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Fiberwise shape theory

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Abstract

We shall describe a modification of fiberwise homotopy theory which we call fiberwise shape theory. This is accomplished by constructing the fiberwise shape category $\mathcal{F}s_B$. The category $\mathcal{F}s_B$ is built using multi-valued functions. Its objects are fiberwise topological spaces while its morphisms are fiberwise homotopy classes of collections of multi-valued functions which we call fiberwise multi-nets. When B is a single-element space, the fiberwise shape category is isomorphic with the shape category. Various authors have previously given other descriptions of fiberwise shape categories under additional assumptions. Our description is intrinsic in the sense that we do not use any outside objects. It is a fiberwise version of the author's extension to arbitrary topological spaces of Sanjurjo's approach to shape theory via small multi-valued functions.

1. Introduction

The book "Fiberwise topology" by I. M. James [13] beautifully exposes the fiberwise point of view in topology. It is the study of the category $\mathcal{F}_B$ of fiberwise topological spaces and fiberwise or fiber-preserving maps. Almost all invariants, notions, and results in topology have their fiberwise analogues. In particular, we can define fiberwise homotopy theory which studies the fiberwise homotopy category $\mathcal{F}h_B$ whose objects are fiberwise topological spaces and whose morphisms are fiberwise homotopy classes of fiber-preserving maps (see Chapter IV of [13]).

On the other hand, the classical homotopy theory has been modified by the introduction of shape theory. The modification was invented by K. Borsuk [6] with the desire to handle more successfully spaces with bad local properties. The new improved homotopy theory that he named shape theory agrees with the old on spaces with nice local properties, for example on absolute neighborhood retracts. The key idea in Borsuk's approach was to replace homotopy classes of maps with homotopy classes of sequences of maps that he calls fundamental sequences.

The aim of this paper is to define shape theory for fiberwise topological spaces. In other words, we shall describe a fiberwise shape category $\mathcal{F}s_B$ whose objects are fiberwise topological spaces and whose morphisms will be fiberwise homotopy classes of collections of multi-valued functions that we call fiberwise multi-nets. Therefore, our approach can be regarded as a fiberwise version of the author's description [7] of the shape category $\mathcal{S}h$ which is an extension of Sanjurjo's method [17] from compact metric spaces to arbitrary topological spaces.

When B is a single-element topological space, the fiberwise multi-net and fiberwise homotopy for fiberwise multi-nets agree with multi-net and homotopy for multi-nets from [7]. Hence, the main result in [8] implies that in this case the category $\mathcal{F}s_B$ is isomorphic to the shape category $\mathcal{S}h$ [15].

The following features of our description of fiberwise shape category deserve to be emphasized. It is intrinsic in the sense that we do not use any outside objects (like fiberwise ANR's, fiberwise embeddings, fiberwise resolutions, and/or fiberwise expansions). There are no restrictions on the topological space B. Finally, it is extremely simple and the only tricky part is in defining the composition of fiberwise homotopy classes of fiberwise multi-nets between fiberwise spaces.

There are several previous attempts to get fiberwise versions of shape theory but they all make some assumptions on spaces under consideration. We note papers by H. Kato [14], S. C. Metcalf [16], Y. Yagasaki [19], M. Clapp and L. Montejano [9], and V. H. Baladze [2], [3], [4], [5]. In [5], which provides the most general setting, Baladze exhibits fiberwise shape theory for maps of arbitrary topological spaces into a metrizable space B. He follows the method of ANR_B-resolutions, i.e., resolutions of spaces over B consisting of fiberwise absolute neighborhood retracts ([11], [12], [18], [19]) providing only an outline without any proofs.

The natural questions of relationship of our fiberwise shape category with those mentioned above are deferred to another paper. We only discuss the problem of identifying fiberwise topological spaces on which fiberwise shape theory and fiberwise homotopy theory coincide.

2. Basic notions of fiberwise homotopy theory

Throughout this paper B denotes a topological space which we keep fixed and call a base space. We follow also the convention that a map is a short name for a continuous single-valued function.

By a *fiberwise topological space* we mean a pair (X, f) consisting of a topological space X and a map $f : X \to B$. Of course, the map f is often dropped from the notation so that we talk about a fiberwise topological space or a *fiberwise space* X. We shall use letters X, Y, Z, and W to denote fiberwise spaces $(X,\, f)$, $(Y,\, g)$, $(Z,\, h)$, and $(W,\, k)$, respectively.

Fiberwise topology is a part of topology which studies fiberwise spaces. In order to do this more successfully we use fiberwise maps to compare them.

Let X and Y be fiberwise spaces. A map $m : X \to Y$ is *fiberwise* or *fiber-preserving* provided $f = g \circ m$. Let $\mathcal{F}_B$ denote the category whose objects are fiberwise topological spaces and whose morphisms are fiberwise maps.

The definition of fiberwise homotopy involves the *fiberwise cylinder* $(X \times I,\, f \times I)$ of the fiberwise space $(X,\, f)$, where I denotes the unit closed segment and the map $f \times I : X \times I \to B$ is defined by $f \times I(x,\, t) = f(x)$ for every $x \in X$ and every $t \in I$.

Let p and q be fiberwise maps between fiberwise spaces X and Y. We shall say that p and q are *fiberwise homotopic* and write $p \simeq_B q$ provided there is a fiberwise map $m : X \times I \to Y$ called *fiberwise homotopy* such that $m(x,\, 0) = p(x)$ and $m(x,\, 1) = q(x)$ for every $x \in X$. The relation of fiberwise homotopy is an equivalence relation and we denote the fiberwise homotopy class of a fiberwise map p by $[\,p\,]_B$. Since the fiberwise homotopy relation $\simeq_B$ is also compatible with the composition, one can define the composition of fiberwise homotopy classes of fiberwise maps by composing representatives, i.e., $[\,q\,]_B \circ [\,p\,]_B = [\,q \circ p\,]_B$, where $p : X \to Y$ and $q : Y \to Z$. In this way one obtains the fiberwise homotopy category $\mathcal{F}h_B$, whose objects are fiberwise topological spaces and whose morphisms are fiberwise homotopy classes of fiberwise maps. There is a homotopy functor H_B from $\mathcal{F}_B$ to $\mathcal{F}h_B$ which keeps the objects fixed and takes a fiberwise map p into its fiberwise homotopy class $[\,p\,]_B$.

For further information concerning fiberwise topology and fiberwise homotopy theory see excellent books [12] and [13].

3. Normal covers and multi-valued functions

In this section we shall introduce notions and results on normal covers and multi-valued functions that are required for our theory.

