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P–adic continuously differentiable functions of several variables

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Abstract

Let K be a non–Archimedean field containing $\mathbb{Q}_p$, the field of the p–adic numbers and let $\mathbb{Z}_p$ denote the ring of p–adic integers. In this paper, we construct the Mahler and van der Put base for $C^n\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$, the space of n–times continuously differentiable functions from $\mathbb{Z}_p \times \mathbb{Z}_p$ to K.

1. Introduction

Let K be a non–Archimedean field containing $\mathbb{Q}_p$, the field of the p–adic numbers. As usual, we denote the ring of p–adic integers by $\mathbb{Z}_p$. For the moment, we are well acquainted with the following bases for $C\big(\mathbb{Z}_p \longrightarrow K\big)$ the Banach space of continuous functions from $\mathbb{Z}_p$ to K. On one hand, we have the Mahler base $\binom{x}{n}(n \in \mathbb{N})$, consisting of polynomials of degree n (see [5] p. 149 or [3]) and on the other hand we have the van der Put base $\{e_n \big| n \in \mathbb{N}\}$ (see [5] p. 189 or [6] p. 61) consisting of locally constant functions. e_n is defined as follows : $e_0(x) = 1$ and for $n > 0$, e_n is the characteristic function of the ball $\{\alpha \in \mathbb{Z}_p \big| \, |\alpha - n| < 1/n\}$. There also exists a generalization of these bases to the space of C^n–functions (i.e. the n–times continuously differentiable functions). For details see [5] in case $n = 1$ and [2] for a more general treatment. We will now construct the similar bases for C^n–functions of several variables. Since the case $n = 0$ has already been treated before ([1], [5]), we can restrict our attention to the case $n \neq 0$. For simplicity we reduce to two variables but everything that follows can be done for an arbitrary number of variables. Let's start by defining the C^n–functions.

Definition. For $f : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K$, the first difference quotients $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ are defined as

$$\phi_1^{(1)} f(x, x', y) = \frac{f(x,y) - f(x',y)}{x - x'} \quad \text{and} \quad \phi_1^{(2)} f(x, y, y') = \frac{f(x,y) - f(x,y')}{y - y'} .$$

If $\phi_1^{(1)} f$ and $\phi_1^{(2)} f$ can be extended to continuous functions on $\mathbb{Z}_p^3$ then f is called a C^1–function.

The space of all C^1–functions, will be denoted by $C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$. For the difference quotients of second order, we get

$$\begin{aligned}
\phi_2^{(11)} f(x, x', x'', y) &= \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x'', y)}{x' - x''} \\
\phi_2^{(21)} f(x, x', y, y') &= \frac{\phi_1^{(1)} f(x, x', y) - \phi_1^{(1)} f(x, x', y')}{y - y'} \\
\phi_2^{(12)} f(x, x', y, y') &= \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x', y, y')}{x - x'} \\
\phi_2^{(22)} f(x, y, y', y'') &= \frac{\phi_1^{(2)} f(x, y, y') - \phi_1^{(2)} f(x, y, y'')}{y' - y''}
\end{aligned}$$

and f is a C^2–function if those four functions can be extended to continuous functions on $\mathbb{Z}_p^4$.

Following the notations above, we denote $C^2\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$ for the space of all C^2–functions.

Remark: $\phi_2^{(21)} f(x, x', y, y') = \phi_2^{(12)} f(x, x', y, y')$.

Continuing in the same way, we define the difference quotients of n–th order and the C^n–functions. Using these definitions, we have the following proposition for the difference quotient of a function f from $\mathbb{Z}_p$ to K .

Proposition

Let $f \in C^2\big(\mathbb{Z}_p \longrightarrow K\big)$, *then* $\overline{\phi_1 f} \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$.

Proof. Recall that

$$\overline{\phi_1 f}\,(x,y) = \begin{cases} \phi_1 f(x,y) = \dfrac{f(x)-f(y)}{x-y} & \text{if} \quad x \neq y \\ f'(x) & \text{if} \quad x = y\,. \end{cases}$$

Now,

$$\phi_1^{(1)}\big(\overline{\phi_1 f}\big)(x,x',y) = \frac{\overline{\phi_1 f}(x,y) - \overline{\phi_1 f}(x',y)}{x-x'} = \tilde{\phi}_2 \; f(x,x',y)$$

which is continuous for $x \neq x'$ ([4], p. 84).

If $x \neq y$ and $x' \neq y$ then

$$\phi_1^{(1)}\big(\overline{\phi_1 f}\big)(x,x',y) = \frac{\phi_1 f(x,y) - \phi_1 f(x',y)}{x-x'} = \phi_2 \, f(x,x',y)$$

and this can be extended to a continuous function on $\mathbb{Z}_p^3$.

If $x = y$ and $x' \neq y$ then

$$\phi_1^{(1)}\big(\overline{\phi_1 f}\big)(x,x',y) = \frac{f'(x) - \phi_1 f(x',x)}{x-x'} = \frac{f(x') - f(x) - (x'-x)f'(x)}{(x-x')^2}$$

and this is continuously extendible, since $f \in C^2$ implies that $f(x') = f(x) + (x'-x)f'(x) + (x'-x)^2 R_2(x,x')$ with R_2 continuous.

If $x \neq y$ and $x' = y$ then

$$\phi_1^{(1)}\big(\overline{\phi_1 f}\big)(x,x',y) = \frac{\phi_1 f(x,x') - f'(x')}{x-x'} = \frac{f(x) - f(x') - (x-x')f'(x')}{(x-x')^2}$$

and this is continuously extendible.

