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Real interpolation for families of Banach spaces (II)

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Abstract

In this paper, we study the dual space and reiteration theorems for the real method of interpolation for infinite families of Banach spaces introduced in [2]. We also give examples of interpolation spaces constructed with this method.

1. Introduction

In [2], we studied a new method of real interpolation for families of Banach spaces. This method recovers the previous real methods of Sparr, Fernández and Cobos-Peetre (see [10], [7] and [5]). The notion of interpolation family was introduced by the St. Louis group in [6] and it is in this setting in which the K and J functionals were defined (see [2]).

This paper is a natural continuation of the previous one. Here, we generalize the definition of the K and J functional in the sense of the appendix of [2]. This generalization turns out to be very useful not only for the identification of concrete examples of interpolation spaces and for the identification of dual spaces as we shall see in this paper, but also to study compactness (see [4]), weakly compactness (see [3]) and uniform convexity (see [9]).

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The paper is organized as follows. In section 2 we, very briefly, recall the definitions and notations necessary for the sequel and introduce the interpolation spaces we shall work with. Section 3 is devoted to the relation of these spaces with the complex method. This is a continuation of section 5 in [2] where this relation was carefully studied for the original K and J functionals. In section 4 we give several examples, some of those already studied in [2]. In section 5 we study duality theorems and section 6 is devoted to reiteration results. Throughout this paper, $\sum'$ indicates a finite sum, the symbol $f \sim g$ is used to indicate the existence of two positive constants a, b such that $af(\cdot) \leq g(\cdot) \leq bf(\cdot)$ and $\equiv$ indicates equivalence of norms.

2. $K-$ and $J-$ functionals for families

Let D denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and Γ its boundary. We say that $\overline{A} = \{A(\gamma) : \gamma \in \Gamma;\ \mathcal{A},\ \mathcal{U}\}$ is a complex interpolation family (i.f.) on Γ with $\mathcal{U}$ as the containing Banach space and $\mathcal{A}$ as the log–intersection space, in the sense of [6], if:

(a) the complex Banach spaces $A(\gamma)$ are continuously embedded in $\mathcal{U}$ ($\|\cdot\|_\gamma$ will be the norm on $A(\gamma)$ and $\|\cdot\|_{\mathcal{U}}$ the norm on $\mathcal{U}$),

(b) for every $a \in \cap_{\gamma\in\Gamma} A(\gamma)$, $\gamma \in \Gamma \longrightarrow \|a\|_\gamma$ is a measurable function on Γ,

(c) $\mathcal{A} = \{a \in A(\gamma)\, a.e.\ \gamma \in \Gamma : \int_\Gamma \log^+ \|a\|_\gamma \ d\gamma < +\infty\}$, and there exists a measurable function P on Γ such that

$$\int_\Gamma \log^+ P(\gamma)\ d\gamma < +\infty \quad \text{and} \quad \|a\|_{\mathcal{U}} \leq P(\gamma)\|a\|_\gamma, \qquad \text{a.e. } \gamma \ \ (\ a \in \mathcal{A}\ )\cdot$$

Let

$$\mathcal{L} = \{\alpha : \Gamma \longrightarrow \mathbb{R}^+ :\ \text{measurable},\ \log\ \alpha \in L^1(\Gamma)\},$$

and let

$$\mathcal{G} = \left\{ b = \sum{}' b_j \chi_{E_j} : b_j \in \mathcal{A} \text{ and } E_j \text{ pairwise disjoint measurable sets in } \Gamma \right\},$$

where χ_E denotes the characteristic function of E. We shall write $a(\cdot) \in \overline{\mathcal{G}}$ whenever $a(\cdot)$ is a Bochner integrable function in $\mathcal{U}$, such that $a(\gamma) \in A(\gamma)$ a.e. $\gamma \in \Gamma$ and such that $a(\cdot)$ can be a.e. approximated in the $A(\cdot)$–norm by functions $a_n(\cdot)$ belonging to $\mathcal{G}$.

Definition 2.1. Let $\alpha \in \mathcal{L}$.
(a) For each $a \in \mathcal{U}$ and $1 \le q \le \infty$, we define the K_q–functional with respect to the i.f. $\overline{A}$ by

$$K_q(\alpha, a) = \inf \left\{ \left(\int_\Gamma \left(\alpha(\gamma) \|a(\gamma)\|_\gamma \right)^q \, d\gamma \right)^{1/q} \right\},$$

where the infimum extends over all representations $a = \int_\Gamma a(\gamma)\, d\gamma$ (convergence in $\mathcal{U}$), with $a(\cdot) \in \overline{\mathcal{G}}$.
(b) For each $a \in \mathcal{A}$ and $1 \le q \le \infty$, we also define the J_q–functional by

$$J_q(\alpha, a) = \left(\int_\Gamma \left(\alpha(\gamma) \|a\|_\gamma \right)^q d\gamma \right)^{1/q}.$$

For $\alpha \in \mathcal{L}$ and $z \in D$, we write

$$\alpha(z) = \exp\left(\int_\Gamma \log \alpha(\gamma) P_z(\gamma)\, d\gamma \right),$$

where P_z is the Poisson kernel and, for $a \in \mathcal{A}$,

$$\varphi_a(z) = \exp\left(\int_\Gamma \log \|a\|_{A(\gamma)} P_z(\gamma)\, d\gamma \right).$$

Finally, $\tilde{\alpha}$ or $\tilde{\varphi}_a$ mean that we are using the Herglotz kernel instead of the Poisson kernel in the previous formulas.

Definition 2.2. Let $S \subset \mathcal{L}$ and $0 < p \le \infty$. Let $\overline{A} = \{A(\gamma) : \ \gamma \in \Gamma; \ \mathcal{A}, \ \mathcal{U}\}$ be an i.f.
(a) The space $[A]^S_{z_0,p,q}$ consists of all $a \in \mathcal{U}$ for which

$$\left(\frac{K_q(\alpha, a)}{\alpha(z_0)} \right)_{\alpha \in S} \in l^p(S),$$

endowed with the quasi-seminorm

$$\|a\|_{[A]^S_{z_0,p,q}} = \left(\sum_{\alpha \in S} \left(\frac{K_q(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p}.$$

When $q = 1$, we simply write $[A]^S_{z_0,p}$.

