i \cap K_j$ is trivial if $j \neq \tau(i)$ and has finite index in both H_i and $K_{\tau(i)}$ if $j = \tau(i)$.

Proof. Without loss we may suppose that $m \leq n$. Put $G = H \cap K$, which has finite index in both H and K, and define $L_i = G \cap K_i\big(= H \cap K_i\big)$, and $L = L_1 \circ \ldots \circ L_n$. Each L_i has finite index in K_i, so that L has finite index in K. Since $L \subset G, L$ also has finite index in H.

Fix $\mu \in \{1, \ldots, m\}$ and let $\pi_\mu : H \longrightarrow H_\mu$ denote the projection map. Since H_μ is irreducible and $\pi_\mu(L_i)$ centralises $\pi_\mu(L_j)$ for $i \neq j$, it follows that there exists a unique element $\tau(\mu) \in \{1, \ldots, n\}$ such that $\pi_\mu(L_i) = \{1\}$ for $i \neq \tau(\mu)$ and $\pi_\mu(L_{\tau(\mu)})$ has finite index in H_μ. τ defines a function $\tau : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$; since each L_i is nontrivial, for each $i \in \{1, \ldots, n\}$ there exists $\mu \in \{1, \ldots, m\}$ such that $\pi_\mu(L_i) \neq \{1\}$; that is, τ is surjective. Thus $m = n$ and hence τ is also bijective. It is easy to check that, for all $\mu, L_{\tau(\mu)}$ is contained in H_μ, with finite index.

If $H_i \cap K_j \neq \{1\}$, then, since K_j is torsion free, $H_i \cap K_j$ must be infinite; thus $j = \tau(i)$, otherwise our previous claim that $L_{\tau(i)}$ is contained in H_i with finite index is contradicted. Thus $H_i \cap K_j$ is trivial for $j \neq \tau(i)$. This completes the proof. □

Two $\mathcal{C}$-product structures $\mathcal{P} = (G_\lambda)_{\lambda \in \Lambda}, Q = (H_\omega)_{\omega \in \Omega}$ on a group G are said to be *equivalent* when there exists a bijection $\sigma : \Lambda \longrightarrow \Omega$ such that for all $\lambda \in \Lambda, G_\lambda \cong H_{\sigma(\lambda)}$, and *strongly equivalent* when, in addition, we have equality $G_\lambda = H_{\sigma(\lambda)}$ for all $\lambda \in \Lambda$. If $(H_\omega)_{\omega \in \Omega}$ is a product structure, we denote the complementary factor to H_μ by

$$\hat{H}_\mu = \prod_{\lambda \neq \mu} H_\lambda \,.$$

Let $G = H_1 \circ \ldots \circ H_m = K_1 \circ \ldots \circ K_n$ be $\mathcal{L}_0$-product structures with $H_i, K_j \in \mathcal{L}_0$. It follows from (6.2) that $m = n$. After re-indexing the K_i factors, if necessary, we can suppose that $H_i \cap K_j$ is trivial if $j \neq i$, and that $H_i \cap K_i$ has finite index in both H_i and K_i. Let $\hat{p}_i$ denote the projection of G onto the complementary factor $\hat{H}_i$ of $H_i; \hat{p}_i$ imbeds the finite group $K_i/\big(H_i \cap K_i\big)$ into the torsion free group $\hat{H}_i$. Thus $K_i = H_i \cap K_i$, and, by symmetry, we also see that $H_i = H_i \cap K_i$. Hence $H_i = K_i$, and we have proved:

Theorem 6.3

Any two $\mathcal{L}_0$-product structures on a group are strongly equivalent.

A somewhat longer proof yields the conclusion of (6.3) when the class $\mathcal{L}_0$ is replaced by the larger class of infinite groups which have trivial centre, and which are indecomposable as nontrivial direct products.

Theorem 6.4

An $\mathcal{L}$ group G contains a characteristic subgroup G_0 of finite index such that

$$G_0 \cong H_1 \times \ldots \times H_m \,,$$

where $H_1, \ldots, H_m$ are $\mathcal{L}_0$-groups.