Analogous:

$$\phi_1^{(2)}\big(\overline{\phi_1 f}\big)(x,y,y') = \frac{\overline{\phi_1 f}(x,y) - \overline{\phi_1 f}(x,y')}{y-y'} = \tilde{\phi}_2 \; f(x,y,y')$$

is continuously extendible. Thus $\overline{\phi_1 f}$ is a C^1–function. □

2. Mahler's base

Given a continuous function $f : \mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K$, we have the following necessary and sufficient condition for f to be C^1. But first, let us recall that for $n = n_0 + n_1 p + \ldots + n_s p^s \in \mathbb{N}_0 = \mathbb{N}\backslash\{0\}$ with $n_s \neq 0$, n_- is defined to be $n_0 + n_1 p + \ldots + n_{s-1} p^{s-1}$ and $n - n_-$ is denoted by γ_n. We further put $\gamma_0 = 1 = \delta_0$ and $\delta_n = p^s$.

Theorem

$f(x,y) = \sum_{n,m} a_{n,m} \binom{x}{n}\binom{y}{m}$ is a C^1–function if and only if $\left|\frac{a_{i+j+1,k}}{j+1}\right| \longrightarrow 0$ and $\left|\frac{a_{i,j+k+1}}{k+1}\right| \longrightarrow 0$ as $i+j+k$ approach infinity or equivalently $\left|\frac{a_{n,m}}{\gamma_n}\right| \longrightarrow 0$ and $\left|\frac{a_{n,m}}{\gamma_m}\right| \longrightarrow 0$ as $n+m$ approach infinity.

Proof. $\binom{x}{i}\binom{y}{j}\binom{z}{k}$ is an orthonormal base for $C\big(\mathbb{Z}_p^3 \longrightarrow K\big)$ and also $\binom{u}{i}\binom{v}{j}\binom{w}{k}$ with $u = x,\ v = y - x - 1$ and $w = z$.

Then, we can write $g(x,y,z) = \sum_{i,j,k} \beta_{ijk} \binom{u}{i}\binom{v}{j}\binom{k}{k}$ with

$$\beta_{ijk} = \sum_{l,m,n} (-1)^{i+j+k-l-m-n} \binom{i}{l}\binom{j}{m}\binom{k}{n} g\big(l, l+m+1, n\big)$$

and $|\beta_{ijk}| \longrightarrow 0$ as $i+j+k$ approach infinity.

Take $g(x,y,z) = \phi_1^{(1)} f(x,y,z) = \frac{f(x,z)-f(y,z)}{x-y}$ for $x \neq y$ then

$$\begin{aligned} g\big(l, l+m+1, n\big) &= \frac{f(l,n) - f(l+m+1,n)}{-(m+1)} \\ &= -\sum_{\alpha,\beta=1}^{\infty} \frac{a_{\alpha,\beta}}{m+1} \left(\binom{l}{\alpha}\binom{n}{\beta} - \binom{l+m+1}{\alpha}\binom{n}{\beta}\right) \\ &= \sum_{\beta,\gamma=1}^{\infty} \sum_{\alpha=\gamma}^{\infty} \frac{a_{\alpha,\beta}}{\gamma} \binom{l}{\alpha-\gamma}\binom{m}{\gamma-1}\binom{n}{\beta} \end{aligned}$$

So,

$$\begin{aligned} \beta_{ijk} &= \sum_{\gamma,\beta=1}^{\infty} \sum_{\alpha=\gamma}^{\infty} \frac{a_{\alpha,\beta}}{\gamma} \sum_{l,m,n} (-1)^{i+j+k-l-m-n} \binom{i}{l}\binom{j}{m}\binom{k}{n}\binom{l}{\alpha-\gamma}\binom{m}{\gamma-1}\binom{n}{\beta} \\ &= \sum_{\gamma,\beta=1}^{\infty} \sum_{\alpha=\gamma}^{\infty} \frac{a_{\alpha,\beta}}{\gamma} \delta_{i,\alpha-\gamma}\ \delta_{j,\gamma-1}\ \delta_{k,\beta} \end{aligned}$$

since for example $\sum_l (-1)^{i-l} \binom{i}{l}\binom{l}{\alpha-\gamma} = \delta_{i,\alpha-\gamma}$.

So $\beta = k,\ \gamma = j+1$ and $\alpha = i+j+1$ and finally $\beta_{ijk} = \frac{a_{i+j+1,k}}{j+1}$.

Doing the same calculations with $u = x,\ v = y$ and $w = z-x-1$ for $\phi_1^{(2)} f(x,y,z)$ gives us the second condition. □

As a corollary, we get that if $\left|\frac{a_{n,m}}{\gamma_n\gamma_m}\right| \longrightarrow 0$ as $n+m$ approach infinity then $f(x,y) = \sum_{n,m} a_{n,m}\binom{x}{n}\binom{y}{m}$ is a C^1–function, but the converse is not necessarily true.

On $C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$ we now put the following norm $\|f\|_1 = \max\big\{\|f\|_s, \|\phi_1^{(1)} f\|_s, \|\phi_1^{(2)} f\|_s\big\}$, where $\|\ \|_s$ denotes the sup norm.

This is indeed a norm as can be easily verified and $C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$, $\|\ \|_1$ is a Banach space with base $\binom{x}{n}\binom{y}{m}$ $(n, m \in \mathbb{N})$ as we will prove now.

Theorem

The sequence $\max\big\{\gamma_n, \gamma_m\big\} \cdot \binom{x}{n} \cdot \binom{y}{m}(n, m \in \mathbb{N})$ *forms an orthonormal base for* $C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$.

