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On a boundary value problem for quasi-linear differential inclusions of evolution

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Abstract

In the present paper we prove two theorems concerning the existence of mild solutions of quasi-linear differential inclusions of evolution. The existence problem is reduced to a fixed point problem and then there are used the multivalued version of Banach fixed point theorem and the Himmelberg-Bohnenblust-Karlin theorem.

1. Introduction

The goal of this paper is to find conditions guaranteeing the existence of a mild solution of the linear boundary value problem for the following quasi-linear differential inclusion:

$$(BP)\qquad \begin{cases} dx(t)/dt \in A\big(t, x(t)\big)x(t) + F\big(t, x(t)\big), & t \in I\,, \\ Lx = 0\,, \end{cases}$$

where $I = [0, T]$, $T > 0$, $A\big(t, w\big)$ is a linear operator in a separable Banach space X, depending on t, and w varies on an open set, say, $\emptyset \neq 0 \subset X$, [22]. L is a linear bounded operator from $C\big(I,X\big)$ (the Banach space of continuous functions defined on I with values in X, endowed with the topology of uniform convergence) in X.

If operator A does not depend on neither t nor w, then the differential inclusion in (BP) is said to be *linear*; if operator A depends only on t, then the differential inclusion in (BP) is said to be *semi-linear*, while if it depends both on t and w it is said to be *quasi-linear*, [4], [22], [24].

The importance of the above problem consists in the fact that it includes many boundary value problems for ODE, PDE and linear or semi-linear differential inclusions of evolution. Let's say that an interesting paper on a boundary value problem of a semi-linear differential inclusion is [21]. Particularly, by a suitable choice of operator L, we get information on the existence of periodic solutions. The problem of existence of periodic mild solutions for linear differential inclusions is studied in [13] by the fixed points index theory of condensing multivalued maps.

The existence of mild solutions of an initial value problem for a quasi-linear differential inclusion, was studied in several papers, e.g., [24], [15], [18], [19], [1]. In the case of linear or semi-linear differential inclusions results on initial value problem may be found in [27], [10], [9].

Let Z be a linear topological space. We will use the following notations: $P(Z) = \big\{S \subset Z \,|\, S \neq \emptyset\big\}$, $C(Z) = \big\{S \in P(Z) \,|\, S \text{ is closed}\big\}$, $Co(Z) = \big\{S \in P(Z) \,|\, S \text{ is convex}\big\}$, $CCo(Z) = \big\{S \in C(Z) \,|\, S \in Co(Z)\big\}$.

Let M be a measurable space with σ-algebra $\mathcal{A}$, and X is a separable metrizable space, a multifunction $F : M \longrightarrow P(X)$ is said to be *measurable* (*weakly measurable*) iff $F^{-1}(E) = \big\{t \in M \mid F(t) \cap E \neq \emptyset\big\}$ is measurable for each closed (open) subset E of X. If F have closed values, F is measurable iff F is weakly measurable, provided the measure is complete. This result together with other equivalences may be found in [12] or [31]. If $F : Y \longrightarrow P(X)$ is a multifunction, where Y is a topological space, then the assertion that F is measurable means that F is measurable when Y is assigned the σ-algebra $\mathcal{B}$ of Borel subsets of Y. If $F : M \times Y \longrightarrow P(X)$, and if the measurability of F is defined in terms of the product σ-algebra $\mathcal{A} \times \mathcal{B}$ on $M \times Y$ generated by the sets $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then F is said to be *product-measurable.* If $F : M \times Y \longrightarrow P(X)$ and for each single valued measurable function $G : M \longrightarrow Y$, the multifunction $t \longrightarrow F\big(t, G(t)\big)$ is measurable, then it is said to be *superpositionally measurable.*

Denote by $C(I,X)$ the Banach space of continuous functions from I to X with the norm $\|x\| = \sup_{t\in I} \|x(t)\|$ and by $L_1(I,X)$ the Banach space of Bochner integrable functions from I to X with the norm $\|x\|_1 = \int_I \|x(t)\| dt$. Set $L_1(I) := L_1(I, \mathbb{R}_+)$, [8].