Proof. We first establish the existence of some subgroup H, not necessarily characteristic, of finite index in G of the form $H \cong H_1 \times \ldots \times H_m$.

Let $\delta = \delta(G)$ denote the cohomological dimension of G. If $\delta = 1, G$ is free of rank ≥ 2, and so is itself an irreducible $\mathcal{L}$-group. Suppose that $\delta \geq 2$; if G is irreducible there is nothing to prove. Otherwise, by (6.1), G has a subgroup G_1 of finite index which is isomorphic to a product

$$G_1 \cong K_1 \times K_2$$

where K_1, K_2 are $\mathcal{L}$-groups of dimensions δ_1, δ_2, with $1 \leq \delta_i \leq n-1,\ \delta_1 + \delta_2 = n$. By induction, K_i has a subgroup L_i of finite index which is a product of irreducible $\mathcal{L}$-groups,

$$L_1 \cong H_1 \times \ldots \times H_k$$
$$L_2 \cong H_{k+1} \times \ldots \times H_m\,;$$

$H = L_1 \times L_2$ has the required properties.

For each automorphism α of $G, H \sim \alpha(H) = \alpha(H_1) \circ \ldots \circ \alpha(H_m)$. By (6.2), α gives rise to a unique permutation $\alpha_* : \{1, \ldots, m\} \longrightarrow \{1, \ldots, m\}$ with the property that $H_i \cap \alpha(H_j)$ is trivial if $j \neq \alpha_*(i)$, and $H_i \cap \alpha\big(H_{\alpha_*(i)}\big)$ has finite index in both H_i and $\alpha\big(H_{\alpha_*(i)}\big)$. For each i, let

$$G_i = \bigcap_{\alpha \in \mathrm{Aut}(G)} \alpha\big(H_{\alpha_*(i)}\big)\,.$$

Put $G_0 = G_1 \circ \ldots \circ G_m$; G_0 is a characteristic subgroup of G. It remains to show that G_0 has finite index in G.

In any finitely generated group there are only finitely many subgroups of a given finite index; thus the set $\{\alpha(H) : \alpha \in \mathrm{Aut}\}$ is finite; from (6.2), the set $\{\alpha(H_i) : \alpha \in \mathrm{Aut}(G), 1 \leq i \leq n\}$ is also finite, showing that G_i has finite index in H_i. Furthermore, since $H = H_1 \circ \ldots \circ H_m$ has finite index in G, and G_i has finite index in H_i, then G_0 has finite index in G. $\square$

Now suppose that G is a poly-$\mathcal{F}_+$ group. From (3.1), (4.1), (4.2) and (4.5) we see that G contains a characteristic strongly poly-$\mathcal{F}_0$-subgroup G' of finite index. However, $\mathcal{F}_0$ is a subclass of $\mathcal{L}$, so that we may apply (6.4) to conclude that G' contains a *characteristic* subgroup G_0 of finite index such that

$$G_0 \cong H_1 \times \ldots \times H_m$$

where $H_1, \ldots, H_m$ are irreducible $\mathcal{L}$-groups. Observe that G_0 is itself characteristic in G. However, by (5.6), each H_i is itself a strongly poly-$\mathcal{F}_0$ group. Combined with the uniqueness result (6.2), we see we have proved:

Theorem 6.5

An $\mathcal{F}_+$-group G contains a characteristic subgroup G_0 of finite index such that

$$G_0 \cong H_1 \times \ldots \times H_m$$

where $H_1, \ldots, H_m$ are irreducible strongly poly-$\mathcal{F}_0$-groups; up to commensurability, this product structure is uniquely determined by the commensurability class of G.

