*Collect. Math.* **45**, 2 (1994), 177– 182


# Monotonicity of the period map of a non–Hamiltonian center

Jorge Billeke

*Universidad de Santiago, Depto. de Matemática y C.C, Casilla 5659, Santiago, Chile*

Eduardo Saez

*Universidad Técnica Federico Santa María, Depto. Matemática, Casilla 110-V, Valparaíso, Chile*

Received December 11, 1993. Revised April 6, 1994

Abstract

In this paper we prove that the period map of

$$\ddot{x} - ax\,\dot{x} + x^3 = 0\,, \quad \text{with} \quad a^2 < 8\,,$$

is monotonically decreasing. As an application, it is obtained that the respective Dirichlet boundary problem for two points has either a unique solution or no solution at all while the Neumann boundary value problem has a unique solution.

## Introduction

The differential equation

$$\ddot{x} - ax\,\dot{x} + x^3 = 0 \tag{1}$$

is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x^3 + axy \end{aligned}$$

or equivalent to the vector field

$$X_a(P, Q_a), \quad \text{with} \quad P(x,y) = y, \quad Q_a(x,y) = -x^3 + axy\,. \tag{2}$$

If $a = 0$ , $X_0$ is a Hamiltonian vector field and if $a \neq 0$ , $X_a$ is a non-Hamiltonian vector field.

For $a^2 < 8$, the function $V : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$ such that

$$V(x,y) =$$

$$\begin{cases} \log(x^4 - ax^2y + 2y^2) + \dfrac{2a}{\sqrt{8-a^2}} \arctan\left(\dfrac{-ax^2+4y}{x^2\sqrt{8-a^2}}\right) & \text{if } x \neq 0. \\ \log(2y^2) + \dfrac{\pi a}{\sqrt{8-a^2}} & \text{if } x = 0,\ y > 0. \\ \log(2y^2) - \dfrac{\pi a}{\sqrt{8-a^2}} & \text{if } x = 0,\ y < 0\,. \end{cases} \quad (3)$$

is a first integral of (2).

It is clear that the vector field $X_a$ satisfies the symmetry

$$P(-x,y) = P(x,y) \quad \text{y} \quad Q_a(-x,y) = -Q_a(x,y)\,.$$

The orbits of (2) are found on the level curves (closed curves) of the first integral (3) and $X_a$ has a center at the origin which extends to infinity, since the origin is the only singularity of vector field (2).

To make the study of the periods of the orbits of the above center, we consider the Hamiltonian system of [3]

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x^{2n-1},\ n \in \mathbb{N}\,. \end{aligned} \quad (4)$$

The Hamiltonian is given by

$$H_n(x,y) = \frac{1}{2}y^2 + \frac{1}{2n}x^{2n}\,.$$

Let $C(\theta)$ and $S(\theta)$ be the analytic functions defined by the following Cauchy problem:

$$\begin{aligned} \frac{d}{d\theta}C(\theta) &= -S(\theta) \qquad C(0) = 1 \\ \frac{d}{d\theta}S(\theta) &= C^{2n-1}(\theta) \quad S(0) = 0\,. \end{aligned}$$

These functions are described in [7]. We note that $nS^2(\theta) + C^{2n}(\theta) \equiv 1$ and that $S(\theta)$ and $C(\theta)$ are $p$-periodic, with

$$p = 4\sqrt{n}\int_0^1 \frac{dx}{\sqrt{1-x^{2n}}}\,.$$

Now, the coordinates $(r,\theta)$ defined by the equations $x = rC(\theta),\ y = -r^n S(\theta)$, with $\theta \in [0,p]$, $r > 0$, parameterize the hyperellipse $x^{2n} + ny^2 = r^{2n}$ which are the level curves of $H_n(x,y)$.