Let $\hat{Y}$ denote the collection of all normal covers of a topological space Y [1]. With respect to the refinement relation $>$ the set $\hat{Y}$ is a directed set. Two normal covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a normal cover and it's equivalence class by the same symbol. Consequently, $\hat{Y}$ also stands for the associated quotient set.

If σ is a normal cover of a space Y, let σ^+ be the collection of all normal covers of Y which refine σ while σ^* denotes the set of all normal covers τ of Y such that the star $st(\tau)$ of τ refines σ. Similarly, for a natural number n, σ^{*n} denotes the set of all normal covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Let $\tilde{Y}$ denote the collection of all finite subsets c of $\hat{Y}$ which have a unique (with respect to the refinement relation) maximal element which we denote either by $\tilde{c}$ or by $[c]$. We consider $\tilde{Y}$ ordered by the inclusion relation and regard $\hat{Y}$ as a subset of single-element subsets of $\hat{Y}$. Notice that $\tilde{Y}$ is a cofinite directed set.

We shall repeatedly use the following lemma (see [15, p. 9]). Let us agree that an *increasing* function $f : P \to P$ of a partially ordered set $(P, <)$ into itself is a function which satisfies $x < f(x)$ for every $x \in P$ and $x < y$ in P implies $f(x) < f(y)$. In the case when the domain and the codomain of a function f are different, the first requirement is dropped.

Lemma 1

Let $\{f_1, \ldots, f_n\}$ be functions from a cofinite directed set $(M, <)$ into a directed set $(L, <)$. Then there is an increasing function $g : M \to L$ such that $g(x) > f_1(x), \ldots, f_n(x)$ for every $x \in M$.

Let X and Y be topological spaces. By a *multi-valued function* $F : X \to Y$ we mean a rule which associates a non-empty subset $F(x)$ of Y to every point x of X.

For our approach to shape theory the following notion of size for multi-valued functions will play the most important role.

Let $F : X \to Y$ be a multi-valued function and let $\alpha \in \hat{X}$ and $\sigma \in \hat{Y}$. We shall say that F is an *(α, σ)-function* provided for every $A \in \alpha$ there is an $S_A \in \sigma$ with $F(A) \subset S_A$. On the other hand, F is a *σ-function* provided there is an $\alpha \in \hat{X}$ such that F is an (α, σ)-function.

Also important will be the following concept of closeness for two multi-valued functions.

Let F, $G : X \to Y$ be multi-valued functions, let $\sigma \in \hat{Y}$, and let $\alpha \in \hat{X}$. We shall say that F and G are (α, σ)-*close* and we write $F \overset{(\alpha,\sigma)}{=} G$ provided for every A in α there is an $S_A \in \sigma$ with $F(A) \cup G(A) \subset S_A$.

4. Fiberwise normal covers and fiberwise multi-valued functions

In this section we shall define fiberwise versions of some notions from the previous section.

By a *fiberwise normal cover* of a fiberwise space (X, f) we mean a pair (σ, σ_B) where σ is a normal cover of the space X and σ_B is a normal cover of B such that σ refines the cover $f^{-1}(\sigma_B)$. We shall again make a simplification of our notation by dropping σ_B so that fiberwise normal covers are denoted by small Greek letters which name a normal cover of the total space X while the part in the base space has index B.

Let $\hat{Y}$ denote the collection of all fiberwise normal covers of a fiberwise space Y. We order $\hat{Y}$ by the refinement relation $>$ defined by $\sigma > \tau$ if and only if $\sigma > \tau$ and $\sigma_B > \tau_B$ for fiberwise normal covers σ and τ of Y. With respect to the relation $>$ the set $\hat{Y}$ is a directed set. Two fiberwise normal covers σ and τ of Y are equivalent provided $\sigma > \tau$ and $\tau > \sigma$. In order to simplify our notation we denote a fiberwise normal cover and it's equivalence class by the same symbol. Consequently, $\hat{Y}$ also stands for the associated quotient set.

If σ is a fiberwise normal cover of a fiberwise space Y, let σ^+ be the collection of all fiberwise normal covers of Y which refine σ while σ^* denotes the set of all fiberwise normal covers τ of Y such that the star $st(\tau)$ of τ refines σ. Here, we define the star $st(\sigma)$ of a fiberwise normal cover $\sigma = (\sigma, \sigma_B)$ as a fiberwise normal cover $(st(\sigma), st(\sigma_B))$. Similarly, for a natural number n, σ^{*n} denotes the set of all fiberwise normal covers τ of Y such that the n-th star $st^n(\tau)$ of τ refines σ.

Let Y be a fiberwise space. Let $\tilde{Y}$ denote the collection of all finite subsets c of $\hat{Y}$ which have a unique (with respect to the refinement relation) maximal element which we denote either by $\tilde{c}$ or by $[c]$. We consider $\tilde{Y}$ ordered by the inclusion relation and regard $\hat{Y}$ as a subset of single-element subsets of $\hat{Y}$. Notice that $\tilde{Y}$ is a cofinite directed set.

For our approach to fiberwise shape theory the following class of fiberwise multi-valued functions will play the most important role.

Let $F : X \to Y$ be a multi-valued function between fiberwise spaces and let α and σ be fiberwise normal covers of X and Y. We shall say that F is a *fiberwise* (α, σ)-*function* provided F is an (α, σ)-function and the functions f and $g \circ F$ are

$(\alpha,\, \sigma_B)$-close. On the other hand, F is a *fiberwise σ-function* provided there is an $\alpha \in \hat{X}$ such that F is a fiberwise $(\alpha,\, \sigma)$-function.

5. Fiberwise σ-homotopy

Now we a ready to introduce an important notion of fiberwise σ-homotopy for multi-valued functions of fiberwise spaces. We shall also prove in Lemma 2 a useful technical result.

Let F and G be multi-valued functions between fiberwise spaces X and Y and let σ be a fiberwise normal cover of Y. We shall say that F and G are *fiberwise σ-homotopic* and write $F \overset{\sigma}{\simeq}_B G$ provided there is a fiberwise σ-function H from the fiberwise cylinder $X \times I$ into Y such that $F(x) = H(x,\, 0)$ and $G(x) = H(x,\, 1)$ for every $x \in X$. We shall say that H is a *fiberwise σ-homotopy* that *joins* F and G or that it *realizes* the relation (or fiberwise σ-homotopy) $F \overset{\sigma}{\simeq}_B G$.

The following lemma is crucial because it provides an adequate substitute for the transitivity of the relation of fiberwise σ-homotopy.

Lemma 2

Let F, G, and H be multi-valued functions between fiberwise spaces X and Y. Let σ be a fiberwise normal cover of Y and let $\tau \in \sigma^*$. If $F \overset{\tau}{\simeq}_B G$ and $G \overset{\tau}{\simeq}_B H$, then $F \overset{\sigma}{\simeq}_B H$.