A set-valued $G : X \longrightarrow P(X)$ is called *L-Lipschitz* on $K \subset X$ if for all $x \in K$, $G(x) \neq \emptyset$ and for every $x, y \in K$, $G(x) \subset G(y) + L\|x - y\|B$, where B denotes the closed unit ball in X.

A set-valued $G : I \longrightarrow 2^X$ is called *integrably bounded* if there exists $m \in L_1(I)$ such that $G(t) \subseteq m(t)B$, a.e. on I.

If $F : I \times X \longrightarrow C(X)$ is a multifunction, then by $S^1_{F(\cdot,x(\cdot))}$ we denote the set of integrable selections of $F\big(\cdot, x(\cdot)\big)$, $x : I \longrightarrow X$. A sufficient condition for $S^1_{F(\cdot,x(\cdot))} \neq \emptyset$ is that F has a measurable selection and $F\big(\cdot, x(\cdot)\big)$ is integrably bounded. The existence of a measurable selection may be obtained by the Kuratowski-Ryll-Nardzewski theorem, [12], [31], while conditions implying the superpositionally and product measurability there are in, e.g., [20], [29].

A multifunction $F : X \longrightarrow P(Y), X$ and Y being topological spaces, is said to be *upper semicontinuous* (usc) on X iff $F^{-1}(E)$ is closed for every closed $E \subset Y$, and it is said to be *lower semicontinuous* (lsc) on X iff $F^{-1}(E)$ is open for every open $E \subset Y$, [2], [3], [5].

In the sequel we assume the followings:

(X) X is a separable reflexive Banach space, $0 \subset X$, 0 is nonempty and open.

(L) L is a bounded linear operator from the Banach space $C(I, X)$ onto $X. D = \ker L$. Hence, D is nonempty, closed and convex in $C(I, X)$.

A two family of bounded linear operators $\mathcal{U}(t, s)$, $0 \leq s \leq t \leq T$ on I is said to be an *evolution system* if the following two conditions are satisfied:

(i) $\mathcal{U}(s, s) = 1$ (identity), $\mathcal{U}(t, r)\mathcal{U}(r, s) = \mathcal{U}(t, s)$, $0 \leq s \leq r \leq t \leq T$;

(ii) $(t, s) \longrightarrow \mathcal{U}(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

(A) For every $v \in D$ the family of linear operators $\big\{A(t, v),\ t \in I\big\}$ generates a unique strongly continuous evolution system $\mathcal{U}_v(t, s)$, $0 \leq s \leq t \leq T$.

(U_1) If $u \in D$ has values in 0, then the evolution system $\mathcal{U}_u(t, s)$, $0 \leq s \leq t \leq T$, satisfies:

(i) $\|\mathcal{U}_u(t, s)\| \leq C_1$, for $0 \leq s \leq t \leq T$, uniformly in u;

(ii) there is a positive constant C_2 such that for every $u, v \in D$ with values in 0 and every $w \in 0$ we have:

$$\|\mathcal{U}_u(t, s)w - \mathcal{U}_v(t, s)w\| \leq C_2\|w\| \int_s^t \|u(\tau) - v(\tau)\| d\tau\,.$$

(U_2) If $u \in D$ has values in 0, and $0 \leq s < t \leq T$, then $\mathcal{U}_u(t, s)$ is a compact operator, i.e., it maps bounded sets in relatively compact sets. From [22], it follows that $\mathcal{U}_u(t, s)$ is continuous in the uniform operator topology.

(U_3) If t, $t + \delta \in I$, $\delta > 0$, then $\lim\limits_{\delta \to 0} \mathcal{U}_u(t + \delta, 1) = 1$, uniformly in u and t.

(F_1) $F : I \times X \longrightarrow CCo(X)$ such that: multifunction $t \longrightarrow F(t,x)$ is measurable for every $x \in X$; $x \longrightarrow F(t,x)$ from X to X is lsc and from X in X_w(X endowed with the weak topology) is usc.

(F_2) F satisfies (F_1) and, moreover, it is $k(t)$-Lipschitz, $k \in L_1(I)$, i.e., $d\big(F(t,x), F(t,y)\big) \leq k(t)\|x-y\|$, $t \in I$, x, $y \in X$, d being the Hausdorff-Pompeiu pseudo-metric.

(F_3) F is integrably bounded by a function $\alpha \in L_1(I)$.