(b) The space $(A)^S_{z_0,p,q}$ is the set of all elements $a \in \mathcal{U}$ such that there exists $\{u(\alpha)\}_{\alpha\in S}$ in $\mathcal{A}$ satisfying $a = \sum_{\alpha\in S} u(\alpha)$ (in the $\mathcal{U}$–norm) and

$$\left(\sum_{\alpha}\left(\frac{J_q(\alpha, u(\alpha))}{\alpha(z_0)}\right)^p\right)^{1/p} < +\infty \cdot$$

This space will be endowed with the quasi-seminorm

$$\|a\|_{(A)^S_{z_0,p,q}} = \inf\left\{\left(\sum_{\alpha}\left(\frac{J_q(\alpha, u(\alpha))}{\alpha(z_0)}\right)^p\right)^{1/p}\right\}$$

where the infimum extends to all possible representations of a. When $q = +\infty$, we write $(A)^S_{z_0,p}$.

We observe that although in the classical cases of Sparr, Fernández and Cobos-Peetre (see §3) these spaces do not depend on q, in general they are not equivalent for different $q's$. A trivial example of this is to take $S = \{1\}$ and the family $A(\gamma) = \big(\mathbb{C}, w(\gamma)\big)$ where $w \in L^{q_0}(\Gamma)$ but $w \notin L^{q_1}(\Gamma)$. In this case, $(A)^S_{z_0,p,q_0} = \big(\mathbb{C}, \|w\|_{q_0}\big)$ while $(A)^S_{z_0,p,q_1} = \{0\}$. In §3 we shall see another non trivial example of this fact. Also, it is known (see [3], [4]) that for the K–method the weakly compactness and the compactness properties are preserved by interpolation for every $q > 1$ but not in the case $q = 1$. Same for the J–method and $q = +\infty$.

Although no restrictions are needed on S for the definition of these spaces, they are necessary to have good properties of these spaces and, hence, one can easily see modifying in a slight way the arguments given in [2] that the natural conditions we need on S to have that the spaces $[A]^S_{z_0,p,q}$ and $(A)^S_{z_0,p,q}$ are Banach spaces with the intermediate property ($\mathcal{A} \subset [A]^S_{z_0,p,q} \subset \mathcal{U}$, $\mathcal{A} \subset (A)^S_{z_0,p,q} \subset \mathcal{U}$) and the usual embedding $(A)^S_{z_0,p} \subset [A]^S_{z_0,p}$ are the following:
(i) For every $\alpha \in S$ there exists a constant C_α such that $P(\gamma) \leq C_\alpha \alpha(\gamma)$ a.e. $\gamma \in \Gamma$, where P is the function in the definition of i.f.
(ii) For every $z_0 \in D$, there exists a compact set $K \subset D$ such that

$$\sum_{\alpha\in S} \frac{\inf_{z\in K} \alpha(z)}{\alpha(z_0)} < +\infty \cdot \tag{1}$$

(iii) S is a multiplicative group; that is, for every $\alpha, \beta \in S$, $\alpha\beta \in S$, $1 \in S$ and $\alpha^{-1} \in S$.

However, we want to insist on the fact that we do not need all that conditions together to prove many of the results in the paper (see also [2]) but it is a good way of simplifying the theory and be sure we have all what we need at any moment. Therefore, unless we specify the contrary, we shall assume, in the sequel, that the set S satisfies conditions (i), (ii) and (iii).

Once we have the intermediate property, we would like to also get the density condition of $\mathcal{A}$ in the interpolated spaces. However, this density property does not hold even in the case of a finite family of three spaces (see [10]). For this reason, we introduce a new K-functional,

$$\widetilde{K}_q(\alpha, a) = \inf\left\{ \left(\int_\Gamma \left(\alpha(\gamma) \|a(\gamma)\|_\gamma \right)^q d\gamma \right)^{1/q} : \ a(\cdot) \in \mathcal{G}, \ \int_\Gamma a(\gamma)\, d\gamma = a \right\},$$

for $a \in \mathcal{A}$, and a new space $\{A\}^S_{z_0,p,q}$ defined as the completion of $\mathcal{A}$ with respect to the norm

$$\|a\|_{\{A\}^S_{z_0,p,q}} = \left(\sum_{\alpha \in S} \left(\frac{\widetilde{K}_q(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p}.$$

For $q = 1$, we write $\{A\}^S_{z_0,p}$.

It is worth to mention the relation between this space and the space $A\{z_0\}$ in the complex interpolation method for families in the same way that $[A]^S_{z_0,p}$ can be related to $A[z_0]$ (see [6] and [2]). We see in the following counterexample that the new space does not coincide in general with $[A]^S_{z_0,p}$ and, similarly to what happens in the complex method, this new space will not always be embedded in $\mathcal{U}$.

Counterexample 2.3. Let $\overline{A} = \{A(\gamma) = A_j : \ \gamma \in \Gamma_j, j = 0, 1, 2\}$ where

$$A_0 = \left\{ \lambda = (\lambda_n)_n : \ \|\lambda\|_{A_0} = \sum_{n \in \mathbb{Z}} |\lambda_n| \min(1, 2^{-n}) < +\infty \right\},$$

$$A_1 = \left\{ \lambda = (\lambda_n)_n : \ \|\lambda\|_{A_1} = \sum_{n \in \mathbb{Z}} |\lambda_n| \min(1, 2^{n}) < +\infty \right\},$$

$$A_2 = \left\{ \lambda = (\lambda_n)_n : \ \|\lambda\|_{A_2} = \sum_{n \in \mathbb{Z}} |\lambda_n| < +\infty \right\}$$

as in Appendix 1 of [6]. Then, we get that for the set $S = \{1\}$,

$$\|a\|_{[A]^S_{z_0,p}} = K(1, a) = \inf\left\{ \sum_{j=0}^{2} \|a_j\|_{A_j} : \ a = \sum_{j=0}^{2} a_j \right\}$$

$$\|a\|_{\{A\}^S_{z_0,p}} = \widetilde{K}(1, a) = \inf\left\{ \sum_{j=0}^{2} \|a_j\|_{A_j} : \ a = \sum_{j=0}^{2} a_j, a_j \in \cap_{j=0}^2 A_j \right\}$$

and, as in Remark A1.1, (iii) of [6], there exists a sequence $(\beta^k)_k$ such that, as $k \to \infty$, $\|\beta^k\|_{[A]^S_{z_0,p}} \to 0$, while $\|\beta^k\|_{\{A\}^S_{z_0,p}} \geq 1/6$ for each k. $\square$