Proof. By assumption there are fiberwise normal covers α and β of $X \times I$, a fiberwise $(\alpha,\, \tau)$-function $K : X \times I \to Y$, and a fiberwise $(\beta,\, \tau)$-function $L : X \times I \to Y$ such that $F(x) = K(x,\, 0)$, $G(x) = K(x,\, 1)$, $G(x) = L(x,\, 0)$, and $H(x) = L(x,\, 1)$ for every $x \in X$. Let a fiberwise normal cover γ of $X \times I$ be a common refinement of α and β. Observe first that K and L are both fiberwise $(\gamma,\, \tau)$-functions. Define a multi-valued function $M : X \times I \to Y$ by the rule

$$M(x,\, t) = \begin{cases} K(x,\, 2t), & x \in X, \quad 0 \le t \le 1/2 \\ L(x,\, 2t-1), & x \in X, \quad 1/2 \le t \le 1\,. \end{cases}$$

Since $F(x) = M(x,\, 0)$ and $H(x) = M(x,\, 1)$ for every $x \in X$, it remains to see that M is a fiberwise σ-function.

By [10, p. 358], there is a normal cover δ of X and a function $r : \delta \to \{2, 3, 4, \ldots\}$ such that the set $D \times D_i$ is contained in a member $C_{D,i}$ of γ for every $D \in \delta$ and every $i = 1, \ldots, rD - 1$, where $D_i = \left[\frac{i-1}{rD}, \frac{i+1}{rD}\right]$ $(i = 1, \ldots, rD - 1)$. Let

$$E_i = \begin{cases} \left[0, \dfrac{1}{2rD}\right), & \text{i} = 0, \\ \left(\dfrac{i}{4rD}, \dfrac{i+2}{4rD}\right), & \text{i} = 1, \ldots, \text{4rD-3}, \\ \left(1 - \dfrac{1}{2rD}, 1\right], & \text{i} = \text{4rD-2} . \end{cases}$$

The collection $\varepsilon = \left\{D \times E_i \,|\, D \in \delta, i = 0, 1, \ldots, 4rD - 2\right\}$ is a normal cover of $X \times I$. Let ϱ be a fiberwise normal cover of $X \times I$ such that ϱ refines ε. In order to prove that M is a fiberwise (ϱ, σ)-function, we shall first show that it is an (ε, σ)-function.

Let $E = D \times E_i$ be a member of ε. We must find an $S_E \in \sigma$ such that $M(E) \subset S_E$.

Case I ($i = 2k$ for $0 \leq k \leq rD - 1$). Then $E_i = E_{2k}$ is below 1/2 so that we get $M(E) = K(D \times 2E_i) = K(V_k) \subset K(C_{D,k}) \subset T \subset S_E$, where $V_k = D \times D_k$, the open set T is a member of τ which we obtain with respect to $C_{D,k}$ from the fact that K is a (γ, τ)-function, and S_E is a member of σ which contains T.

Case II ($i = 2k$ for $rD \leq k \leq 2rD - 1$). Then $E_i = E_{2k}$ is above 1/2 so that we get $M(E) = L(D \times (2E_i - 1)) \subset L(V_{k-rD}) \subset L(C_{D,k-rD}) \subset T \subset S_E$, where T is a member of τ which we obtain with respect to $C_{D,k-rD}$ from the fact that L is a (γ, τ)-function, and S_E is a member of σ which contains T.

Case III ($i = 2k + 1$ for $0 \leq k \leq rD - 2$). Then $E_i = E_{2k+1}$ is below 1/2 so that we get

$$M(E) = K(D \times 2E_i) \subset K(V_k) \cup K(V_{k+1}) \subset K(C_{D,k}) \cup K(C_{D,k+1}) \subset T_1 \cup T_2 \subset S_E,$$

where T_1 and T_2 are members of τ which we obtain with respect to $C_{D,k}$ and $C_{D,k+1}$ from the fact that K is a (γ, τ)-function, and S_E is a member of σ which contains the union $T_1 \cup T_2$. Such an S_E exists because $D \times \{(k + 1)/rD\} \subset C_{D,k} \cap C_{D,k+1}$ so that $\emptyset \neq K(D \times \{(k + 1)/rD\}) \subset T_1 \cap T_2$.

Case IV ($i = 2k + 1$ for $rD \leq k \leq 2rD - 1$). This case is analogous to the Case III. This time E_i is above 1/2 and we must use L instead of K.

Case V ($i = 2k + 1$ and $k = rD - 1$). Then $E_i = E^- \cup E^+$, where $E^- = \left(\frac{2rD-1}{4rD}, \frac{1}{2}\right]$ and $E^+ = \left[\frac{1}{2}, \frac{2rD+1}{4rD}\right)$ so that we get

$$M(E) = K(D \times 2E^-) \cup L\big(D \times (2E^+ - 1)\big) \subset K(C_{D,rD-1}) \cup L(C_{D,1}) \subset T_1 \cup T_2,$$

where T_1 and T_2 are members of τ which we obtain with respect to $C_{D,rD-1}$ and $C_{D,1}$ from the fact that K and L are (γ, τ)-functions.

Observe that $K(D \times \{1\}) = G(D) \subset T_1$ and $L(D \times \{0\}) = G(D) \subset T_2$. We conclude that $T_1 \cap T_2 \neq \emptyset$ so that there is a member S_E of σ which contains both T_1 and T_2. It follows that $M(E) \subset S_E$.

Finally, it remains to check that functions $f \times I$ and $g \circ M$ are (ε, σ_B)-close. Let $E = D \times E_i$ be a member of ε. We must find a member S_E of σ_B such that both $f \times I(E)$ and $g \circ M(E)$ are contained in S_E.

Once again we shall distinguish five cases considered above.

Case I. Let A be a member of α which contains the set $C_{D,k}$. Since $f \times I$ and $g \circ K$ are (α, τ_B)-close, there is a member T of τ_B which contains both $f \times I(A)$ and $g \circ K(A)$. Since τ_B is a star-refinement of σ_B, there is an $S_E \in \sigma_B$ with $T \subset S_E$. Our choices imply $g \circ M(E) \subset g \circ M(C_{D,k} \subset g \circ M(A)$ because $C_{D,k} \subset A$ and $D \times 2E_i \subset V_k$ and $f \times I(E) \subset f \times I(A)$ because $E \subset A$. It follows that $f \times I(E)$ and $g \circ M(E)$ are both subsets of S_E.

Case II. This is similar to the previous case. We have to deal with L instead of with K.