Proof.

$$\Big\|\binom{x}{n}\binom{y}{m}\Big\|_s = \Big\|\binom{x}{n}\Big\|_s\Big\|\binom{y}{m}\Big\|_s = 1$$

$$\Big\|\phi_1^{(1)}\binom{x}{n}\binom{y}{m}\Big\|_s = \Big\|\phi_1\binom{x}{n}\Big\|_s\Big\|\binom{y}{m}\Big\|_s = \frac{1}{|\gamma_n|}$$

$$\Big\|\phi_1^{(2)}\binom{x}{n}\binom{y}{m}\Big\|_s = \Big\|\binom{x}{n}\Big\|_s\Big\|\phi_1\binom{y}{m}\Big\|_s = \frac{1}{|\gamma_m|}.$$

So

$$\Big\|\binom{x}{n}\binom{y}{m}\Big\|_1 = \max\Big\{1, \frac{1}{|\gamma_n|}, \frac{1}{|\gamma_m|}\Big\} = \frac{1}{|\max\{\gamma_n, \gamma_m\}|}.$$

Let us now consider an arbitrary (finite) linear combination $\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}$. Then,

$$\Big\|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\Big\|_1 \geq \Big\|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\Big\|_s \geq \Big|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{k}{i}\binom{l}{j}\Big| = |a_{k,l}|$$

but also

$$\Big\|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\Big\|_1 \geq \Big\|\phi_1^{(1)}\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\Big\|_s \geq \Big|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\phi_1\binom{x}{i}(k, k_-)\binom{l}{j}\Big| = \Big|\frac{a_{k,l}}{\gamma_k}\Big|$$

and

$$\left\|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\right\|_1 \geq \left\|\phi_1^{(2)}\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\right\|_s$$

$$\geq \left|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{k}{i}\,\phi_1\binom{y}{j}(l,l_-)\right| = \left|\frac{a_{k,l}}{\gamma_l}\right| .$$

Thus

$$\left\|\sum_{i=k}^{n}\sum_{j=l}^{m} a_{i,j}\binom{x}{i}\binom{y}{j}\right\|_1 \geq \left|\frac{a_{k,l}}{\max\{\gamma_k,\gamma_l\}}\right| = |a_{k,l}|\left\|\binom{x}{k}\binom{y}{l}\right\|_1 .$$

So, $\max\{\gamma_n,\gamma_m\}\cdot\binom{x}{n}\cdot\binom{y}{m}$ $(n,m\in\mathbb{N})$ is orthonormal ([5], proposition 50.4). Let $f\in C^1\big(\mathbb{Z}_p\times\mathbb{Z}_p\longrightarrow K\big)\subset C\big(\mathbb{Z}_p\times\mathbb{Z}_p\longrightarrow K\big)$.

As continuous function, we can write $f(x,y)=\sum a_{n,m}\binom{x}{n}\binom{y}{m}$.

The previous theorem tells us that $\left|\frac{a_{n,m}}{\gamma_n}\right|\longrightarrow 0$ and $\left|\frac{a_{n,m}}{\gamma_m}\right|\longrightarrow 0$ as $n+m$ approach infinity and thus also $\left|\frac{a_{n,m}}{\max\{\gamma_n,\gamma_m\}}\right|\longrightarrow 0$ as $n+m$ approach infinity.

So $f(x,y)=\sum\frac{a_{n,m}}{\max\{\gamma_n,\gamma_m\}}\max\{\gamma_n,\gamma_m\}\cdot\binom{x}{n}\cdot\binom{y}{m}$ in $C^1\big(\mathbb{Z}_p\times\mathbb{Z}_p\longrightarrow K\big)$, $\|\ \|_1$ and $\|f\|_1=\max_{n,m}\left|\frac{a_{n,m}}{\max\{\gamma_n,\gamma_m\}}\right|$. □

3. The van der Put base

In the sequel, we will use the following notation.

For $m,x\in\mathbb{Q}_p,\ x=\sum_{j=-\infty}^{+\infty}a_jp^j : m\lhd x$ if $m=\sum_{j=-\infty}^{i}a_jp^j$ for some $i\in\mathbb{Z}$.

We sometimes refer to the relation $\lhd$ between m and x as "m is an initial part of x" or "x starts with m".

Lemma 1

Let $f\in C\big(\mathbb{Z}_p\times\mathbb{Z}_p\longrightarrow K\big)$, B a ball in $\mathbb{Z}_p$ and S a ball in K.

Suppose $\phi_1^{(1)}f(n,n_-,m)=\frac{f(n,m)-f(n_-,m)}{n-n_-}\in S$ for $n,n_-\in B, n,m\in\mathbb{N}_0$ then $\phi_1^{(1)}f(x,x',y)=\frac{f(x,y)-f(x'-y)}{x-x'}\in S$ for $x,x'\in B$, $x\neq x'$, $y\in\mathbb{Z}_p$.

Proof. It suffices to prove this for $x,x'\in B\cap\mathbb{N},\ y\in\mathbb{N}$ since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, f is continuous and S is closed in K.

S is "convex" i.e. if $x_1,x_2,\dots,x_n\in S$ and $\lambda_1,\lambda_2,\dots,\lambda_n\in K$ with $|\lambda_i|\leq 1$ for all i and $\sum\lambda_i=1$ then $\sum\lambda_ix_i\in S$.