(L_1) For every $v \in D$ with values in 0 the linear mapping L_{1v} is considered and it is the same with L_1 in [17], p. 18. We suppose it is onto.

(S_v) For every $v \in D$ $S_v : X \longrightarrow \ker L_{1v}$ is the unique pseudo-inverse of the restriction of L to $\ker L_{1v}$, [17]. Suppose there exist the constants $c \geq \|S_v\|$, $v \in D$ and p with $\|S_u - S_v\| \leq p\|u-v\|$, u, $v \in D$.

(B) Let B be the closed ball in $C(I,X)$ centered in the origin and with radius $b, b = \big(c\|L\|+1\big)C_1\|\alpha\|_1$.

(P) For every $v \in D$ we define the linear bounded projector P_{1v} by $P_{1v}(x) = \mathcal{U}_v\big(\cdot,0\big)x(0)$. For every $v \in D$ let P_{3v} be a linear bounded projector from $\ker L_{1v}$ in $\ker L_{1v}$ defined by $P_{3v}\big(\mathcal{U}_v(\cdot,0)c\big) = \mathcal{U}_v(\cdot,0)c_1$ such that Im $P_{3v} = \ker\big(L|_{\ker L_{1v}}\big)$.

Remark 1.1. If the operator A does not depend on w, the differential inclusion in (BP) is linear or semi-linear, then (A) has to read as: $\big\{A(t),\, t \in I\big\}$ generates a unique strongly continuous evolution system $\mathcal{U}(t,s)$, $0 \leq s \leq t \leq T$. Also, L_{1v} is L_1 and S_v is S. In this case $C_2 = 0$ and $p = 0$.

We will need the following fixed point theorems for multifunctions:

Theorem 1.1 [26, Theorem 1]

Let D be a nonvoid, convex and closed subset of a locally convex space X. Let $\psi : D \longrightarrow CCo(D)$ be an upper semicontinuous multifunction such that $\overline{\psi(D)}$ is compact. Then ψ has a fixed point i.e., there exists an $x \in D$ such that $x \in \psi(x)$.

Theorem 1.2 [7, Theorem 11.1]

Let $D \neq \emptyset$ be a closed subset of a Banach space X and $F : D \longrightarrow C(D)$ be a contraction, with closed values. Then $Fix(F) \neq \emptyset$.

2. Existence result

We will prove two existence theorems, one based on a multivalued version of the Banach fixed point theorem, Theorem 1.2., the other based on the Himmelberg-Bohnenblust-Karlin fixed point theorem, Theorem 1.1.

A function $x \in C(I,X)$ is said to be a *mild solution* of the boundary value problem (BP) if it satisfies:

$$x(t) = \mathcal{U}_x(t,0)x(0) + \int_0^t \mathcal{U}_x(t,s)f(s)ds\,,\ t \in I, \quad \text{and} \quad Lx = 0\,,$$

where $f(\cdot) \in S^1_{F(\cdot,x(\cdot))}$.

For $v \in 0$ let us consider the following semi-linear differential inclusion:

$$\begin{cases} dx(t)/dt \in A(t,v)x(t) + F\big(t,x(t)\big)\,, & t \in I, \\ Lx = 0\,. \end{cases}$$

Remark 2.1. The above problem has a mild solution due to the Theorem 1 [21] or the Theorem 2.1 or 2.2 below. The difference between the two approaches lies in the fact that we get the weak compactness of the set S^1_F using the reflexivity of the space X (which it is not assumed in [21]) while in [21] it is used the assumption that the values of F are weakly compact (which it is not assumed here). Our approach appears in [18], [19] too.

Remark 2.2. As it is shown in [17] or [16] the existence of the solutions of the problem (BP) is equivalent to the existence of the fixed points of the operator $\psi : D \longrightarrow P(D)$, $\psi(v) = C_v(v)$ defined by:

$$C_v(x) = \Big\{ y \in D \,\big|\, y(t) = P_{3v}\Big(P_{1v}(x)\Big) - S_v L \int_0^t \mathcal{U}_v(t,s)f(s)ds + \int_0^t \mathcal{U}_v(t,s)f(s)ds\,, \quad f \in S^1_{F(\cdot,x(\cdot))} \Big\}\,.$$

From [6] we have that it is possible to consider the first term in the expression of $y(t)$ as zero, what we will do in the sequel.