Case III. The set $g \circ M(E)$ is now a subset of the union of sets $g \circ K(C_{D,k})$ and $g \circ K(C_{D,k+1})$. Let A_1 and A_2 be members of α which contain $C_{D,k}$ and $C_{D,k+1}$, respectively. Since $f \times I$ and $g \circ K$ are (α, τ_B)-close, there are members T_1 and T_2 of τ_B such that T_1 contains $f \times I(A_1)$ and $g \circ K(A_1)$ while T_2 contains $f \times I(A_2)$ and $g \circ K(A_2)$. The sets T_1 and T_2 both contain the set $f(D)$. It follows that some member S_E of σ_B contains their union. This is the required open set.

Case IV. This is similar to the previous case.

Case V. The set $g \circ M(E)$ is now a subset of the union of sets $g \circ K(C_{D,rD-1})$ and $g \circ L(C_{D,1})$. Let A_1 be a member of α and let B_2 be a member of β such that A_1 contains $C_{D,rD-1}$ and B_2 contains $C_{D,1}$. Since $f \times I$ is (α, τ_B)-close to $g \circ K$ and (β, τ_B)-close to $g \circ L$, there are members T_1 and T_2 of τ_B such that T_1 contains $f \times I(A_1)$ and $g \circ K(A_1)$ while T_2 contains $f \times I(B_2)$ and $g \circ L(B_2)$. But, the sets T_1 and T_2 both contain the set $f(D)$ so that some member S_E of σ_B contains their union. It follows that $f \times I(E)$ and $g \circ M(E)$ are both subsets of S_E. □

Since we shall be using [10, p. 358] quite often, for a space Y and a normal cover α of the product $Y \times I$, we let $D(Y, \alpha)$ denote all normal covers β of Y such that some stacked normal cover over β refines α.

6. Fiberwise multi-nets

The following two definitions correspond to Borsuk's definitions of fundamental sequence and homotopy for fundamental sequences.

Let X and Y be fiberwise spaces. By a *fiberwise multi-net* from X into Y we shall mean a collection $\varphi = \{F_c \mid c \in \tilde{Y}\}$ of multi-valued functions $F_c : X \to Y$ such that for every $\sigma \in \hat{Y}$ there is a $c \in \tilde{Y}$ with $F_d \overset{\sigma}{\simeq}_B F_c$ for every $d > c$. We use functional notation $\varphi : X \to Y$ to indicate that φ is a fiberwise multi-net from X into Y. Let $\mathcal{F}m_B(X, Y)$ denote all fiberwise multi-nets $\varphi : X \to Y$ from X into Y.

Two fiberwise multi-nets $\varphi = \{F_c\}$ and $\psi = \{G_c\}$ between fiberwise spaces X and Y are said to be *fiberwise homotopic* and we write $\varphi \simeq_B \psi$ provided for every $\sigma \in \hat{Y}$ there is a $c \in \tilde{Y}$ such that $F_d \overset{\sigma}{\simeq}_B G_d$ for every $d > c$.

It follows from Lemma 2 that the relation of fiberwise homotopy is an equivalence relation on the set $\mathcal{F}m_B(X, Y)$. The fiberwise homotopy class of a fiberwise multi-net φ is denoted by $[\varphi]_B$ and the set of all fiberwise homotopy classes by $\mathcal{F}s_B(X, Y)$.

7. Composition of fiberwise homotopy classes

Our first goal is to define a composition for fiberwise homotopy classes of fiberwise multi-nets and to establish it's associativity.

Let X and Y be fiberwise spaces. Let $\varphi = \{F_c\} : X \to Y$ be a fiberwise multi-net. Let $\varphi : \tilde{Y} \to \tilde{Y}$ be an increasing function such that for every $c \in \tilde{Y}$ the relation $d, e > \varphi(c)$ implies the relation $F_d \overset{[c]}{\simeq}_B F_e$.

Let $\mathcal{C}_\varphi = \{(c, d, e) \mid c \in \tilde{Y},\ d, e > \varphi(c)\}$. Then $\mathcal{C}_\varphi$ is a subset of $\tilde{Y} \times \tilde{Y} \times \tilde{Y}$ that becomes a cofinite directed set when we define that $(c, d, e) > (c', d', e')$ iff $c > c'$, $d > d'$, and $e > e'$.

We shall use the same notation φ for an increasing function $\varphi : \mathcal{C}_\varphi \to \widehat{X \times I}$ such that F_d and F_e are joined by a fiberwise $(\varphi(c, d, e), [c])$-homotopy whenever $(c, d, e) \in \mathcal{C}_\varphi$.

Let $\bar{\varphi} : \mathcal{C}_\varphi \to \tilde{X}$ be an increasing function such that the fiberwise normal cover $[\bar{\varphi}(c, d, e)]$ belongs to the set $D(X, \varphi(c, d, e))$ for every $(c, d, e) \in \mathcal{C}_\varphi$.

Claim 1. *There is an increasing function $\varphi^* : \tilde{Y} \to \tilde{X}$ such that*

(1) $\varphi^*(c) > \bar{\varphi}(c, \varphi(c), \varphi(c))$ for every $c \in \tilde{Y}$, and

(2) φ^* is cofinal in $\bar{\varphi}$, i.e., for every $(c, d, e) \in \mathcal{C}_\varphi$ there is an $m \in \tilde{Y}$ with $\varphi^*(m) > \bar{\varphi}(c, d, e)$.

Proof. Let $\mathcal{D} = \{\bar{\varphi}(c,\, d,\, e) \,|\, (c,\, d,\, e) \in \mathcal{C}_\varphi\}$.

If $\tilde{Y}$ is a finite set, then $\mathcal{D}$ is a finite collection of elements of $\tilde{X}$. Let $a \in \tilde{X}$ be greater than all members of $\mathcal{D}$. Let $\varphi^* : \tilde{Y} \to \tilde{X}$ be a constant function into a.

If $\tilde{Y}$ is an infinite set, then the cardinality of $\mathcal{D}$ does not exceed the cardinality of $\tilde{Y}$. Hence, there is a surjection $g : \tilde{Y} \to \mathcal{D}$. Let $\varphi^* : \tilde{Y} \to \tilde{X}$ be an increasing function such that $\varphi^*(c) > g(c),\, \bar{\varphi}(c,\, \varphi(c),\, \varphi(c))$ for every $c \in \tilde{Y}$. □

The above discussion shows that every fiberwise multi-net $\varphi : X \to Y$ determines four functions denoted by φ, $\bar{\varphi}$, and φ^*. In notation there are only three but φ has two meanings. With the help of these functions we shall define the composition of fiberwise homotopy classes of fiberwise multi-nets as follows.