Let t be the common initial part of x and x'

$$\begin{aligned} t &= t_0 + t_1 p + \ldots + t_n p^n \\ x &= t_0 + t_1 p + \ldots + t_n p^n + x_{n+1} p^{n+1} + \ldots \\ x' &= t_0 + t_1 p + \ldots + t_n p^n + x'_{n+1} p^{n+1} + \ldots \quad \text{with} \quad x_{n+1} \neq x'_{n+1} \\ &\big(x_0 \neq x'_0 \Longrightarrow t = 0\big) \\ &\phi_1^{(1)} f(x, x', y) = \phi_1^{(1)} f(x, t, y) \frac{x - t}{x - x'} + \phi_1^{(1)} f(t, x', y) \frac{t - x'}{x - x'} \end{aligned}$$

and thus $\phi_1^{(1)} f(x, x', y) \in S$ as soon as $\phi_1^{(1)} f(x, t, y)$ and $\phi_1^{(1)} f(t, x', y) \in S$.

So there remains to prove: $\phi_1^{(1)} f(x, x', y) \in S$ for $x' \triangleleft x$.

There exist $t_1 \triangleleft t_2 \triangleleft \ldots \triangleleft t_n$ with $t_1 = x', t_n = x$ and $(t_j)_- = t_{j-1}$.

$$\phi_1^{(1)} f(x, x', y) = \sum_{j=2}^{n} \lambda_j \, \phi_1^{(1)} f(t_j, t_{j-1}, y) \quad \text{with} \quad \lambda_j = \frac{t_j - t_{j-1}}{x - x'}\,.$$

Now $|\lambda_j| \leq 1$, $\sum_{j=2}^{n} \lambda_j = 1$ and $\phi_1^{(1)} f(t_j, t_{j-1}, y) \in S$ for all j by assumption. So $\phi_1^{(1)} f(x, x', y) \in S$. □

Similarly, we can prove:

Lemma 2

Let $f \in C\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$, B a ball in $\mathbb{Z}_p$ and S a ball K.

Suppose $\phi_1^{(2)} f(n, m, m_-) = \frac{f(n,m) - f(n,m_-)}{m - m_-} \in S$ for $m, m_- \in B$, m, $n \in \mathbb{N}$ then $\phi_1^{(2)} f(x, y, y') = \frac{f(x,y) - f(x,y')}{y - y'} \in S$ for $y, y' \in B$, $y \neq y'$, $x \in \mathbb{Z}_p$.

Theorem

Let $f(x, y) = \sum a_{nm} \, e_n(x) e_m(y) \in C\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$, then $f \in N^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big) = \Big\{f \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big) \big| \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0\Big\}$ if and only if $\lim_{n \to \infty} \frac{a_{nm}}{\gamma_n} = 0$ for all m and $\lim_{m \to \infty} \frac{a_{nm}}{\gamma_m} = 0$ for all n.

Proof. Let $f \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$, then $f(x, y) = f(x_0, y) + \frac{\partial f}{\partial x}(x_0, y)(x - x_0) + (x - x_0)^2 R_1(x, x_0, y)$ for all y and $f(x, y) = f(x, y_0) + \frac{\partial f}{\partial y}(x, y_0)(y - y_0) + (y - y_0)^2 R_2(x, y, y_0)$ for all x where R_1 and R_2 are continuous functions.

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}\,.$$

So $f(x,y) = f(x_0,y) + (x-x_0)^2 R_1(x,x_0,y)$ for all y and $f(x,y) = f(x,y_0) + (y-y_0)^2 R_2(x,y,y_0)$ for all x.

Or equivalently,

$$\begin{cases} \dfrac{f(x,y)-f(x_0,y)}{x-x_0} = (x-x_0)R_1(x,x_0,y) \quad \text{for all} \quad y \\ \dfrac{f(x,y)-f(x,y_0)}{y-y_0} = (y-y_0)R_2(x,y,y_0) \quad \text{for all} \quad x\,. \end{cases}$$

In particular,

$$\begin{cases} \dfrac{f(n,m)-f(n_-,m)}{\gamma_n} = \gamma_n R_1(n,n_-,m) \\ \dfrac{f(n,m_-)-f(n_-,m_-)}{\gamma_n} = \gamma_n R_1(n,n_-,m_-) \quad \text{for all} \quad \text{m} \\ \dfrac{f(n,m)-f(n,m_-)}{\gamma_m} = \gamma_m R_2(n,m,m_-) \\ \dfrac{f(n_-,m)-f(n_-,m_-)}{\gamma_m} = \gamma_m R_2(n_-,m,m_-) \quad \text{for all} \quad \text{n}\,. \end{cases}$$

And thus

$$\frac{f(n,m)-f(n_-,m)-f(n,m_-)+f(n_-,m_-)}{\gamma_n} = \gamma_n\big(R_1(n,n_-,m)-R_1(n,n_-,m)\big) \quad \text{for all} \quad m$$

$$\frac{f(n,m)-f(n,m_-)-f(n_-,m)+f(n_-,m_-)}{\gamma_m} = \gamma_m\big(R_2(n,m,m_-)-R_2(n_-,m,m_-)\big) \quad \text{for all} \quad n\,.$$

So, $\lim\limits_{n\to\infty} \frac{a_{nm}}{\gamma_n} = 0$ for all m and $\lim\limits_{n\to\infty} \frac{a_{nm}}{\gamma_m} = 0$ for all n since taking $\lim\limits_{n\to\infty}$ for $a \in \mathbb{Z}_p (n \neq a)$ coincides with taking $\lim\limits_{n\to\infty}$ in the classical case and $\lim\limits_{n\to\infty} \gamma_n = 0$.