The *t-section* of $\psi(D)$ is:

$$C(t) := \big\{ y(t) \,|\, y \in C_v(v)\,, \quad v \in D \big\}\,.$$

(S_1) When A depends on t and w we suppose that for every $t \in I$, $C(t)$ is relatively compact.

If A depends on t only this will be proved in Lemma 2.3.

Theorem 2.1

If the following assumptions hold: $(X), (U_1), (F_{1-3}), (L), (L_1), (S_v), (P)$, *and* $0 < C_3 = \big(c\|L\| + 1\big)\big(C_1\|k\|_1 + C_2 T\|\alpha\|_1\big) + p\|L\|C_1\|\alpha\|_1 < 1$, *then boundary value problem* (BP) *has a mild solution in* D.

Theorem 2.2

If there hold the assumptions (X), (A), (U_1), (U_3), (F_{1-3}), *and* (U_2) *or* (S_1), *then there exists a mild solution of* (BP) *in* D.

In the next lemmata we suppose that there are fulfilled all the necessary assumptions listed at the end of the first paragraph.

Lemma 2.1

If $v \in D$, *then for every* $\psi(v) \in CCo\big(C(I,X)\big)$.

Proof. $\psi(v)$ is nonempty. The convexity of $\psi(v)$ follows from the convexity of the values of F and from the linearity of the operators S_v and L. To prove that $C_v(v) \in C\big(C(I,X)\big)$ let us consider $(y_n)_{n\geq 1} \subset C_v(v)$ a sequence converging uniformly to an element $y \in D$. We have to show that $y \in C_v(v)$, i.e., there exists an element $f \in S^1_{F(\cdot,v(\cdot))}$ such that:

$$y(t) = -S_v L \int_0^t \mathcal{U}_v(t,s)f(s)ds + \int_0^t \mathcal{U}_v(t,s)f(s)ds\,.$$

If $y_n \in C_v(v)$, then it results that there exists $f_n(\cdot) \in S^1_{F(\cdot,v(\cdot))}$ such that for every $n \in \mathbb{N}$:

$$y_n(t) = -S_v L \int_0^t \mathcal{U}_v(t,s)f_n(s)ds + \int_0^t \mathcal{U}_v(t,s)f_n(s)ds\,.$$

Since F is integrably bounded, $\{f_n\}_{n\geq 1}$ is a bounded set in $L_1(I,X)$. By Pettis's theorem ([11], Theorem 2.11.2) taking into account the reflexivity of X it results that $\cup_{n\geq 1}\{f_n(t)\}$ is sequentially weakly compact, $t \in I$. From [27] Proposition 1.2 we have that $\{f_n\}_{n\geq 1}$ is a metrizable relatively weak compact subset in $L_1(I,X)$. It means that (taking a subsequence if necessary and keeping the same notations) $(f_n)_{n\geq 1}$ converges weakly in $L_1(I,X)$ to some $f \in L_1(I,X)$. It remains to show that $f(\cdot) \in S^1_{F(\cdot,v(\cdot))}$. By Mazur lemma ([28], p. 199), ([23], p. 65) there exists a sequence $(g_n)_{n\geq 1}$ formed by convex combinations of $\{f_n\}_{n\geq 1}$ tending to f in $L_1(I,X)$. It is clear that $g_n(\cdot) \in F(\cdot,v(\cdot))$, and, moreover $g_n(\cdot) \in S^1_{F(\cdot,v(\cdot))}$, $n \in \mathbb{N}$. It follows that $g(t) \in F(t,v(t))$ a.e. on I and $f(\cdot) \in S^1_{F(\cdot,v(\cdot))}$.