For $n \geq 2$, the free group F_n imbeds as a subgroup of index $(n-1)$ in F_2, whilst if Σ_g denotes the fundamental group of an orientable surface of genus $g \geq 2$, Σ_g imbeds as a subgroup of index $(g-1)$ in Σ_2. Since every group in $\mathcal{F}$ is commensurable with a torsion free Fuchsian group, $\mathcal{F}$ contains precisely two commensurability classes of abstract groups. For poly-$\mathcal{F}$ groups, the situation is quite different. In a subsequent paper [13], we shall show the following, using a variation on the arguments of [10]:

Theorem 6.6

For each $n \geq 1$, the irreducible poly-$\mathcal{F}$ groups of dimension 4n represent infinitely many distinct commensurability classes of abstract groups.

This result is consistent with the situation of irreducible semisimple lattices; for each $n \geq 2$, there are infinitely many commensurability classes of irreducible lattices in the n-fold product $\mathrm{PSL}_2(\mathbb{R})^{(n)}$. In fact, it can be shown using results of [8], [11] that if G is any connected linear semisimple Lie group which is $\mathbb{C}$-isotypic and non-simple, then G contains infinitely many commensurability classes of irreducible lattices.

§ 7. Automorphisms

Let K, Q be groups and let G be given as a group extension of the form

$$\mathcal{E} = (1 \longrightarrow K \longrightarrow G \xrightarrow{p} Q \longrightarrow 1)\,.$$

In general, there is no easy relationship between the automorphism groups of G, K and Q. However, in the case which interests us, when K and Q are poly-Fuchsian groups, there is a nontrivial relation as we shall see. We start by considering the group $\mathrm{Aut}(\mathcal{E})$ of automorphisms which preserve $\mathcal{E}$; to be precise, $\mathrm{Aut}(\mathcal{E})$ consists of those automorphisms $\alpha : G \longrightarrow G$ for which there exist automorphisms α_K, α_Q making the following commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\ & & \Big\downarrow \alpha_K & & \Big\downarrow \alpha & & \Big\downarrow \alpha_Q & & \\ 1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1\,. \end{array}$$

$\mathrm{Aut}(\mathcal{E})$ has a distinguished normal subgroup $C(\mathcal{E})$, the group of self-congruences of $\mathcal{E}$; that is, the automorphisms γ of $\mathcal{E}$ which make the following commute;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G_1 & \longrightarrow & Q & \longrightarrow & 1 \\ & & \Big\downarrow 1_K & & \Big\downarrow \alpha & & \Big\downarrow 1_Q & & \\ 1 & \longrightarrow & K & \longrightarrow & G_2 & \longrightarrow & Q & \longrightarrow & 1\,. \end{array}$$

There is a homomorphism $\rho : \mathrm{Aut}(\mathcal{E}) \longrightarrow \mathrm{Aut}(K) \times \mathrm{Aut}(Q), \rho(\alpha) = (\alpha_k, \alpha_Q)$, giving rise to an exact sequence

(7.1) $$1 \longrightarrow C(\mathcal{E}) \longrightarrow \mathrm{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathrm{Aut}(K) \times \mathrm{Aut}(Q)\,.$$

The homomorphism $c : G \longrightarrow \mathrm{Aut}(K)$ obtained from conjugation, $c(g)(k) = g\,k\,g^{-1}$, induces the so-called "operator homomorphism" $\varphi : Q \longrightarrow \mathrm{Out}(K) = \mathrm{Aut}(K)/\mathrm{Inn}(K)$; the centre $\mathcal{Z}(K)$ of K is naturally a module over $\mathrm{Out}(K)$, and becomes a module over Q via the operator homomorphism. It is easy to check that for $\alpha \in C(\mathcal{E})$, the assignment $\bar{z}_\alpha(x) = \alpha(x)x^{-1}$ is a function on G taking values in $\mathcal{Z}(K)$. Let $z_\alpha : Q \longrightarrow \mathcal{Z}(K)$ be the function defined by $z_\alpha(p(y)) = \bar{z}_\alpha(y)$. Then z_α is an element of $Z^1(Q, \mathcal{Z}(K))$, the (abelian) group of 1-cocycles of Q with values in $\mathcal{Z}(K)$. Moreover, the mapping $C(\mathcal{E}) \longrightarrow Z^1(Q, \mathcal{Z}(K)), \alpha \longmapsto z_\alpha$, is an isomorphism of groups. When K has trivial centre, matters simplify to give:

(7.2) when K has trivial centre, the group of congruences $C(\mathcal{E})$ is trivial, so that the exact sequence $1 \longrightarrow C(\mathcal{E}) \longrightarrow \mathrm{Aut}(\mathcal{E}) \xrightarrow{\rho} \mathrm{Aut}(K) \times \mathrm{Aut}(Q)$ reduces to an injection $\mathrm{Aut}(\mathcal{E}) \overset{\rho}{\rightarrowtail} \mathrm{Aut}(K) \times \mathrm{Aut}(Q)$.

Theorem 7.3

Let $\mathcal{E} = (1 \to K \longrightarrow G \xrightarrow{p} Q \to 1)$ be an exact sequence of groups where K is finitely generated and $Q \in \mathcal{F}_0$; then $\mathrm{Aut}(\mathcal{E})$ is a subgroup of finite index in $\mathrm{Aut}(G)$.

Proof. Since $Q \in \mathcal{F}_0$, each nontrivial normal finitely generated subgroup of Q has finite index. If $\alpha \in \mathrm{Aut}(G)$ then either $p\alpha(K)$ is a finitely generated nontrivial normal subgroup of Q, and so has finite index in Q, or $\alpha(K) = K$, so that $\alpha \in \mathcal{E}$. Let $\mathcal{K}$ denote the following set of normal subgroups of Q:

$$\mathcal{K} = \{N \triangleleft Q : \exists\, \alpha \in \mathrm{Aut}(G) \quad \text{such that} \quad p\alpha(K) = N\}\,.$$

Let $\rho(H)$ denote the *rank,* that is, the minimal number of generators, of a group H: for each nontrivial element N of $\mathcal{K}$ one has;

$$\rho(N) \le \rho(K)\,.$$

Since $Q \in \mathcal{F}_0$, the rank $\rho(H)$ of a subgroup H of finite index is related to its index $j(H)$ by means of the Riemann-Hurwitz formula

$$j(H) = \Big(\frac{\rho(H) - \delta}{\rho(Q) - \delta}\Big)$$

where δ denotes the cohomological dimension (either 1 or 2) of Q. In particular, for any nontrivial element H of $\mathcal{K}$ we have

$$j(H) \le \Big(\frac{\rho(H) - \delta}{\rho(Q) - \delta}\Big)\,.$$

A finitely generated group has only finitely many subgroups of a given finite index, so that Q has only finitely many subgroups of index $\le \Big(\frac{\rho(H)-\delta}{\rho(Q)-\delta}\Big)$: hence $\mathcal{K}$ has only finitely many *nontrivial* elements and so is finite.

$\mathrm{Aut}(G)$ acts transitively on the left of $\mathcal{K}$ as follows; let $N \in \mathcal{K}$ be represented thus : $N = p(\beta(K))$, and define $\alpha \circ N = p(\alpha\beta(K))$. We obtain an action $\circ : \mathrm{Aut}(G) \times \mathcal{K} \longrightarrow \mathcal{K}$ in which $\mathrm{Aut}(\mathcal{E})$ is the isotropy subgroup of the trivial element of $\mathcal{K}$. Since $\mathcal{K}$ is finite, it follows that $\mathrm{Aut}(\mathcal{E})$ has finite index in $\mathrm{Aut}(G)$. This completes the proof. $\square$

If $\mathcal{G} = (G_r)_{0 \le r \le n}$ is a poly-$\mathcal{C}$ filtration of length n on G_n, we define

$$\mathrm{Aut}(\mathcal{G}) = \{\alpha \in \mathrm{Aut}(G_n) : \alpha(G_r) \quad \text{for all} \quad r, 1 \le r \le n\}\,.$$

Theorem 7.4

If $\mathcal{G}$ is a poly-$\mathcal{F}_0$ filtration on G, then $Aut(\mathcal{G})$ is a subgroup of finite index in $Aut(G)$.