Definition. We let $(x,y) \mapsto X(x,y,\lambda)$ be a family of analytic fields, $\lambda \in \mathbb{R}^N$, such that the origin is a center for all $\lambda$. Let $\Sigma$ be a transversal section to the flow of $X$ which is contained in the $x$-axis such that the orbit $\varphi(\xi,t),\ \xi \in \Sigma$, is periodic. The function $\xi \mapsto P(\xi,\lambda)$, where $P(\xi,\lambda)$ is the minimum period of $\varphi(\xi,t)$, is called *period maps.*

If the linear part of $X(x,y,\lambda)$ is a linear center, it is known [4] that $P$ is analytic at the origin and

$$P(\xi,\lambda) = 2\pi + \sum_{k=2}^{\infty} p_k(\lambda)\xi^k\,.$$

The coefficients $p_k(\lambda) = \frac{P^{(k)}(0,\lambda)}{k!}$ are called periodic coefficients and play an analogous role like the Poincaré return transform in the center-focus problem.

In general, if a vector field has a center at the origin and $(r,\theta)$ are the form (in hipereliptic coordinates)

$$\begin{aligned} \dot{r} &= R(r,\theta,\lambda) \\ \dot{\theta} &= A(r,\theta,\lambda),\ \text{with } \theta \in [0,p]\,, \end{aligned}$$

then

$$P(\xi,\lambda) = \int_0^p \frac{d\theta}{A(r,\theta,\lambda)} \tag{5}$$

is an integral representation of its period map, where $r = r(\theta,\xi)$ is a periodic solution of the Cauchy problem

$$\frac{dr}{d\theta} = \frac{R(r,\theta,\lambda)}{A(r,\theta,\lambda)}\,; \quad r(0,\xi) = \xi\,.$$

## Main results

There are many results on the critical points and monotonicity of the period maps of Hamiltonian vector fields (see, e.g., [1],[2],[3],[4] and [5]). When applicable, some results are used to prove existence and uniqueness of Dirichlet or Neumann boundary value problems.

In [3], the authors proved that the periodic maps of the Hamiltonian vector fields on the plane

$$-y\frac{\partial}{\partial x} + x^{2n-1}\frac{\partial}{\partial y}, \quad n \in \mathbb{N},\ n > 1$$

are strictly decreasing monotonic functions.

A generalization of this result for vector fields (2) is given in Theorem 1, and Theorem 2 is an application to Dirichlet or Neumann boundary value problems. This type of planar dynamical system comes from the study of behavior of three-dimensional viscous fluid flowing along a flat wall (see [6]).

**Theorem 1**

*The period map $P(\xi, a)$ of the family of vector fields on the plane*

$$y\frac{\partial}{\partial x} + (-x^3 + axy)\frac{\partial}{\partial y}, \quad \textit{with} \quad a^2 < 8$$

*is* $P(\xi, a) = K(a)/\xi, \quad K(a) > 0\ , \quad \xi \in (0, \infty)\,.$

**Theorem 2**

*We consider the second order differential equation*

$$\ddot{x} - ax\,\dot{x} + x^3 = 0\,, \quad \textit{with} \qquad a^2 < 8.$$

*The Dirichlet boundary value problem*

$$x(0) = x(L) = \xi_0 > 0\,, \quad L > 0$$

*has either a unique solution or no solution at all.*

*The Neumann boundary value problem*

$$\dot{x}(0) = \dot{x}(L) = 0\,, \quad L > 0$$

*has a unique solution.*

## Proofs of the Results

*Proof of Theorem 1.* To avoid cumbersome calculus we study the period maps from infinity. For $\xi \in (0,\infty)$ let $\eta = 1/\xi$ . If $P_\infty(\eta, a)$ denote the period maps from infinity, then $P(\xi, a) = P_\infty(1/\xi, a) = P_\infty(\eta, a)$ .