For $\overline{A} = \{A(\gamma); \mathcal{A}, \mathcal{U}\}$ and $\overline{B} = \{B(\gamma); \mathcal{B}, \mathcal{V}\}$ two i.f., we say that $T : \overline{A} \longrightarrow \overline{B}$ is an interpolation operator if $T : \mathcal{U} \longrightarrow \mathcal{V}$ is bounded and $T : A(\gamma) \longrightarrow B(\gamma)$ with norm $\|T\|_{A(\gamma)\to B(\gamma)} \leq M(\gamma) \in \mathcal{L}$. Then, the following theorem was essentially proved in [2].

Theorem 2.4

Let $\overline{A}$ and $\overline{B}$ be two i.f. and let $T : \overline{A} \longrightarrow \overline{B}$ be an interpolation operator with norm $\|T\|_{A(\gamma)\to B(\gamma)} \leq M(\gamma) \in \mathcal{L}$. Then
(a) if $\|M\|_\infty < +\infty$, we get that, for every $S \subset \mathcal{L}$,

$T : [A]^S_{z_0,p,q} \longrightarrow [B]^S_{z_0,p,q}$, $T : \{A\}^S_{z_0,p,q} \longrightarrow \{B\}^S_{z_0,p,q}$ and $T : (A)^S_{z_0,p,q} \longrightarrow (B)^S_{z_0,p,q}$

are bounded with norm less than or equal to $\|M\|_\infty$.
(b) If $MS = \{M\alpha : \ \alpha \in S\} \subset S$, then they are bounded with norm less than or equal to $M(z_0)$ (that is, with convexity).
(c) In fact, one can easily check, using the hypothesis (iii) on S that the norm of any of these three operators can be bounded by

$$\inf_{\beta\in S} \frac{\|M\beta\|_\infty}{\beta(z_0)} .$$

Obviously, we have the following chain of embeddings:

$$(A)^S_{z_0,p,\infty} \subset \cdots (A)^S_{z_0,p,q} \subset \cdots (A)^S_{z_0,p,1} ,$$
$$[A]^S_{z_0,p,\infty} \subset \cdots [A]^S_{z_0,p,q} \subset \cdots [A]^S_{z_0,p,1} ,$$

and with respect to the embedding $(A)^S_{z_0,p} \subset [A]^S_{z_0,p}$ we have the following result:

Proposition 2.5

Under the hypotheses assumed on S we get:
(a) If $p \geq 1$, then $(A)^S_{z_0,p,\infty}$ is embedded in $[A]^S_{z_0,p,1}$; and if

$$\sum_n \left(\frac{\operatorname{ess\,inf} \alpha_n(\gamma)}{\alpha_n(z_0)}\right)^p < +\infty ,$$

then the same holds for $p < 1$.
(b) Let $p \geq 1$. Then, $(A)^S_{z_0,p,1}$ is embedded in $[A]^S_{z_0,p,\infty}$; and if

$$\sum_n \left(\frac{\inf_{z\in K} \alpha_n(z)}{\alpha_n(z_0)}\right)^p < +\infty ,$$

then the same result holds for $p < 1$.
(c) Under the same hypotheses of (a) and assuming that $\{A\}^S_{z_0,p,1}$ is embedded in $\mathcal{U}$, we get that $(A)^S_{z_0,p,\infty}$ is embedded in $\{A\}^S_{z_0,p,1}$.
(d) Under the same hypotheses of (b) and assuming that $\{A\}^S_{z_0,p,\infty}$ is embedded in $\mathcal{U}$, we get that $(A)^S_{z_0,p,1}$ is embedded in $\{A\}^S_{z_0,p,\infty}$.

Proof.
(a) See [2].
(b) We get, using Szegö Theorem, (see also [2], 2.6) that, if $a = \sum_{\beta \in S} a_\beta$, $a_\beta \in \mathcal{A}$ and $C(z) = \|P_z\|_\infty$, then

$$\begin{aligned} K_\infty(\alpha, a) &\le \sum_{\beta \in S} K_\infty(\alpha, a_\beta) \le \sum_{\beta \in S} \alpha(z_\beta)\varphi_{a_\beta}(z_\beta) \\ &= \sum_{\beta \in S} (\alpha\beta^{-1})(z_\beta)\beta(z_\beta)\varphi_{a_\beta}(z_\beta) \le \sum_{\beta \in S} C(z_\beta)(\alpha\beta^{-1})(z_\beta) J_1(\beta, a_\beta), \end{aligned}$$

for every $\{z_\beta\}_\beta \subset D$, and hence, choosing $\{z_\beta\} \subset K$ such that $(\alpha\beta^{-1})(z_\beta) = \inf_{z \in K}(\alpha\beta^{-1})(z)$, we obtain

$$K_\infty(\alpha, a) \le C \sum_\beta J_1(\beta, a_\beta) \inf_{z \in K} (\alpha\beta^{-1})(z)$$

and we proceed as in (a); that is, if $p \ge 1$

$$\begin{aligned} \|a\|_{[A]^S_{z_0,p,\infty}} &= \left(\sum_{\alpha \in S} \left(\frac{K_\infty(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} \\ &\le C \left(\sum_\alpha \left(\sum_\beta \frac{J_1(\beta, a_\beta)}{\beta(z_0)} \, \frac{\operatorname{ess\,inf}_{z \in K}(\alpha\beta^{-1})(z)}{(\alpha\beta^{-1})(z_0)} \right)^p \right)^{1/p} \\ &\le C' \left(\sum_\alpha \sum_\beta \left(\frac{J_1(\beta, a_\beta)}{\beta(z_0)} \right)^p \frac{\operatorname{ess\,inf}_{z \in K}(\alpha\beta^{-1})(z)}{(\alpha\beta^{-1})(z_0)} \right)^{1/p} \\ &\le C'' \|a\|_{(A)^S_{z_0,p,1}} + \varepsilon \cdot \end{aligned}$$