Now suppose that $\lim\limits_{n\to\infty} \frac{a_{nm}}{\gamma_n} = 0$ for all $m \in \mathbb{N}$ holds.

$$\begin{aligned} \frac{a_{nm}}{\gamma_n} + \frac{a_{nm_-}}{\gamma_n} &= \frac{f(n,m)-f(n_-,m)-f(n,m_-)+f(n_-,m_-)}{\gamma_n} \\ &\quad + \frac{f(n,m_-)-f(n_-,m_-)-f\big(n,(m_-)_-\big)+f\big(n_-,(m_-)_-\big)}{\gamma_n} \\ &= \frac{f(n,m)-f(n_-,m)-f\big(n,(m_-)_-\big)+f\big(n_-,(m_-)_-\big)}{\gamma_n}\,. \end{aligned}$$

Repeating this, gives

$$\frac{a_{nm}}{\gamma_n} + \frac{a_{nm_-}}{\gamma_n} + \frac{a_{n(m_-)_-}}{\gamma_n} + \ldots + \frac{a_{n0}}{\gamma_n} = \frac{f(n,m) - f(n_-,m)}{\gamma_n}$$

since

$$a_{n0} = \frac{f(n,0) - f(n_-,0)}{\gamma_n} .$$

On the left side, there are only a finite number of terms, so

$$\lim_{n\to\infty} \frac{f(n,m) - f(n_-,m)}{\gamma_n} = \lim_{n\to\infty} \left(\frac{a_{nm}}{\gamma_n} + \frac{a_{nm_-}}{\gamma_n} + \ldots + \frac{a_{n0}}{\gamma_n} \right) = 0 \text{ for all } m$$

and thus

$$\lim_{n\to a} \frac{f(n,m) - f(n_-,m)}{\gamma_n} = 0 \quad \big(a \in \mathbb{Z}_p (n \neq a),\ m \in \mathbb{N}\big) .$$

So for all $\varepsilon > 0$, there exists an $\delta_1 > 0$ such that $0 < |n - a| < \delta_1$ implies $\big|\phi_1^{(1)} f(n, n_-, m)\big| < \varepsilon$.

Let $a \in \mathbb{N}_0$, then $|a - a_-| > 0$ and thus $|a - a_-| > \delta_2$ for a certain δ_2 .

Now take $\delta = \min\{\delta_1, \delta_2\}$, we then have:

If $0 < |n-a| < \delta$ then $\big|\phi_1^{(1)} f(n, n_-, m)\big| < \varepsilon$ but also $|a-a_-| > \delta$. Lemma 1 with $B_1 = \{x \in \mathbb{Z}_p \big| |x-a| < \delta\}$ and $S = \{x \in \mathbb{Z}_p \big| |x| < \varepsilon\}$ assures that $\big|\phi_1^{(1)} f(x, x', y)\big| < \varepsilon$ for all $x, x' \in B_1$ and all $y \in \mathbb{Z}_p$.

In the same way, we can prove that $\big|\phi_1^{(2)} f(x, y, y')\big| < \varepsilon$ for all $x \in \mathbb{Z}_p$ and all $y, y' \in B_2 = \{x \in \mathbb{Z}_p \big| |y - b| < \delta\}$.

So f is C^1 in (a, b) with $\frac{\partial f}{\partial x}(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b) = 0$. □

Theorem

Let $f(x, y) = \sum f_{nm} e_n(x) e_m(y) \in C\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$ *, then* $f \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$ *if and only if for all* $a \in \mathbb{Z}_p$, $\lim\limits_{n\to a} \frac{f_{nm}}{\gamma_n}$ *exists for all* m *and* $\lim\limits_{m\to a} \frac{f_{nm}}{\gamma_m}$ *exists for all* n .

Proof.

$$f \in C^1 \Longleftrightarrow \phi_1^{(1)} f(x, x', y) = \frac{f(x, y) - f(x', y)}{x - x'} \quad \text{and}$$

$$\phi_1^{(2)} f(x, y, y') = \frac{f(x, y) - f(x, y')}{y - y'}$$

are extendible to continuous functions.

Thus $\lim\limits_{(x,x')\to(a,a)} \phi_1^{(1)} f(x, x', y)$ exists for all y .

In particular, $\lim\limits_{(n,n_-)\to(a,a)} \phi_1^{(1)} f(n,n_-,y)$ exists for all y or equivalently

$$\lim_{(n,n_-)\to(a,a)} \sum_k \frac{f_{nk}e_k(y)}{\gamma_n}$$

exists for all y.

So, $\lim\limits_{(n,n_-)\to(a,a)} \sum_{k\triangleleft m} \frac{f_{nk}}{\gamma_n}$ and $\lim\limits_{(n,n_-)\to(a,a)} \sum_{k\triangleleft m_-} \frac{f_{nk}}{\gamma_n}$ exists for all m and thus $\lim\limits_{n\to a} \frac{f_{nm}}{\gamma_n}$ exists for all m.

Similarly: $\lim\limits_{m\to a} \frac{f_{nm}}{\gamma_m}$ exists for all n.

Now assume this to be the case.

$$\frac{f(n,m)-f(n_-,m)}{\gamma_n} = \frac{f_{nm}}{\gamma_n} + \frac{f_{nm_-}}{\gamma_n} + \frac{f_{n(m_-)_-}}{\gamma_n} + \ldots + \frac{f_{n0}}{\gamma_n}.$$

So

$$\lim_{(n,n_-)\to(a,a)} \frac{f(n,m)-f(n_-,m)}{\gamma_n} = \lim_{n\to a}\frac{f_{nm}}{\gamma_n} + \lim_{n\to a}\frac{f_{nm_-}}{\gamma_n} + \ldots + \lim_{n\to a}\frac{f_{n0}}{\gamma_n}$$

exists for all $m \in \mathbb{N}$, which implies that

$$\lim_{(x,x')\to(a,a)} \frac{f(x,y)-f(x'-y)}{x-x'}$$

exists for all $y \in \mathbb{Z}_p$.