Proof. The proof goes by induction on the length n of a poly-$\mathcal{F}_0$ filtration; for $n = 1$, there is nothing to prove; for $n = 2$, the result follows from (7.3). Suppose proved for $n - 1$, and let $\mathcal{G} = (G_r)_{0 \le r \le n}$ be a poly-$\mathcal{C}$ filtration of length n on G_n. For each r, let $\mathcal{E}_r$ denote the extension

$$\mathcal{E}_r = (1 \to G_{r-1} \longrightarrow G_r \longrightarrow G_r/G_{r-1} \to 1)\,,$$

and let $\mathcal{G}_r$ denote the filtration $\mathcal{G}_r = (G_s)_{0 \le s \le r}$. Under the imbedding

$$\mathrm{Aut}(\mathcal{E}_n) \hookrightarrow \mathrm{Aut}(G_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1})\,,$$

$\mathrm{Aut}(\mathcal{G}_n)$ corresponds to $\mathrm{Aut}(\mathcal{E}_n) \cap (\mathrm{Aut}(\mathcal{G}_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1}))$; by induction, $\mathrm{Aut}(\mathcal{G}_{n-1})$ has finite index in $\mathrm{Aut}(G_{n-1})$, so that $\mathrm{Aut}(\mathcal{G}_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1})$ has finite index in $\mathrm{Aut}(G_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1})$, and $\mathrm{Aut}(\mathcal{G}_n) = \mathrm{Aut}(\mathcal{E}_n) \cap (\mathrm{Aut}(\mathcal{G}_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1}))$ is a subgroup of finite index in $\mathrm{Aut}(\mathcal{E}_n) = \mathrm{Aut}(\mathcal{E}_n) \cap (\mathrm{Aut}(G_{n-1}) \times \mathrm{Aut}(G_n/G_{n-1}))$. However, by (7.3), $\mathrm{Aut}(\mathcal{E}_n)$ has finite index in $\mathrm{Aut}(G_n)$, so that $\mathrm{Aut}(\mathcal{G}_n)$ has finite index in $\mathrm{Aut}(G_n)$. $\square$

Since groups in $\mathcal{F}_0$ all have trivial centre, we see inductively that:

Theorem 7.5

Let $\mathcal{G}$ be a poly-$\mathcal{F}_0$ filtration on $G = G_n$; then $Aut(\mathcal{G})$ imbeds as a subgroup of $Aut(Q_1) \times \ldots \times Aut(Q_m)$ where $Q_1, \ldots, Q_m$ are the successive quotients, $Q_r = G_r/G_{r-1}$.

If G and H are commensurable finitely generated groups, it is easy to see that $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ are also commensurable. From (3.1), (7.4), (7.5) and the above observation, we conclude that:

Corollary 7.6

If G is a group with poly-Fuchsian filtration $\mathcal{G} = (G_r)_{0 \le r \le n}$, then $Aut(G)$ is commensurable with a subgroup of $Aut(Q_1) \times \ldots \times Aut(Q_m)$ where $Q_1, \ldots, Q_m$ are the successive quotients, $Q_r = G_r/G_{r-1}$.

This result relates the linearity problem for poly-Fuchsian groups to the corresponding problem for the automorphism groups $\mathrm{Aut}(Q_i)$. If each $\mathrm{Aut}(Q_i)$ has a faithful finite dimensional linear representation then so also does $\mathrm{Aut}(Q_1) \times \ldots \times \mathrm{Aut}(Q_m)$; (7.6) would allow us, by taking a suitable induced representation, to construct a faithful finite dimensional linear representation for $\mathrm{Aut}(G)$. Since G has trivial centre, it imbeds in $\mathrm{Aut}(G)$, and the restriction would be a faithful finite dimensional linear representation for G. However, it is not known under what conditions the automorphism group $\mathrm{Aut}(Q)$ of a $\mathcal{F}_0$-group Q admits a faithful finite dimensional linear representation.

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