If $p < 1$ we just need to do the obvious changes in the second inequality.
(c) and (d): To see this, we only have to notice that if $a = \sum_n a_{\alpha_n}$ and we call $a^N = \sum_{n=-N}^N a_{\alpha_n}$, then, as in the proof of (b), one can easily see that $(a^N)_N$ is a Cauchy sequence in $\{A\}^S_{z_0,p}$ and therefore converges to an element $b \in \{A\}^S_{z_0,p}$. But, if $\{A\}^S_{z_0,p}$ is embedded in $\mathcal{U}$ we get that necessarily $b = a$. $\square$

The following proposition will be very useful for the examples.

Proposition 2.6

Let $\overline{A} = (A_0, A_1)$ be an i.f. corresponding to the partition of Γ, $\{\Gamma_0, \Gamma_1\}$; that is, $A(\gamma) = A_j$, for every $\gamma \in \Gamma_j$. Let $\alpha \in \mathcal{L}$ such that $\operatorname{ess\,inf}_{\gamma\in\Gamma_j} \alpha(\gamma) \neq 0$ for $j = 0, 1$ and define

$$\alpha_j^{(q)} = \left(\int_{\Gamma_j} \alpha(\gamma)^{-q'} \, d\gamma \right)^{-1/q'} \cdot$$

Then, for every $a \in A_0 \cap A_1$,

$$K_q(\alpha, a) \sim \inf \left\{ \alpha_0^{(q)} \|a_0\|_{A_0} + \alpha_1^{(q)} \|a_1\|_{A_1} : a = a_0 + a_1 \right\},$$

and if $A_0 \cap A_1$ is dense in A_j for $j = 0, 1$, then the same holds for every $a \in A_0 + A_1$.

Proof. First, we observe that

$$\alpha_j^{(q)} = \inf \left\{ \left(\int_{\Gamma_j} (\alpha(\gamma)\varphi(\gamma))^q \, d\gamma \right)^{1/q} : \int_{\Gamma_j} \varphi(\gamma) \, d\gamma = 1 \right\} \cdot$$

Now, let $a = a_0 + a_1$ with $a_j \in A_j$ and let us consider $a(\gamma) = a_0\varphi_0(\gamma) + a_1\varphi_1(\gamma)$ where φ_j are arbitrary measurable functions such that

$$(*) \qquad \operatorname{supp} \varphi_j \subset \Gamma_j \qquad \text{and} \qquad \int_{\Gamma_j} \varphi_j(\gamma) \, d\gamma = 1 \cdot$$

Then $\int_\Gamma a(\gamma) \, d\gamma = a$ and

$$\begin{aligned} K_q(\alpha, a) &\leq \inf \left\{ \left(\sum_{j=0}^{1} \int_{\Gamma_j} \left(\|a_j\|_{A_j} \alpha(\gamma) |\varphi_j(\gamma)| \right)^q d\gamma \right)^{1/q} : \varphi_j \text{ satisfies } (*) \right\} \\ &\leq \left(\|a_0\|_{A_0}^q (\alpha_0^{(q)})^q + \|a_1\|_{A_1}^q (\alpha_1^{(q)})^q \right)^{1/q} \sim \alpha_0^{(q)} \|a_0\|_{A_0} + \alpha_1^{(q)} \|a_1\|_{A_1} \cdot \end{aligned}$$

Conversely, given $\varepsilon > 0$, if $a = \int_\Gamma a(\gamma) \, d\gamma$ with

$$\left(\int_{\Gamma_0} (\alpha(\gamma) \|a(\gamma)\|_0)^q \ d\gamma + \int_{\Gamma_1} (\alpha(\gamma) \|a(\gamma)\|_1)^q \ d\gamma \right)^{1/q} \leq K_q(\alpha, a) + \varepsilon\,,$$

we get, defining $a_j = \int_{\Gamma_j} a(\gamma) d\gamma \in A_j$, that $a = a_0 + a_1$ and

$$\begin{aligned} &\alpha_0^{(q)} \|a_0\|_{A_0} + \alpha_1^{(q)} \|a_1\|_{A_1} \\ &\qquad \leq \sum_{j=0}^{1} \inf \left\{ \left(\int_{\Gamma_j} (\alpha(\gamma)\varphi_j(\gamma))^q \, d\gamma \right)^{1/q} : \int_{\Gamma_j} \varphi(\gamma) \, d\gamma = 1 \right\} \int_{\Gamma_j} \|a(\gamma)\|_{A(\gamma)} d\gamma \cdot \end{aligned}$$

Taking $\varphi_j(\gamma) = \|a(\gamma)\|_{A(\gamma)}\chi_{\Gamma_j}(\gamma)\left(\int_{\Gamma_j}\|a(\gamma)\|_{A(\gamma)}d\gamma\right)^{-1}$ we get that the last term is less than or equal to

$$\sum_{j=0}^{1}\left(\int_{\Gamma_j}\left(\alpha(\gamma)\|a(\gamma)\|_{A(\gamma)}\right)^q d\gamma\right)^{1/q} \sim \left(\int_{\Gamma}\left(\alpha(\gamma)\|a(\gamma)\|_{A(\gamma)}\right)^q d\gamma\right)^{1/q},$$
$$\leq K_q(\alpha, a) + \varepsilon$$

and, thus, letting ε go to zero and taking the appropriate infimum we get the result. □

3. Examples

In this section we shall give some examples of interpolation spaces with the method introduced above. Some of these examples were already studied in [2]; however, we shall be here more precise about the q parameter. In the next examples, we assume that the intersection space $\cap_j A_j$ is dense in every A_j.