Let X, Y, and Z be fiberwise spaces and let $\varphi = \{F_c\} : X \to Y$ and $\psi = \{G_s\} : Y \to Z$ be fiberwise multi-nets. Let $\chi = \{H_s\}$, where $H_s = G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}$ for every $s \in \tilde{Z}$. Observe that each H_s is a multi-valued function because the composition of two multi-valued functions is a multi-valued function.

Claim 2. *The collection χ is a fiberwise multi-net from X into Z.*

Proof. Let $\sigma \in \hat{Z}$. We must find a $u \in \tilde{Z}$ such that

$$H_v \overset{\sigma}{\simeq}_B H_u \qquad \text{for every} \quad v > u. \tag{3}$$

Let $\tau \in \sigma^{*2}$ and $\xi \in \tau^*$. Let $u = \{\xi\} \in \tilde{Z}$.

Consider an index $v > u$. We shall find an index $e \in \tilde{Y}$ so that

$$H_v \overset{\tau}{\simeq}_B G_y \circ F_e, \tag{4}$$

$$G_y \circ F_e \overset{\tau}{\simeq}_B G_x \circ F_e, \tag{5}$$

and

$$G_x \circ F_e \overset{\tau}{\simeq}_B H_u, \tag{6}$$

where $x = \psi(u)$, $y = \psi(v)$, $a = \psi^*(u)$, $b = \psi^*(v)$, $c = \varphi(a)$, and $d = \varphi(b)$. Repeated use of Lemma 2 will give (3) from the relations (4) – (6).

Add (4). Since $(v,\, y,\, y) \in \mathcal{C}_\psi$, we see that there is a fiberwise $(\alpha,\, \xi)$-homotopy $K : Y \times I \to Z$ such that $\alpha = \psi(v,\, y,\, y)$, $K_0 = G_y$, and $K_1 = G_y$. Let $s = \bar{\psi}(v,\, y,\, y)$ and $\pi = [s]$. Observe that $\pi \in D(Y,\, \alpha)$. We claim that G_y is a fiberwise $(\pi,\, \xi)$-function from Y into Z.

Indeed, let P be a member of π. Then there is a $t > 0$ such that the product $P \times [0, t)$ lies in a member A of α. Since K is an (α, ξ)-function, there is a $T \in \xi$ such that $K(A) \subset T$. It follows that $G_y(P) = K_0(P) = K(P \times \{0\}) \subset K(A) \subset T$ which proves that G_y is a (π, ξ)-function. Finally, it remains to check that functions g and $h \circ G_y$ are (π, ξ_B)-close. But, functions $g \times I$ and $h \circ K$ are (α, ξ_B)-close so that the restrictions $g = g \times I|_{Y \times \{0\}}$ and $h \circ G_y = h \circ K|_{Y \times \{0\}}$ are (π, ξ_B)-close.

Once we know that G_y is a fiberwise (π, ξ)-function, we see that it suffices to take $e > d$ because then F_d and F_e are joined by a fiberwise $(\varepsilon, [b])$-homotopy $L : X \times I \to Y$, for some fiberwise normal cover ε of $X \times I$, so that $G_y \circ L$ is a fiberwise (ε, τ)-homotopy which realizes the relation (4).

Indeed, by construction, the normal cover $[b]$ refines π so that the composition $G_y \circ L$ is a multi-valued (ε, ξ)-homotopy joining H_v and $G_y \circ F_e$. On the other hand, since the functions $f \times I$ and $g \circ L$ are $(\varepsilon, [b]_B)$-close, the functions g and $h \circ G_y$ are (π, ξ_B)-close, the normal cover $[b]_B$ refines π_B, and $st(\xi_B)$ refines τ_B, it follows that the functions $f \times I$ and $h \circ G_y \circ L$ are (ε, τ_B)-close.

Add (5). Since $(u, x, y) \in \mathcal{C}_\psi$, it follows that G_x and G_y are joined by a fiberwise (α, ξ)-homotopy $K : Y \times I \to Z$, where α denotes the fiberwise normal cover $\psi(u, x, y)$ of $Y \times I$. Choose a normal cover β of Y and a function $r : \beta \to \{4, 5, 6, \ldots\}$ such that every set $V \times \left[\frac{i-1}{rV}, \frac{i+1}{rV}\right]$, where $V \in \beta$ and $i = 1, \ldots, rV - 1$, is contained in a member $A_{V,i}$ of α. Pick a fiberwise normal cover κ of Y such that κ refines β and κ_B refines ξ_B. Let $k = \{\kappa\}$ and $e = \varphi(k)$. Since $(k, e, e) \in \mathcal{C}_\varphi$, the function F_e is a fiberwise (π, κ)-function from X into Y for some fiberwise normal cover π of X. It follows that for every $P \in \pi$ there is a $V_P \in \beta$ and an $L_P \in \kappa_B$ such that $F_e(P) \subset V_P$ and L_P contains both $f(P)$ and $g \circ F_e(P)$.

For every $P \in \pi$, let

$$\nu_P = \left\{\Big[0, \frac{2}{rV_P}\Big), \Big(\frac{rV_P - 2}{rV_P}, 1\Big]\right\} \bigcup \left\{\Big(\frac{i}{rV_P}, \frac{i+2}{rV_P}\Big) \,|\, i = 1, \ldots, rV_P - 3\right\}.$$

Put $\varrho = \{P \times N \,|\, P \in \pi,\, N \in \nu_P\}$. Observe that ϱ is a normal cover of $X \times I$. Let ω be a fiberwise normal cover of $X \times I$ such that the normal cover ω refines ϱ. We claim that the composition $H = K \circ (F_e \times id_I)$ is a fiberwise (ω, τ)-homotopy which joins $G_x \circ F_e$ and $G_y \circ F_e$.

Indeed, let S be a member of ω. Pick a member $R = P \times N$ of ϱ which contains S, where $P \in \pi$ and $N \in \nu_P$. Then

$$H(S) \subset H(R) = K(F_e(P) \times N) \subset K(V_P \times N) \subset K(A_{V_P,j}),$$

where j is such that N is an interior of the segment $\left[\frac{j}{rV_P}, \frac{j+2}{rV_P}\right]$. Since K is a fiberwise (α, ξ)-homotopy, we obtain that the last set in the chain of inclusions above is a subset of a member of ξ. Hence, H is an (ω, τ)-function.