For every $t \in I$ the map $\bar{f} \longrightarrow \int_0^t \mathcal{U}_v(t,s)\bar{f}(s)ds$ from $L_1(I,X)$ into X is continuous and linear and, by Theorem IV.7.4 in [25], it remains continuous as a map from $L_1(I,X)_w$ in X_w. Hence, for every $t \in I$, the sequence $\big(y_n(t)\big) \longrightarrow y(t)$ in X_w. But $y_n(\cdot) \longrightarrow y(\cdot)$, and this implies that $y \in C_v(v)$. $\square$

Proof of Theorem 2.1. Let us find an upper bound for the Hausdorff-Pompeiu distance of the sets $\psi(u)$ and $\psi(v)$, $u, v \in D\, d\big(\psi(u), \psi(v)\big)$. Our desire is to show that ψ is a contraction. In order to do this let be $y \in \psi(u)$, $z \in \psi(v)$. If so, there are $f \in S^1_{F(\cdot,v(\cdot))}$, $g \in S^1_{F(\cdot,u(\cdot))}$ such that:

$$y(t) = -S_u L \int_0^t \mathcal{U}_u(t,s) g(s) ds + \int_0^t \mathcal{U}_u(t,s) g(s) ds\,.$$
$$z(t) = -S_v L \int_0^t \mathcal{U}_v(t,s) f(s) ds + \int_0^t \mathcal{U}_v(t,s) f(s) ds\,.$$

Then:

$$\begin{aligned}
\|y(t) - z(t)\| &= \Big\| S_u L \int_0^t \mathcal{U}_u(t,s) g(s) ds - S_v L \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\quad + \Big\| \int_0^t \mathcal{U}_u(t,s) g(s) ds - \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\le \Big\| S_u L \int_0^t \mathcal{U}_u(t,s) g(s) ds - S_u L \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\quad + \Big\| S_u L \int_0^t \mathcal{U}_u(t,s) f(s) ds - S_v L \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\quad + \Big\| \int_0^t \mathcal{U}_u(t,s) g(s) ds - \int_0^t \mathcal{U}_v(t,s) g(s) ds \Big\| \\
&\quad + \Big\| \int_0^t \mathcal{U}_u(t,s) g(s) ds - \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\le \|S_u\| \|L\| \, \Big\| \int_0^t \mathcal{U}_u(t,s) g(s) ds - \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\quad + \|S_u - S_v\| \, \Big\| L \int_0^t \mathcal{U}_v(t,s) f(s) ds \Big\| \\
&\quad + C_2 \int_0^t \|g(s)\| \int_s^t \|u(\tau) - v(\tau)\| d\tau ds + C_1 \|k\|_1 \|u - v\| \\
&\le \|S_u\| \|L\| \Big[C_2 \int_0^t \|g(s)\| \int_s^t \|u(\tau) - v(\tau)\| d\tau ds + C_1 \|k\|_1 \|u - v\| \Big] \\
&\quad + p \|L\| C_1 \|\alpha\|_1 \|u - v\| + \big[C_1 \|k\|_1 + C_2 T \|\alpha\|_1 \big] \|u - v\| \\
&\le C_3 \|u - v\|\,.
\end{aligned}$$

Hence, $\|y - z\| \le C_3 \|u - v\|$ and it follows that:

$$d\big(\psi(u), \psi(v)\big) \le C_3 \|u - v\|\,. \ \square$$

Lemma 2.2

There holds the inclusion $\psi(D) \subset B \cap D$.

Proof. For any $v \in D$ and $y \in \psi(v)$ we have:

$$\|y(t)\| \leq \|S_v\| \|L\| C_1 \|\alpha\|_1 + C_1 \|\alpha\|_1 \leq \big(c\|L\| + 1\big) C_1 \|\alpha\|_1 \,. \ \square$$

Lemma 2.3

$C(t)$, *the* t*-section of* $\psi(D)$, *is relatively compact in* X.

Proof. If A depends on t and w this is (S_1). If not then following [21] we note that:

$$C(t) \subseteq (1 - SL) \int_0^t \mathcal{U}(t,s) P(s) ds \,,$$

where $P(s) = \{x \in X \mid \|x\| = \sup \{|F(s,z)| : \|z\| \leq b\}\}$, and S is the unique pseudo-inverse of L to $\ker L_1$. Since $\mathcal{U}(t,s)$ is compact, we have that $\overline{\mathcal{U}(t,s)P(s)}$ is a convex and compact subset in X. Also $s \longrightarrow \mathcal{U}(t,s)P(s)$ is measurable. So by the Radstrom embedding theorem $\int_0^t \mathcal{U}(t,s)P(s)ds$, [14], is a compact and convex subset of X. We get that $\overline{C(t)}$ is compact in X. $\square$