So $\phi_1^{(1)} f(x,x',y)$ is extendible to a continuous function.

Analogous: $\phi_1^{(2)} f(x,y,y')$ is extendible to a continuous function. Thus f is a C^1–function. □

With the theorems above, we can finally construct the van der Put base for $C^1\big(\mathbb{Z}_p \times Z_p \longrightarrow K\big)$.

Theorem

The sequence $e_n(x)e_m(y)$, $(x-n)e_n(x)e_m(y)$, $(y-m)e_n(x)e_m(y)$ $(n,m\in\mathbb{N})$ *forms an orthogonal base for* $C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$.

Proof.

$$\begin{aligned}
&\|e_n(x)e_m(y)\|_1\\
&\quad = \max\left\{\|e_n(x)e_m(y)\|_s,\ \|\phi_1^{(1)}e_n(x)e_m(y)\|_s,\ \|\phi_1^{(2)}e_n(x)e_m(y)\|_s\right\}\\
&\quad = \max\left\{\|e_n(x)\|_s\|e_m(y)\|_s,\ \|\phi_1 e_n(x)\|_s\ \|e_m(y)\|_s,\ \|e_n(x)\|_s\ \|\phi_1 e_m(y)\|_s\right\}\\
&\quad = \max\left\{1,\ \frac{1}{|\gamma_n|}\,,\ \frac{1}{|\gamma_m|}\right\}\\
&\quad = \frac{1}{|\max\{\gamma_n,\gamma_m\}|}\\
&\|(x-n)e_n(x)e_m(y)\|_1\\
&\quad = \max\left\{\|(x-n)e_n(x)e_m(y)\|_s,\ \|\phi_1^{(1)}(x-n)e_n(x)e_m(y)\|_s,\right.\\
&\qquad\quad \left.\|\phi_1^{(2)}(x-n)e_n(x)e_m(y)\|_s\right\}\\
&\quad = \max\left\{\|(x-n)e_n(x)\|_s\ \|e_m(y)\|_s,\ \|\phi_1(x-n)e_n(x)\|_s\ \|e_m(y)\|_s,\right.\\
&\qquad\quad \left.\|(x-n)e_n(x)\|_s\ \|\phi_1 e_m(y)\|_s\right\}\\
&\quad = \max\left\{\frac{|\gamma_n|}{p},\ 1\,,\ \frac{|\gamma_n|}{p|\gamma_m|}\right\}\\
&\quad = \max\left\{1,\frac{|\gamma_n|}{p|\gamma_m|}\right\}\ \text{for } n\neq 0\,.\ \left(\text{In case } n=0, \text{ we have } \frac{1}{|\gamma_m|}\right).
\end{aligned}$$

Similarly: $\|(y-m)e_n(x)e_m(y)\|_1 = \max\left\{1,\ \frac{|\gamma_m|}{p|\gamma_n|}\right\}$ for $m \neq 0$. $\Big($In case $m = 0$, we have $\frac{1}{|\gamma_n|}\Big)$.

Let us consider now a finite linear combination

$$\sum_{i=k}^{n}\sum_{j=l}^{m} a_{ij}e_i(x)e_j(y) + b_{ij}(x-i)e_i(x)e_j(y) + c_{ij}(y-j)e_i(x)e_j(y)$$

which we will denote $f(x,y)$

$$\|f(x,y)\|_1 \geq \|f(x,y)\|_s \geq |f(k,l)| = |a_{kl}|$$

but also

$$\|f(x,y)\|_1 \geq \left\|\phi_1^{(1)}f(x,y)\right\|_s \geq \left|\phi_1^{(1)}f(k,k-\delta_k,l)\right| = \frac{|a_{kl}|}{|\delta_k|}$$

and

$$\|f(x,y)\|_1 \geq \left\|\phi_1^{(12)}f(x,y)\right\|_s \geq \left|\phi_1^{(2)}f(k,l,l-\delta_l)\right| = \frac{|a_{kl}|}{|\delta_l|}\,.$$

Thus

$$\|f(x,y)\|_1 \geq \frac{|a_{kl}|}{|\max\{\gamma_k,\gamma_l\}|} = |a_{kl}|.\|e_k(x)e_l(y)\|_1 \,.$$

This can also be done for the other two kind of elements.

$$\|f(x,y)\|_1 \geq \|f(x,y)\|_s \geq |f(k+p\delta_k,l)| = \frac{|b_{kl}|.|\delta_k|}{p}$$

but also

$$\|f(x,y)\|_1 \geq \left\|\phi_1^{(1)}f(x,y)\right\|_s \geq \left|\phi_1^{(1)}f(k,k+p\delta_k,l)\right| = |b_{kl}|$$

and

$$\|f(x,y)\|_1 \geq \left\|\phi_1^{(2)}f(x,y)\right\|_s \geq \left|\phi_1^{(2)}f(k+p\delta_k,l,l-\delta_l)\right| = \frac{|b_{kl}|.|\delta_k|}{p|\delta_l|} \,.$$

Thus $\|f(x,y)\|_1 \geq |b_{kl}|.\|(x-k)e_k(x)e_l(y)\|_1$.

Similarly $\|f(x,y)\|_1 \geq |c_{kl}|.\|(y-l)e_k(x)e_l(y)\|_1$.

So, the $e_n(x)e_m(y)$, $(x-n)e_n(x)e_m(y)$, $(y-m)e_n(x)e_m(y)$ are orthogonal. To prove that it is also a base, we first calculate the coefficients of an $f \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$.