On the other hand, in order to check that $h \circ H$ and $f \times I$ are (ω, τ_B)-close, let S be a member of ω. Choose sets N, P, and R as above. Observe that the sets $g \circ F_e(P)$ and $f(P)$ both lie in a member L_P of κ_B. But, $(g \times I) \circ (F_e \times id_I)(R) = g \circ F_e(P)$ and $f \times I(R) = f(P)$. Also, $(F_e \times id_I)(R) \subset V_P \times N$ so that

$$(g \times I) \circ (F_e \times id_I)(R) \subset (g \times I)(V_P \times N).$$

Since $V_P \times N$ lies in a member $A_{V_P,j}$ of α and K is a fiberwise (α, ξ)-homotopy, some member V of the cover ξ_B contains sets $(g \times I)(V_P \times N)$ and $h \circ K(V_P \times N)$. Hence, V contains $g \circ F_e(P)$ and $h \circ H(R)$. Since ξ_B is a star-refinement of τ_B it follows that $f \times I(S)$ and $h \circ H(S)$ lie in some member of τ_B.

Add (6). This is analogous to the proof of (4). □

We now define the composition of fiberwise homotopy classes of fiberwise multi-nets by the rule $[\{G_s\}]_B \circ [\{F_c\}]_B = [\{G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}\}]_B$.

Claim 3. *The composition of fiberwise homotopy classes of fiberwise multi-nets is well-defined.*

Proof. Let $\kappa = \{K_c\}$ and $\lambda = \{L_s\}$ be fiberwise multi-nets fiberwise homotopic to φ and ψ, respectively, and let $\mu = \{M_s\}$, where $M_s = L_{\lambda(s)} \circ K_{\kappa(\lambda^*(s))}$ for every $s \in \tilde{Z}$. We must show that fiberwise multi-nets χ and μ are fiberwise homotopic. In other words, that for every $\sigma \in \hat{Z}$ there is an $s \in \tilde{Z}$ such that

$$H_t \overset{\sigma}{\simeq}_B M_t \quad \text{for every} \quad t > s. \tag{7}$$

Let $\sigma \in \hat{Z}$. Let $\tau \in \sigma^{*4}$ and $\xi \in \tau^*$. Let $s = \{\xi\} \in \tilde{Z}$. In order to prove (7), we shall argue that for every $t > s$ we can find indices $e \in \tilde{Y}$ and $u \in \tilde{Z}$ such that

$$H_t \overset{\tau}{\simeq}_B G_x \circ F_e, \tag{8}$$

$$G_x \circ F_e \overset{\tau}{\simeq}_B G_u \circ F_e, \tag{9}$$

$$G_u \circ F_e \overset{\tau}{\simeq}_B L_u \circ F_e, \tag{10}$$

$$L_u \circ F_e \overset{\tau}{\simeq}_B L_u \circ K_e, \tag{11}$$

(12) $$L_u \circ K_e \overset{\tau}{\simeq}_B L_y \circ K_e,$$

(13) $$L_y \circ K_e \overset{\tau}{\simeq}_B M_t,$$

where we put $x = \psi(t)$, $a = \psi^*(t)$, $b = \varphi(a)$, $y = \lambda(t)$, $c = \lambda^*(t)$, and $d = \kappa(c)$. From the relations (8) – (13) with the help of Lemma 2 we shall get (7).

We shall now describe how big e and u must be chosen for relations (8), (9), (10), and (11) to hold separately. The relations (12) and (13) are analogous to relations (9) and (8), respectively. We leave to the reader the task of making a cumulative choice for e and u which accomplishes our goal. It is important to notice that u is selected first while e is selected only once u is already known.

Add (8). We know from the proof of Claim 2 that G_x is a fiberwise (π, ξ)-function, where $\pi = [s]$ and $s = \bar{\psi}(t, x, x)$. Since $a > s$ by the property (1) of Claim 1, it suffices to take $e > b$.

Add (9). If $u > x$, then G_x and G_u are joined by a fiberwise (α, ξ)-homotopy $K : Y \times I \to Z$, where $\alpha = \bar{\psi}(u, m, m)$ and $m = \psi(u)$. Choose a normal cover β of Y and a function $r : \beta \to \{4, 5, 6, \ldots\}$ such that every set $V \times [\frac{i-1}{rV}, \frac{i+1}{rV}]$, where $V \in \beta$ and $i = 1, \ldots, rV - 1$, is contained in a member of α. Pick a fiberwise normal cover η of Y such that η refines β and η_B refines ξ_B. Let $k = \{\eta\}$ and $e > \varphi(k)$. Just as in the proof of (5) we can see that $K \circ F_e \times id_I$ is a fiberwise τ-homotopy joining the left and the right side of the relation (9).

Add (10). Since fiberwise multi-nets ψ and λ are fiberwise homotopic, there is a $u \in \tilde{Z}$, a fiberwise normal cover α of $Y \times I$, and a fiberwise (α, ξ)-homotopy $S : Y \times I \to Z$ joining G_u and L_u. Choose a β, an r, and an e as above. Then $S \circ (F_e \times id_I)$ is a fiberwise τ-homotopy joining compositions which appear in (10).

Add (11). Let $u > y$. Then L_u is a fiberwise (α, ξ)-function from Y into Z, where $\alpha = [s]$ and $s = \bar{\lambda}(t, y, y)$. Pick a fiberwise normal cover η of Y such that η refines α and η_B refines ξ_B. Since fiberwise multi-nets φ and κ are fiberwise homotopic, there is an index $e \in \tilde{Y}$ so that F_e and K_e are joined by a fiberwise η-homotopy $T : X \times I \to Y$. The composition $L_u \circ T$ realizes the relation (11). □

Theorem 1

The composition of fiberwise homotopy classes of fiberwise multi-nets is associative.

Proof. Let $\varphi = \{F_c\}$, $\psi = \{G_s\}$, and $\chi = \{H_p\}$ be fiberwise multi-nets from X into Y, from Y into Z, and from Z into W, respectively. Let $\mu = \{M_s\}$, $\nu = \{N_p\}$, $\kappa = \{K_p\}$, and $\lambda = \{L_p\}$, where $M_s = G_{\psi(s)} \circ F_{\varphi(\psi^*(s))}$ for every $s \in \tilde{Z}$ and $N_p = H_{\chi(p)} \circ G_{\psi(\chi^*(p))}$, $K_p = H_{\chi(p)} \circ M_{\mu(\chi^*(p))}$, and $L_p = N_{\nu(p)} \circ F_{\varphi(\nu^*(p))}$, for every $p \in \tilde{W}$. We must show that fiberwise multi-nets κ and λ are fiberwise homotopic, i. e., that for every fiberwise normal cover $\pi \in \hat{W}$ there is an index $p \in \tilde{W}$ such that

$$K_q \overset{\pi}{\simeq}_B L_q \quad \text{for every} \quad q < p. \tag{14}$$

Let $\pi \in \hat{W}$. Let $\varrho \in \pi^{*4}$, $\xi \in \varrho^*$, and $\eta \in \xi^*$. Let $p = \{\eta\} \in \tilde{W}$. In order to prove (14), we shall show that for every $q > p$ we can find indices $e \in \tilde{Y}$ and $s \in \tilde{Z}$ such that