(I) Let $\overline{A} = (A_0, A_1)$; that is, $A(\gamma) = A_j$ for $\gamma \in \Gamma_j$, $j = 0, 1$ with $\{\Gamma_0,\ \Gamma_1\}$ a partition of Γ. Then if we take

$$S = \left\{\alpha_n(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0 \\ 2^n & \text{if } \gamma \in \Gamma_1 \end{cases},\ n \in \mathbb{Z}\right\}$$

we get that, $[A]^S_{z_0,p,q} \equiv (A_0, A_1)_{|\Gamma_1|_{z_0},p}$ since by Proposition 2.6

$$K_q(\alpha_n, a) \sim K(2^n, a)$$

with $K(2^n, a)$ the classical K–functional.

(II) If $\overline{A} = (A_0, A_1, \cdots, A_m)$; that is, $A(\gamma) = A_j$ for $\gamma \in \Gamma_j$, $j = 0, \cdots, m$, $\{\Gamma_0, \cdots, \Gamma_m\}$ a partition of Γ and we consider

$$S = \left\{\alpha_{\overline{n}}(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma_0 \\ 2^{n_j} & \text{if } \gamma \in \Gamma_j,\ j = 1, \cdots, m \end{cases},\ \overline{n} = (n_1, \cdots, n_m) \in \mathbb{Z}^m\right\},$$

then

$$[A]^S_{z_0,p,q} \equiv (A_0, A_1, \cdots, A_N)_{(|\Gamma_j|_{z_0}, j=1,\cdots,N),p}$$

(Sparr K–space, see [10]). Analogously for the J–method.

(III) If $\overline{A}=(A_0,A_1,A_2,A_3)$; that is, $A(\gamma)=A_j$ when $\gamma\in\Gamma_j$, $j=0,\cdots,3$ and we consider

$$S=\left\{\alpha_{\overline{n}}(\gamma)=\begin{cases}1 & \text{if } \gamma\in\Gamma_0\\ 2^n & \text{if } \gamma\in\Gamma_1\\ 2^k & \text{if } \gamma\in\Gamma_2\\ 2^n2^k & \text{if } \gamma\in\Gamma_3\end{cases},\ \overline{n}=(n,k)\in\mathbb{Z}^2\right\}$$

then $[A]^S_{z_0,p,q}\equiv(A_0,A_1,A_2,A_3)_{(\theta_1,\theta_2),p;K}$ where $\theta_1=|\Gamma_1\cup\Gamma_3|_{z_0}$ and $\theta_2=|\Gamma_2\cup\Gamma_3|_{z_0}$ (Fernández K–space, see [7]). Analogously for the J–method.

(IV) If $\overline{A}=(A_1,\cdots,A_m)$ and we take

$$S=\left\{\alpha_{n,k}(\gamma)=\big\{(a_j)^n(b_j)^k:\ \gamma\in\Gamma_j,\ j=1,\cdots,m\big\},\ (n,k)\in\mathbb{Z}^2\right\}$$

with $a_j=2^{x_j}$ and $b_j=2^{y_j}$ and (x_j,y_j) the vertices of a polygon in the affine plane $\mathbb{R}^2$, then $[A]^S_{z_0,p,q}=(A_1,\cdots,A_m)_{(\alpha,\beta),p;K}$ where $(\alpha,\beta)=\sum_{j=1}^m|\Gamma_j|_{z_0}(x_j,y_j)$ (Cobos-Peetre interpolation spaces, see [5]). Analogously for the J–method.

(V) Let $1\le q\le\infty$ and let $\overline{A}=\{A(\gamma);\gamma\in\Gamma\}$ be an i.f. with $A(\gamma)=L^q(w(\gamma,\cdot))$ where $L^q(w)=\big\{f;\ fw\in L^q\big\}$ and $w(\gamma,\cdot)$ is a family of weights on a measure space $\mathcal{M}$ such that, $w(\gamma,x)\in\mathcal{L}$ a.e. x. Then

$$\begin{aligned}&\big(K_q(\alpha,f)\big)^q\\ &=\inf\left\{\int_\Gamma\big(\alpha(\gamma)\ \|F(\gamma)\|_{L^q(w(\gamma,\cdot))}\big)^q\,d\gamma;\ \int_\Gamma F(\gamma)=f\right\}\\ &=\inf\left\{\int_{\mathcal{M}}\int_\Gamma\big(\alpha(\gamma)\ |F(\gamma,x)|w(\gamma,x)\big)^q\,d\gamma\,dx;\ \int_\Gamma F(\gamma,x)\,d\gamma=f(x)\right\}\\ &=\int_{\mathcal{M}}|f(x)|^q\left(\inf\left\{\int_\Gamma\big(\alpha(\gamma)\ |\varphi(\gamma,x)|w(\gamma,x)\big)^q\,d\gamma;\ \int_\Gamma\varphi(\gamma,x)\,d\gamma=1\right\}\right)dx\\ &=\int_{\mathcal{M}}|f(x)|^q\left(\int_\Gamma\big(\alpha(\gamma)w(\gamma,x)\big)^{-q'}\,d\gamma\right)^{-q/q'}dx,\end{aligned}$$

and hence, if we write

$$W_q(\alpha,x)=\left(\int_\Gamma\big(\alpha(\gamma)w(\gamma,x)\big)^{-q'}\,d\gamma\right)^{-1/q'},$$

we get $K_q(\alpha,f)=\|f\|_{L^q(W_q(\alpha,\cdot))}$ and thus, if we define

$$W_q(S,z_0)=\left(\sum_{\alpha\in S}\Big(\frac{W_q(\alpha,x)}{\alpha(z_0)}\Big)^q\right)^{1/q},$$

one can easily see that, for every S and every $z_0\in D$, $[A]^S_{z_0,q,q}=L^q(W_q(S,z_0))$.