Let $x = \frac{1}{r}C(\theta)$ and $y = -\frac{1}{r^2}\,S(\theta)$. Then, in the new coordinates $(r,\theta)$, the vector field (2) is given by

$$\begin{aligned} \dot r &= -aS^2(\theta)C(\theta) \\ \dot\theta &= \frac{1}{r}\big(1 + aS(\theta)C^2(\theta)\big)\,, \quad \theta \in [0,p] \end{aligned}$$

where $1 + aS(\theta)C^2(\theta) > 0$ in $[0,p]$, for $a^2 < 8$, since $C^4(\theta) + 2S^2(\theta) \equiv 1$ and the expression $\varphi(\theta) = C^4(\theta) + aC^2(\theta)S(\theta) + 2S^2(\theta)$ has discriminant $\triangle = a^2 - 8 < 0$. At infinity the period map (5) has the integral representation

$$P_\infty(\eta, a) = \int_0^p \frac{r(\theta,\eta)}{1 + aS(\theta)C^2(\theta)}\, d\theta$$

where $r = r(\theta, \eta)$ is solution to the Cauchy problem

$$\frac{dr}{d\theta} = \frac{-arS^2(\theta)C(\theta)}{1 + aS(\theta)C^2(\theta)}\,, \quad r(0,\eta) = \eta > 0\,.$$

Integrating the above equation we obtain

$$r(\theta,\eta) = \eta\, exp\left(-\int_0^\theta \frac{-aS^2(\nu)C(\nu)}{1 + aS(\nu)C^2(\nu)}\, d\nu\right)$$

and the period map is given by

$$P_\infty(\eta, a) = \eta \int_0^p \big(1 + aS(\theta)C^2(\theta)\big)^{-1} exp\left(-\int_0^\theta \frac{-aS^2(\nu)C(\nu)}{1 + aS(\nu)C^2(\nu)}\, d\nu\right) d\theta = K\eta\,,$$

with $K = K(a)$ a constant, which completes the proof. □

*Proof of Theorem 2.* Let $L > 0$ and $\xi_0 > 0$ be fixed.

Hence a solution of (1), with $a^2 < 8$, that satisfies the Dirichlet conditions $x(0) = x(L) = \xi_0$ in the phase plane of (2) is an orbit which, at the initial moment, is found on a point of the line $x = \xi_0$, returning to this point after a time L.

The period map, in $(0,\infty)$, $P(\xi)$, is given by $P(\xi) = P_\infty(\frac{1}{\xi}, a) = \frac{K}{\xi}$, where $P_\infty(\frac{1}{\xi}, a)$ is given by Theorem 1. Hence if $(\xi_0, L)$ is in the graph of map $P_0$, the

Dirichlet boundary value problem has a unique solution; if $(\xi_0, L)$ is not on the graph, its has no solution.

In the case of the Neumann conditions $\dot{x}(0) = \dot{y}(0) = 0$, with $L > 0$ fixed, a solution of (1), with $a^2 < 8$, is an orbit of the phase plane of (2) which, at the initial moment, is found at the point $(x(0), 0)$ (since $\dot{x}(0) = \dot{y}(0) = 0$) and return to it at time $t = L$. By Theorem 1, the only orbit satisfying the above condition is the one which, at initial moment, is found at $(x(0), 0)$ with $x(0) = \frac{K}{L}$ . $\square$

## Acknowledgments

The authors are grateful for support from FONDECYT, Grant 1931140 and UTFSM, Grant 931201.

## References

1. C. Chicone, The Monotonicity of the Period Function for Planar Hamiltonian Vector Fields. *Journal of Differential Equations*, **69** (1987), 310–321.
2. S. N. Chow and J.A. Sanders, On the Number of Critical Points of the Period. *Journal of Differential Equations*, **64** (1986), 51–66.
3. C. Chicone and F. Dumortier, Finiteness for Critical Periods of Planar Analytic Vector Fields. Preprint Department of Mathematics University of Missouri, Columbia, 1992.
4. C. Chicone and M. Jacobs, Bifurcations of Critical Periods for Plane Vector Fields. *Transactions of the American Mathematical Society*. Volume 312, **2** April (1989), 433–486.
5. L. Gavrilov, Remark on the Number of Critical Points of the Period. *Journal of Differential Equations*, **101** (1993), 58–65.
6. R.E. Kooij and P.G. Barkker, Three-dimensional viscous flow structures from bifurcation of a degenerate singularity with three zeros eingenvalues. Report LR-572, Delft University of Technology. Faculty of Aerospace Engeneering, 1989.
7. A. Lyapunov, *Problème général de la stabilité du mouvement*. Annals of Math Studies, **17** Princeton University Press. Princeton, 1949.