$$K_q \overset{\varrho}{\simeq}_B H_x \circ G_y \circ F_e\,, \tag{15}$$

$$H_x \circ G_y \circ F_e \overset{\varrho}{\simeq}_B H_x \circ G_s \circ F_e\,, \tag{16}$$

$$H_x \circ G_s \circ F_e \overset{\varrho}{\simeq}_B H_z \circ G_s \circ F_e\,, \tag{17}$$

$$H_z \circ G_s \circ F_e \overset{\varrho}{\simeq}_B N_w \circ F_e\,, \tag{18}$$

and

$$N_w \circ F_e \overset{\varrho}{\simeq}_B L_q\,, \tag{19}$$

where $x = \chi(q)$, $y = \psi(\mu(\chi^*(q)))$, $z = \chi(\nu(q))$, and $w = \nu(q)$. Repeated use of Lemma 2 will give (14) from the relations (15) – (19).

The method of proof is similar to the proof of Claim 3. We shall only describe for each of the relations (15) – (19) how large the indices u and e must be in order that this fiberwise ϱ-homotopy holds. An easy exercise of putting together all these selections is once again left to the reader. Since relations (18) and (19) are analogous with relations (16) and (15), respectively, it suffices to consider only relations (15) – (17).

Add (15). Observe that H_x is a fiberwise (θ, η)-function, where $\theta = [a]$, $a = \bar{\chi}(q, x, x)$. Let $m = \chi^*(q)$, $n = \mu(m)$, $k = \psi(n)$, $d = \bar{\psi}(n, y, y)$, $c = \psi^*(n)$, and $\omega = [d]$. Then G_y is a fiberwise $(\omega, [n])$-function from Y into Z. Since $n > m$ and by the property (1) from Claim 1, $m > a$ we obtain that $[n]$ refines θ. Let $b = \varphi(c)$.

If $e > b$, then F_e and F_b are joined by a fiberwise $[c]$-homotopy P. But, $c > d$ so that $[c]$ refines ω. Hence, $H_x \circ G_y \circ P$ is a fiberwise ϱ-homotopy between K_q and $H_x \circ G_y \circ F_e$.

Add (16). As above, H_x is a fiberwise (θ, η)-function from Z into W. If we take $s > y$, then $(n, y, s) \in \mathcal{C}_\psi$ so that G_y and G_s are joined by a fiberwise $(\varepsilon, [n])$-homotopy $Q : Y \times I \to Z$, where $\varepsilon = \psi(n, y, s)$. But, since $m > a$, we see that $[n]$ refines θ. Choose a normal cover β of Y and a function $r : \beta \to \{4, 5, 6, \ldots\}$ such that every set $V \times [\frac{i-1}{rV}, \frac{i+1}{rV}]$, where $V \in \beta$ and $i = 1, \ldots, rV - 1$, is contained in a member of ε. Pick a fiberwise normal cover γ of Y such that γ refines β and γ_B refines η_B. Let $t = \{\gamma\}$ and $e > \varphi(t)$. Then $H_x \circ Q \circ (F_e \times id_I)$ realizes the relation (16).

Add (17). Since $\nu(r) > r$ for every $r \in \tilde{W}$, we get $z > x$ so that $(q, x, z) \in \mathcal{C}_\chi$ and H_x and H_z are joined by a fiberwise (β, η)-homotopy $T : Z \times I \to W$, where $\beta = \chi(q, x, z)$. Let $v = \chi^*(z)$ and let $s > \psi(v)$. Then G_s is a fiberwise $([t], [v])$-function from Y into Z, where $t = \bar{\psi}(s, \psi(s), \psi(s))$. Let $u = \psi^*(t)$ and take $e > \varphi(u)$. The composition $T \circ ((G_s \circ F_e) \times id_I)$ realizes the relation (17). □

8. The fiberwise shape category $\mathcal{F}s_B$

For a fiberwise topological space X, let $\iota^X = \{I_a\} : X \to X$ be the identity fiberwise multi-net defined by $I_a = id_X$ for every $a \in \tilde{X}$. It is easy to show that for every fiberwise multi-net $\varphi : X \to Y$, the following relations hold:

$$[\varphi]_B \circ [\iota^X]_B = [\varphi]_B = [\iota^Y]_B \circ [\varphi]_B.$$

We can summarize the above with the following theorem which is the main result of this paper.

Theorem 2

The fiberwise topological spaces as objects together with the fiberwise homotopy classes of fiberwise multi-nets as morphisms and the composition of fiberwise homotopy classes form the category $\mathcal{F}s_B$. When B is a single-element space, the category $\mathcal{F}s_B$ is isomorphic to the shape category $\mathcal{S}h$.

9. Special fiberwise multi-nets

In this section we shall be looking for conditions under which a given fiberwise homotopy class of fiberwise multi-nets has a representative of a special kind. As a corollary of these results we identify a class of fiberwise spaces on which fiberwise shape theory and fiberwise homotopy theory coincide.

A fiberwise space X is *fiberwise internally movable* provided for every $\sigma \in \hat{X}$ there is a $\tau \in \hat{X}$ such that every fiberwise τ-function into X is fiberwise σ-homotopic to a fiberwise map.

A fiberwise space X is *fiberwise internally calm* provided there is a $\sigma \in \hat{X}$ such that fiberwise maps into X which are fiberwise σ-homotopic are fiberwise homotopic.

A fiberwise space X is *fiberwise calm* provided there is a $\sigma \in \hat{X}$ such that for every $\tau \in \hat{X}$ there is a $\varrho \in \hat{X}$ with the property that fiberwise ϱ-functions into X which are fiberwise σ-homotopic are also fiberwise τ-homotopic.

A fiberwise multi-net $\varphi = \{F_c\}_{c \in \tilde{Y}}$ from a fiberwise space X into a fiberwise space Y is *regular* provided each function F_c is a fiberwise map. It is called *simple* when there is a fiberwise map f such that $f = F_c$ for every $c \in \tilde{Y}$.

Theorem 3

If a fiberwise space Y is fiberwise internally movable, then every fiberwise multi-net φ from a fiberwise space X into Y is fiberwise homotopic to a regular fiberwise multi-net.

Proof. Since Y is fiberwise internally movable, for every $c \in \tilde{Y}$ there is a $\chi(c) \in \tilde{c}^+$ such that every fiberwise $\chi(c)$-function into Y is fiberwise $[c]$-homotopic to a fiberwise map. Let $\lambda : \tilde{Y} \to \tilde{Y}$ be an increasing function such that $\lambda(c) > \varphi(\chi(c))$ for every $c \in \tilde{Y}$. Then $F_{\lambda(c)}$ is a fiberwise $\chi(c)$-function so that we can select a fiberwise map $g_c : X \to Y$ with $g_c \overset{[c]}{\simeq}_B F_{\lambda(c)}$, for every $c \in \tilde{Y}$.