(VI) In this example we get a weighted Lorentz space as an interpolation of L^1 and L^∞. However, our set S will not satisfy, in general conditions (i), (ii) and (iii). Let $\overline{A} = (L^1, L^\infty)$. Then, it is well known that, for the classical K–functional $K(t,f) = \int_0^t f^*(s)\,ds$ where f^* is the rearrangement decreasing function of f. Now, let w be a weight such that for some $0 < \lambda < 1$, $\int_{2^n}^{2^{n+1}} w \le 2^{np(1-\lambda)}$. Then, if we take Γ_1 such that $|\Gamma_1|_{z_0} = \lambda$, we can find α_n such that $\inf_{\gamma\in\Gamma_0} \alpha_n(\gamma) = 1$, $\inf_{\gamma\in\Gamma_1} \alpha_n(\gamma) = 2^n$ and $\alpha_n(z_0) = 2^n \big(\int_{2^n}^{2^{n+1}} w \big)^{-1/p}$. Therefore, if $S = \{\alpha_n\}_{n\in\mathbb{Z}}$, we get by 2.6,

$$[A]^S_{z_0,p,1} = \left\{ f \in L^1 + L^\infty ; \left(\sum_{n\in\mathbb{Z}} \left(\frac{K(2^n;f)}{2^n} \right)^p \int_{2^n}^{2^{n+1}} w \right)^{1/p} < +\infty \right\}$$

$$\sim \left\{ f \in L^1 + L^\infty ; \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f^*(t)\,dt \right)^p w(x)\,dx \right)^{1/p} < +\infty \right\}.$$

Thus, if w is in the class of Ariño and Muckenhoupt B_p, then $[A]^S_{z_0,p,1} \equiv \Lambda^p(w)$ is a weighted Lorentz space.

(VII) Let $(A_0, A_1) = (L^1, L^\infty)$. By Proposition 2.6, if we take ($\Gamma \equiv [0,2]$)

$$\alpha(\gamma) = \begin{cases} 1 & \text{if } \gamma \in [0,1] \\ \gamma & \text{if } \gamma \in [1,2] \end{cases},$$

and $S = \{\alpha^n,\ n \in \mathbb{Z}\}$, then one can easily check that

$$K_1(\alpha^n, a) = \begin{cases} \|a\|_{L^1+L^\infty} & \text{if } n > 0 \\ K(2^n, a) & \text{if } n \le 0 \end{cases},$$

and

$$K_\infty(\alpha^n, a) = \begin{cases} \|a\|_{L^1+L^\infty} & \text{if } n > 0 \\ K(|n|2^n, a) & \text{if } n \le 0 \end{cases},$$

where $K(2^n, a)$ and $K(|n|2^n, a)$ are the usual K–functional for a couple of Banach spaces.

Hence, there exists $0 < \theta < 1$ such that

$$[A]^S_{z_0,\infty,\infty} = \left\{ f \in L^1 + L^\infty;\ \big(|n| 2^{n(1-\theta)} f^{**}(|n|2^n) \big)_{n\le 0} \in l^\infty \right\}$$

and

$$[A]^S_{z_0,\infty,1} = \left\{ f \in L^1 + L^\infty;\ \big(2^{n(1-\theta)} f^{**}(2^n) \big)_{n\le 0} \in l^\infty \right\}.$$

These spaces are obviously not equivalent since, for example a function f such that $f^*(s) = s^{\theta-1}$ satisfies that $f \in [A]^S_{z_0,\infty,1}$ but $f \notin [A]^S_{z_0,\infty,\infty}$.

4. Relation with the complex method for families

In this section, we shall use the notation of [6]. Let us recall that for an i.f. $\overline{A}$, the St. Louis group defines the following function spaces:

$$\mathcal{G}(A(\cdot),\Gamma)=\left\{g(z)=\sum{}'\varphi_j a_j:\ \varphi_j\in H^\infty(D),\ a_j\in\mathcal{A},\|g\|_{\mathcal{G}(A(\cdot),\Gamma)}<+\infty\right\}$$

where $\|g\|_{\mathcal{G}(A(\cdot),\Gamma)} = \operatorname{ess\,sup}_{\gamma\in\Gamma}\|g(\gamma)\|_{A(\gamma)}$ and $\mathcal{F}(A(\cdot),\Gamma)$ is the completion of $\mathcal{G}(A(\cdot),\Gamma)$ with respect to the norm $\|\cdot\|_{\mathcal{G}}$. However, as is shown in [6], one can substitute the above norm by $\|g\| = \big(\int_\Gamma \|g(\gamma)\|_\gamma^p\,d\gamma\big)^{1/p}$ for every $p\geq 1$ and the interpolated spaces $A\{z_0\}$ and $A[z_0]$ remains unchanged. In this paper, we shall denote the norm with index p by $\|\cdot\|_{\mathcal{G}^p}$ and the corresponding spaces by $\mathcal{G}^p(A(\cdot),\Gamma)$ and $\mathcal{F}^p(A(\cdot),\Gamma)$.

In [2], we see that the usual embeddings known for the classic method

$$(A_0,A_1)_{\theta,1}\subset[A_0,A_1]_\theta\subset(A_0,A_1)_{\theta,\infty}\,,$$

(see [1]) are now given by

$$(A)^S_{z_0,1,\infty}\subset A[z_0]\subset[A]^S_{z_0,\infty,1}\,.$$

Now, the previous relationship can be improved in the following way.

Proposition 4.1

Let $z_0\in D$ and $1\leq q\leq\infty$. Then the following embeddings hold.
(a) $(A)^S_{z_0,1,q}$ is embedded in $A[z_0]$.
(b) $A[z_0]$ is embedded in $[A]^S_{z_0,\infty,q}$
(c) If, for every $a\in\mathcal{A}$, the function $\|a\|_\gamma\in L^p(\Gamma)$ for some $p>q$. Then, for every $a\in\mathcal{A}$, $\|a\|_{\{A\}^S_{z_0,\infty,q}}\leq C\|a\|_{A\{z_0\}}$.

Proof. The proof of (a) and (b) is similar to that in [2].