Let

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}e_i(x)e_j(y) + b_{i,j}(x-i)e_i(x)e_j(y) + c_{i,j}(y-j)e_i(x)e_j(y)$$

then

$$\begin{aligned} f(0,0) &= a_{0,0} \\ f(n,0) &= \sum_{i \lhd n} a_{i,0} + \sum_{i \lhd n} b_{i,0}(n-i) \\ f(n_-,0) &= \sum_{i \lhd n_-} a_{i,0} + \sum_{i \lhd n_-} b_{i,0}(n_- - i)\,. \end{aligned}$$

So

$$\begin{aligned} f(n,0)-f(n_-,0) &= a_{n,0} + \sum_{i \lhd n_-} b_{i,0}(n-n_-) \\ \frac{\partial f}{\partial x}(n_-,0) &= \sum_{i \lhd n_-} b_{i,0} \end{aligned}$$

and thus

$$f(n,0)-f(n_-,0)-\gamma_n\frac{\partial f}{\partial x}(n_-,0) = a_{n,0}\,.$$

Analogous : $f(0,m) - f(0,m_-) - \gamma_m \dfrac{\partial f}{\partial y}(0,m_-) = a_{0,m}$

$$f(n,m) = \sum_{\substack{i \lhd n \\ j \lhd m}} a_{i,j} + \sum_{\substack{i \lhd n \\ j \lhd m}} b_{i,j}(n-i) + \sum_{\substack{i \lhd n \\ j \lhd m}} c_{i,j}(m-j)$$

$$f(n,m_-) = \sum_{\substack{i \lhd n \\ j \lhd m_-}} a_{i,j} + \sum_{\substack{i \lhd n \\ j \lhd m_-}} b_{i,j}(n-i) + \sum_{\substack{i \lhd n \\ j \lhd m_-}} c_{i,j}(m_- - j)$$

$$f(n_-,m) = \sum_{\substack{i \lhd n_- \\ j \lhd m}} a_{i,j} + \sum_{\substack{i \lhd n_- \\ j \lhd m}} b_{i,j}(n_- - i) + \sum_{\substack{i \lhd n_- \\ j \lhd m}} c_{i,j}(m-j)$$

$$f(n_-,m_-) = \sum_{\substack{i \lhd n_- \\ j \lhd m_-}} a_{i,j} + \sum_{\substack{i \lhd n_- \\ j \lhd m_-}} b_{i,j}(n_- - i) + \sum_{\substack{i \lhd n_- \\ j \lhd m_-}} c_{i,j}(m_- - j)\,.$$

So

$$\begin{aligned} f(n-m) &- f(n_-,m) - f(n,m_-) + f(n_-,m_-) \\ &= a_{n,m} + \sum_{i \lhd n_-} b_{i,m}(n-n_-) + \sum_{j \lhd m_-} c_{n,j}(m-m_-) \end{aligned}$$

$$\frac{\partial f}{\partial x}(x,y) = \sum b_{i,j} e_i(x) e_j(y)$$

$$\frac{\partial f}{\partial y}(x,y) = \sum c_{i,j} e_j(x) e_j(y)\,.$$

Thus

$$\frac{\partial f}{\partial x}(n_-,m) = \sum_{\substack{i \lhd n_- \\ j \lhd m}} b_{i,j} \quad \text{and} \quad \frac{\partial f}{\partial x}(n_-,m_-) = \sum_{\substack{i \lhd n_- \\ j \lhd m_-}} b_{i,j}$$

so that

$$\frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n_-,m_-) = \sum_{i \lhd n_-} b_{i,m}\,.$$

Similarly:

$$\frac{\partial f}{\partial y}(n,m_-) - \frac{\partial f}{\partial y}(n_-,m_-) = \sum_{j \lhd m_-} c_{n,j}\,.$$

So finally,

$$\begin{aligned} a_{n,m} &= f(n,m) - f(n_-,m) - f(n,m_-) + f(n_-,m_-) \\ &\quad - \gamma_n \left(\frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n_-,m_-) \right) - \gamma_m \left(\frac{\partial f}{\partial y}(n,m_-) - \frac{\partial f}{\partial y}(n_-,m_-) \right)\,. \end{aligned}$$

We further also have:

$$\begin{aligned}
b_{0,0} &= \frac{\partial f}{\partial x}(0,0) \\
b_{n,0} &= \frac{\partial f}{\partial x}(n,0) - \frac{\partial f}{\partial x}(n_-,0) \\
b_{0,m} &= \frac{\partial f}{\partial x}(0,m) - \frac{\partial f}{\partial x}(0,m_-) \\
b_{n,m} &= \frac{\partial f}{\partial x}(n,m) - \frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n,m_-) + \frac{\partial f}{\partial x}(n_-,m_-) \\
c_{0,0} &= \frac{\partial f}{\partial y}(0,0) \\
c_{n,0} &= \frac{\partial f}{\partial y}(n,0) - \frac{\partial f}{\partial y}(n_-,0) \\
c_{0,m} &= \frac{\partial f}{\partial y}(0,m) - \frac{\partial f}{\partial y}(0,m_-) \\
c_{n,m} &= \frac{\partial f}{\partial y}(n,m) - \frac{\partial f}{\partial y}(n_-,m) - \frac{\partial f}{\partial y}(n,m_-) + \frac{\partial f}{\partial y}(n_-,m_-) .
\end{aligned}$$

There now remains to prove that a series with these coefficients converges and coincides with f .