Lemma 2.4

The mapping $x \longrightarrow \psi(x)$ *from* D *into* $\psi(D)$ *is uniformly upper semicontinuous.*

Proof. Choose $\varepsilon > 0$ and $u \in D$ arbitrary. We shall determine a positive η such that if $v \in D$ with $\|v - u\| < \eta$ then $\|z - y\| < \varepsilon$, where $y \in \psi(u)$ and $z \in \psi(v)$. If so, there are $f \in S^1_{F(\cdot,v(\cdot))}, g \in S^1_{F(\cdot,u(\cdot))}$ such that:

$$y(t) = -S_u L \int_0^t \mathcal{U}_u(t,s) g(s) ds + \int_0^t \mathcal{U}_u(t,s) g(s) ds \,.$$

$$z(t) = -S_v L \int_0^t \mathcal{U}_u(t,s) f(s) ds + \int_0^t \mathcal{U}_v(t,s) f(s) ds \,.$$

It follows:

$$\|y(t) - z(t)\| \leq \left\| S_u L \int_0^t \mathcal{U}_u(t,s) g(s) ds - S_v L \int_0^t \mathcal{U}_v(t,s) f(s) ds \right\| + \left\| \int_0^t \mathcal{U}_u(t,s) g(s) ds - \int_0^t \mathcal{U}_v(t,s) f(s) ds \right\| .$$

Performing the same estimations as in the proof of Theorem 2.1 we get: $\|y - z\| \leq C_3 \|u - v\|$. C_3 does not depend on u, being a constant, so the lemma is obvious. $\square$

Lemma 2.5

$\psi(D)$ *is a family of equicontinuous maps.*

Proof. To prove that $\psi(D)$ is a family of equicontinuous maps we have to show that for every $\varepsilon > 0$ there is $\mu > 0$ such that for every, $t,\ t+\delta \in I,\ 0 < \delta < \mu, x \in D$, $y \in \psi(v)$ there holds $\|y(t+\delta) - y(t)\| < \varepsilon$. Then we have:

$$
\begin{aligned}
\|y(t+\delta)-y(t)\| \le & \left\| S_v L \Big[\int_0^{t+\delta} \mathcal{U}_x(t+\delta,s)v(s)ds - \int_0^t \mathcal{U}_x(t,s)v(s)ds \Big] \right\| \\
& + \left\| \int_0^{t+\delta} \mathcal{U}_x(t+\delta,s)v(s)ds - \int_0^t \mathcal{U}_x(t,s)v(s)ds \right\| \\
\le & \big(\|S_v\|\|L\|+1\big) \left\| \int_0^{t+\delta} \mathcal{U}_x(t+\delta,s)v(s)ds - \int_0^t \mathcal{U}_x(t,s)v(s)ds \right\| \\
\le & \big(c\|L\|+1\big) \left[\left\| \int_0^t \big[\mathcal{U}_x(t+\delta,s)-1\big]\mathcal{U}_x(t,s)v(s)ds \right\| \right. \\
& \left. + \int_t^{t+\delta} \big\|\mathcal{U}_x(t+\delta,s)v(s)\big\| ds \right] \\
\le & \big(c\|L\|+1\big) \left[C_1 \int_t^{t+\delta} \|v(s)\| ds \right. \\
& \left. + \left\| \big(\mathcal{U}_x(t+\delta,t)-1\big) \int_0^t \mathcal{U}_x(t,s)v(s)ds \right\| \right].
\end{aligned}
$$

By (U_3) and Theorem 9. p. 4 in [8] we conclude that $\psi(S)$ is a family of equicontinuous maps. □

Proof of Theorem 2.2. Consider again $\psi : D \longrightarrow P(D)$. From Lemma 2.3 and 2.5, based on the Ascoli-Arzelá Theorem [30] we have that $\psi(D)$ is relatively compact. From Lemma 2.1–2.5 we observe that all the assumptions of Theorem 1.1 are satisfied, hence (BP) has a mild solution in D. □

Remark 2.3. If L is considered as $Lx = x(0) - x(T)$, then under the above conditions we get periodic mild solution.

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