We shall only prove (c). We observe that, given $g\in\mathcal{G}^q(A(\cdot),\Gamma)$ such that $g(z_0)=a$, we have that $g=\sum_j\varphi_j a_j$ with $\varphi_j\in H^\infty(D)$ and $a_j\in\mathcal{A}$ for all j. Set $g_\alpha=\tilde\alpha(z_0)g/\tilde\alpha$, with $\tilde\alpha(z_0)=\alpha(z_0)$. Then, we can consider simple functions s_j^m such that

$$\int_\Gamma\left(\left|s_j^m(\gamma)-\frac{\tilde\alpha(z_0)\varphi_j(\gamma)}{\tilde\alpha(\gamma)}P_{z_0}(\gamma)\right|\max\Big(1,\alpha(\gamma)\Big)\right)^{(p/q)'}d\gamma$$

converges to zero as m tends to infinity and such that $\int_\Gamma s_j^m(\gamma)\,d\gamma = \varphi_j(z_0)$. If we define $g_m = \sum_j s_j^m a_j$ we get that $g_m \in \mathcal{G}$, $\int_\Gamma g_m(\gamma)\,d\gamma = a$ and

$$\int_\Gamma \big(\alpha(\gamma)\|g_m(\gamma) - g_\alpha(\gamma)P_{z_0}(\gamma)\|_\gamma\big)^q d\gamma \overset{m\to\infty}{\longrightarrow} 0\,.$$

Thus, given $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and constant $C > 0$ such that

$$\begin{aligned}\widetilde{K}_q(\alpha, a) &\le \bigg(\int_\Gamma \big(\alpha(\gamma)\|g_m(\gamma)\|_\gamma\big)^q d\gamma\bigg)^{1/q}\\ &\le \varepsilon + C\bigg(\int_\Gamma \big(\alpha(z_0)\|g(\gamma)\|_\gamma\big)^q d\gamma\bigg)^{1/q} \le \varepsilon + C\alpha(z_0)\|g\|_{\mathcal{G}^q}\,.\end{aligned}$$

Therefore, $\|a\|_{\{A\}^S_{z_0,\infty,q}} \le C\|a\|_{A\{z_0\}}$ for every $a \in \mathcal{A}$. □

Analogously, we could define a space V closely related to $(A)^S_{z_0,p,q}$ as the completion of $\mathcal{A}$ with respect to the norm

$$\|a\|_p = \inf\left\{\bigg(\sum_\alpha \Big(\frac{J_q(\alpha, a_\alpha)}{\alpha(z_0)}\Big)^p\bigg)^{1/p} : \sum{}' a_\alpha = a\right\}.$$

Then, for every $a \in \mathcal{A}$, $\|a\|_{A\{z_0\}} \le \|a\|_1$. We only need to consider $a = \sum' a_n$ and, hence, the function $G(z) = \sum{}' \frac{\tilde\alpha_n(z)}{\tilde\alpha_n(z_0)} a_n$ is already in $\mathcal{G}^q(A(\cdot),\Gamma)$ and the proof follows as in (a).

However, if V is as before and assume that V is embedded in $\mathcal{U}$, then one can easily show that this new space coincides with $(A)^S_{z_0,p,q}$, whenever $(A)^S_{z_0,p,q}$ is a Banach space contained in $\mathcal{U}$.

5. Duality

In this section, we characterize the dual of the spaces $\{A\}^S_{z_0,p,1}$, $[A]^S_{z_0,p,1}$ and $(A)^S_{z_0,p,q}$ for all $1 \le p,\ q < +\infty$. The dual spaces of $\{A\}^S_{z_0,p,1}$ and $[A]^S_{z_0,p,1}$ are constructed in a similar way to the construction of the dual space of $A\{z_0\}$ and $A[z_0]$ (see [6]).

Although a slight modification in the proof of the results for the first two cases would give us some information about the dual of the spaces $\{A\}^S_{z_0,p,q}$ and $[A]^S_{z_0,p,q}$ for $1 < q < +\infty$, we are unable to give a complete characterization of them. The reason is quite simple: for an infinite family of functions h_j, the fact that $\sup_j \|h_j\|_{q'} < +\infty$ does not imply $\sup_j |h_j| \in L^{q'}$ (except for the case $q = 1$).

We will only consider the K_1–functional in the following theorems and we simply write K.

Let $E(\alpha) = \big\{a \in \mathcal{U} :\ K(\alpha, a) < \infty\big\}$. Then $E(\alpha)$ is a locally convex vector space endowed with the seminorm $K(\alpha,\cdot)$.

Theorem 5.1

A linear functional l belongs to the dual space of $E(\alpha)$ if and only if the two following conditions hold.
(i) There exists a constant $C > 0$ such that, for every $a \in \mathcal{A}$, $|(l,a)| \leq C\alpha(\gamma)\|a\|_\gamma$, a.e. $\gamma \in \Gamma$.
(ii) Let us define for $a(\cdot) \in \overline{\mathcal{G}}$ and $\Big(a_n(\cdot)\Big)_n \in \mathcal{G}$ such that $a(\cdot) = \lim_n a_n(\cdot)$ in the usual way, the function $< l, a > (\cdot)$ by

$$< l, a > (\gamma) := \lim_n \Big(l, a_n(\gamma)\Big) \cdot$$

Then $\int_\Gamma < l, a > (\gamma) d\gamma = 0$ whenever $\int_\Gamma a(\gamma) d\gamma = 0$.

Moreover, $\|l\|_{E(\alpha)^*} = \inf \big\{C : \ C$ satisfies (i)$\big\}$.

Proof. First, we prove that $\mathcal{A}$ is dense in $E(\alpha)$: let $a \in E(\alpha)$ and let $a(\cdot) \in \overline{\mathcal{G}}$ such that $a = \int_\Gamma a(\gamma)\, d\gamma$ and $\|a(\cdot)\|_\cdot \in L^1(\alpha)$. Let $\Big(a_n(\cdot)\Big)_n \in \mathcal{G}$ converging to $a(\cdot)$ in the usual way and set $a_n = \int_\Gamma a_n(\gamma)\, d\gamma \in \mathcal{A}$. Then

$$K(\alpha, a - a_n) \leq \int_\Gamma \alpha(\gamma) \|a_n(\gamma) - a(\gamma)\|_\gamma \, d\gamma\,,$$

and this last expression goes to zero when n goes to infinity.