In order to verify that $\psi = \{g_c\}_{c \in \tilde{Y}}$ is a fiberwise multi-net from X into Y, let a $\sigma \in \hat{Y}$ be given. Let $\mu \in \sigma^*$ and put $c = \{\mu\}$. For every $d > c$ we have $g_d \overset{\mu}{\simeq}_B F_{\lambda(d)} \overset{\chi(c)}{\simeq}_B F_{\lambda(c)} \overset{\mu}{\simeq}_B g_c$. Hence, $g_d \overset{\sigma}{\simeq}_B g_c$ for every $d > c$.

It remains to check that fiberwise multi-nets φ and ψ are fiberwise homotopic. Let a fiberwise normal cover $\sigma \in \hat{Y}$ be given. Let $\mu \in \sigma^*$. Choose an index $c_0 \in \tilde{Y}$ such that $F_d \overset{\mu}{\simeq}_B F_e$ for all d, $c > c_0$. Let $c > c_0$, $\{\mu\}$. For every $d > c$, we get $g_d \overset{\mu}{\simeq}_B F_{\lambda(d)}$ by construction, while $F_{\lambda(d)} \overset{\mu}{\simeq}_B F_d$ because $\lambda(d) > d > c_0$. Hence, $g_d \overset{\sigma}{\simeq}_B F_d$ for every $d > c$. □

Theorem 4

If a fiberwise space Y is both fiberwise internally movable and fiberwise calm, then every fiberwise multi-net φ into Y is fiberwise homotopic to a simple fiberwise multi-net.

Proof. Since Y is fiberwise calm, there is a fiberwise normal cover $\gamma \in \hat{Y}$ such that for every $\sigma \in \hat{Y}$ there is a $\tau \in \hat{Y}$ with the property that fiberwise γ-homotopic fiberwise τ-functions into Y are in fact fiberwise σ-homotopic.

Let $\delta \in \gamma^*$. Since Y is also fiberwise internally movable, there is an $\eta \in \delta^+$ such that fiberwise η-functions into Y are fiberwise δ-homotopic to fiberwise maps. Let $c = \varphi(\{\eta\})$. Then F_c is a fiberwise η-function so that it is fiberwise δ-homotopic to a fiberwise map g. Let ψ denote the simple fiberwise multi-net determined by the fiberwise map g.

In order to check that φ and ψ are fiberwise homotopic, let a fiberwise normal cover $\sigma \in \hat{Y}$ be given. Choose a $\tau \in \hat{Y}$ as above. Since φ is a fiberwise multi-net, there is an index $d > c$ such that F_e is a fiberwise τ-function for every $e > d$. Thus, for every $e > d$ we get $F_e \overset{\eta}{\simeq}_B F_c \overset{\delta}{\simeq}_B g$. Hence, $F_e \overset{\gamma}{\simeq}_B g$ so that $F_e \overset{\sigma}{\simeq}_B g$. □

Theorem 5

Let Y be a fiberwise internally calm fiberwise space and let $\varphi = \{f\}$ and $\psi = \{g\}$ be simple fiberwise multi-nets into Y. If φ and ψ are fiberwise homotopic, then the fiberwise maps f and g are fiberwise homotopic.

Proof. Since Y is fiberwise internally calm, there is a $\sigma \in \hat{Y}$ such that fiberwise σ-homotopic fiberwise maps into Y are in fact fiberwise homotopic. But, the assumption that φ and ψ are fiberwise homotopic gives $f \overset{\sigma}{\simeq}_B g$. Hence, f and g are fiberwise homotopic. □

There is an obvious functor J_B from the category $\mathcal{F}h_B$ of fiberwise spaces and fiberwise homotopy classes of fiberwise maps into the category $\mathcal{F}s_B$. On objects the functor J_B is the identity while on morphisms it associates to a fiberwise homotopy class $[\,f\,]_B$ of a fiberwise map $f : X \to Y$ the fiberwise homotopy class $[\,\underline{f}\,]_B$ of a fiberwise multi-net $\underline{f} = \{F_c\} : X \to Y$, where $F_c = f$ for every $c \in \tilde{Y}$.

The last three theorems imply that the functor J_B is an isomorphism of categories when we restrict to spaces that have the above properties.

Let $\mathcal{A}$ denote the collection of all fiberwise spaces that are at the same time fiberwise internally movable, fiberwise internally calm, and fiberwise calm. One can show that every ANR_B is in the class $\mathcal{A}$.

Let $\mathcal{F}h_B^{\mathcal{A}}$ be the full subcategory of $\mathcal{F}h_B$ with objects precisely the members of the collection $\mathcal{A}$. The category $\mathcal{F}s_B^{\mathcal{A}}$ is defined similarly. Let $J_B^{\mathcal{A}} : \mathcal{F}h_B^{\mathcal{A}} \to \mathcal{F}s_B^{\mathcal{A}}$ be the restriction of the functor J_B to the category $\mathcal{F}h_B^{\mathcal{A}}$.

Theorem 6

The functor $J_B^{\mathcal{A}}$ is an isomorphism of categories.

Proof. We shall construct a functor $K_B^{\mathcal{A}} : \mathcal{F}s_B^{\mathcal{A}} \to \mathcal{F}h_B^{\mathcal{A}}$ which satisfies the relations $J_B^{\mathcal{A}} \circ K_B^{\mathcal{A}} = Id$ and $K_B^{\mathcal{A}} \circ J_B^{\mathcal{A}} = Id$. The functor $K_B^{\mathcal{A}}$ leaves the objects unchanged and on morphisms it is defined as follows. Let C be a fiberwise homotopy class of fiberwise multi-nets between two members X and Y of $\mathcal{A}$. Let φ be a representative of C and let $g : X \to Y$ be a fiberwise map such that the simple fiberwise multi-net ψ determined by g is fiberwise homotopic to φ. The functor $K_B^{\mathcal{A}}$ associates to C the fiberwise homotopy class of the fiberwise map g. It follows from the above results that this definition is correct and that $K_B^{\mathcal{A}}$ has the required properties. □

Corollary 1

On fiberwise spaces which are at the same time fiberwise internally movable, fiberwise internally calm, and fiberwise calm the fiberwise homotopy theory and the fiberwise shape theory coincide.

Corollary 2

On ANR_B spaces the fiberwise homotopy theory and the fiberwise shape theory coincide.

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