Let

$$g(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{i,j}e_i(x)e_j(y) + b_{i,j}(x-i)e_i(x)e_j(y) + c_{i,j}(y-j)e_i(x)e_j(y)$$

where the $a_{n,m}$, $b_{n,m}$ and $c_{n,m}$ are given by the formulas above.
$f \in C^1\big(\mathbb{Z}_p \times \mathbb{Z}_p \longrightarrow K\big)$ so f, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous.

$$\begin{aligned}
\lim_{n\to\infty} |a_{n,0}| &= \lim_{n\to\infty} \left| f(n,0) - f(n_-,0) - \gamma_n \frac{\partial f}{\partial x}(n_-,0)\right| \\
&\leq \lim_{n\to\infty} \max\left(|f(n,0) - f(n_-,0)|\, ,\, |\gamma_n| . \left|\frac{\partial f}{\partial x}(n_-,0)\right|\right) = 0 \\
\lim_{m\to\infty} |a_{0,m}| &\leq \lim_{m\to\infty} \max\left(|f(0,m) - f(0,m_-)|\, ,\, |\gamma_m| . \left|\frac{\partial f}{\partial x}(0,m_-)\right|\right) = 0 \\
\lim_{n\to\infty} |a_{n,m}| &\leq \lim_{n\to\infty} \max\Bigg(|f(n,m) - f(n_-,m)|\, ,\, |f(n,m_-) - f(n_-,m_-)|\, , \\
&\qquad |\gamma_n| . \left|\frac{\partial f}{\partial x}(n_-,m) - \frac{\partial f}{\partial x}(n_-,m_-)\right|\, ,\, |\gamma_m| . \left|\frac{\partial f}{\partial y}(n,m_-) - \frac{\partial f}{\partial y}(n_-,m_-)\right|\Bigg) \\
&= 0 \quad \text{for all} \quad m
\end{aligned}$$

and

$$\lim_{m\to\infty} |a_{n,m}| \le \lim_{m\to\infty} \max\Big(|f(n,m)-f(n,m_-)|\,,\ |f(n_-,m)-f(n_-,m_-)|,$$
$$|\gamma_n|.\left|\frac{\partial f}{\partial x}(n_-,m)-\frac{\partial f}{\partial x}(n_-,m_-)\right|\,,\ |\gamma_m|.\left|\frac{\partial f}{\partial y}(n,m_-)-\frac{\partial f}{\partial y}(n_-,m_-)\right|\Big)$$
$$= 0 \quad \text{for all} \quad n\,.$$

The same can be done for the $b_{n,0}, b_{0,m}, b_{n,m}, c_{n,0}, c_{0,m}$ and $c_{n,m}$. Thus the series $g(x,y)$ converges uniformly.

Since $e_i(x)e_j(y)$, $(x-i)e_i(x)e_j(y)$ and $(y-j)e_i(x)e_j(y)$ belong to $C\big(\mathbb{Z}_p\times\mathbb{Z}_p \longrightarrow K\big)$, $g(x,y)$ is also an element of $C\big(\mathbb{Z}_p\times\mathbb{Z}_p \longrightarrow K\big)$. Further on, we have $f(n,m) = g(n,m)$ for all $n,m\in\mathbb{N}$. By continuity, $f=g$ which proves the theorem. □

Generalization

The sequence $(x-i)^k(y-j)^l e_i(x)e_j(y)$ with $0\le k+l\le n,\ i\in\mathbb{N}$ and $j\in\mathbb{N}$ forms an orthogonal base for $C^n\big(\mathbb{Z}_p\times\mathbb{Z}_p\longrightarrow K\big)$ whereby every C^n–function f can be written as

$$f(x,y) = \sum_{i,j=0}^{\infty}\sum_{k+l=0}^{n} a_{i,j}^{k,l}\frac{(x-i)^k}{k!}\frac{(y-j)^l}{l!}e_i(x)e_j(y)$$

with

$$a_{i,j}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k\partial y^l}(i,j) - \sum_{\alpha=0}^{n-k-l}\frac{\partial^{k+l+\alpha} f}{\partial x^{k+\alpha}\partial y^l}(i_-,j)\frac{\gamma_i^\alpha}{\alpha!} - \sum_{\beta=0}^{n-k-l}\frac{\partial^{k+l+\beta} f}{\partial x^k\partial y^{l+\beta}}(i,j_-)\frac{\gamma_j^\beta}{\beta!}$$
$$+\sum_{\alpha+\beta=0}^{n-k-l}\frac{\partial^{k+l+\alpha+\beta} f}{\partial x^{k+\alpha}\partial y^{l+\beta}}(i_-,j_-)\frac{\gamma_i^\alpha\gamma_j^\beta}{\alpha!\beta!} \qquad \text{for} \quad i\neq 0 \quad \text{and} \quad j\neq 0$$
$$a_{i,0}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k\partial y^l}(i,0) - \sum_{\alpha=0}^{n-k-l}\frac{\partial^{k+l+\alpha} f}{\partial x^{k+\alpha}\partial y^l}(i_-,0)\frac{\gamma_i^\alpha}{\alpha!} \qquad \text{for} \quad i\neq 0$$
$$a_{0,j}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k\partial y^l}(0,j) - \sum_{\beta=0}^{n-k-l}\frac{\partial^{k+l+\beta} f}{\partial x^k\partial y^{l+\beta}}(0,j_-)\frac{\gamma_j^\beta}{\beta!} \qquad \text{for} \quad j\neq 0$$

and

$$a_{0,0}^{k,l} = \frac{\partial^{k+l} f}{\partial x^k\partial y^l}(0,0)\,.$$

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