Let us prove the necessary condition. Let $l \in E(\alpha)^*$ and $a \in \mathcal{A}$. Let us define for $\varphi \in L^1(\Gamma) \cap L^1(\Gamma; \alpha\varphi_a)$

$$< L_a, \varphi >= (l, a) \int_\Gamma \varphi(\gamma)\, d\gamma \cdot$$

Then, L_a is a linear operator such that

$$| < L_a, \varphi > | \leq \|l\|_{E(\alpha)^*} \int_\Gamma \alpha(\gamma) \|a\|_\gamma |\varphi(\gamma)|\, d\gamma \cdot$$

Therefore, by the Hahn-Banach Theorem, L_a can be extended to a linear operator in $L^1(\Gamma; \alpha\varphi_a)$. Thus, there exists a measurable function on Γ, h_a, such that
(a) $|h_a(\gamma)| \leq \|l\|_{E(\alpha)^*} \alpha(\gamma) \|a\|_\gamma$ a.e. $\gamma \in \Gamma$,
(b) $< L_a, \varphi >= \int_\Gamma h_a(\gamma)\varphi(\gamma)\, d\gamma$, for every $\varphi \in L^1(\Gamma; \alpha\varphi_a)$.

Since $\alpha(\cdot)$ and $\varphi_a(\cdot)$ belong to $\mathcal{L}$, we can deduce (as in [6] for the dual space of $A[z_0]$) that if $< h(\gamma), a >= h_a(\gamma)$, then, for every $a, b \in \mathcal{A}$,

$$< h(\gamma), a + b >=< h(\gamma), a > + < h(\gamma), b >, \text{ a.e.}\, \gamma \in \Gamma;$$

that is, the above equality holds for every $\gamma \in \Gamma \setminus E$, where $|E| = 0$ but E depends on a and b.

We have that, for every $\varphi \in L^1(\Gamma) \cap L^1(\Gamma; \alpha\varphi_a)$,

$$(l, a) \int_\Gamma \varphi(\gamma) d\gamma = \int_\Gamma < h(\gamma), a > \varphi(\gamma) d\gamma,$$

and this condition implies $h(\gamma) =$ constant a.e. $\gamma \in \Gamma$ and hence $h = l$. Obviously l satisfies (i) by condition (a). To see that it also satisfies (ii) we observe that by condition (i)

$$\left| \Big(l, a_n(\gamma) - a_m(\gamma) \Big) \right| \le C\alpha(\gamma) \|a_n(\gamma) - a_m(\gamma)\|_\gamma,$$

a.e. $\gamma \in \Gamma$, and thus, the function $< l, a >$ (that depends on the sequence $(a_n)_n$) is well defined. Then, if $\int_\Gamma a(\gamma) d\gamma = 0$ for $a(\cdot) \in \overline{\mathcal{G}}$ and $\|a(\cdot)\|_. \in L^1(\alpha)$, we get that if $a_n = \int_\Gamma a_n(\gamma)\, d\gamma$ then,

$$0 = \lim_{n\to\infty} l(a_n) = \lim_{n\to\infty} \int_\Gamma \Big(l, a_n(\gamma) \Big)\, d\gamma = \int_\Gamma < l, a > (\gamma) d\gamma\,,$$

by dominated convergence. Conversely, let l be satisfying the hypotheses. Let $a = \int_\Gamma a(\gamma) d\gamma$ with $a(\cdot) \in \overline{\mathcal{G}}$ and $\|a(\cdot)\|_. \in L^1(\alpha)$.

By (ii) l can be extended to $E(\alpha)$ by $(l, a) = \int_\Gamma < l, a > (\gamma)\, d\gamma$. Now, by condition (i) we have that, a.e. $\gamma \in \Gamma$,

$$| < l, a > (\gamma)| = \lim_n \left| \Big(l, a_n(\gamma) \Big) \right| \le C \lim_n \|a_n(\gamma)\|_\gamma \alpha(\gamma) = C\|a(\gamma)\|_\gamma \alpha(\gamma),$$

and hence,

$$|(l, a)| \le \int_\Gamma | < l, a > (\gamma)|\, d\gamma \le C \int_\Gamma \alpha(\gamma) \|a(\gamma)\|_\gamma\, d\gamma;$$

that is, $|(l, a)| \le CK(\alpha, a)$ and $l \in (E(\alpha))^*$ with $\|l\|_{(E(\alpha))^*} \le C$. $\square$

Remark 5.2. If we define $\widetilde{E(\alpha)}$ as the completion of $\mathcal{A}$ with respect to the seminorm $\widetilde{K}(\alpha, \cdot)$, the dual space is characterized only by condition (i).

Our next goal is to characterize the dual space of $[A]^S_{z_0,p}$ for $1 \le p < +\infty$. We shall use the following general fact: if $(E_i)_{i\in I}$ is a collection of locally convex spaces, then for $1 \le p < +\infty$, we have that if $l^p(E_i) = \Big\{ (a_i)_i \subset (E_i)_i : \ (\|a_i\|_{E_i})_i \in l^p \Big\}$, then $(l^p(E_i))^* = l^{p'}(E_i^*)$.

Theorem 5.3

A linear functional l belongs to the dual space of $[A]^S_{z_0,p}$ if and only if, for every $\alpha \in S$, there exists $l_\alpha \in \big(E(\alpha)\big)^*$ such that
(i) $(l,a) = \sum_{\alpha\in S}\int_\Gamma < l_\alpha, a_\alpha > (\gamma)\,d\gamma$, *with* $\|a_\alpha(\cdot)\|_\cdot \in L^1(\alpha)$, $a_\alpha \in \overline{\mathcal{G}}$ *and* $a = \int_\Gamma a_\alpha(\gamma)\,d\gamma$, *and*
(ii) there exists a constant $C > 0$ *such that*

$$\left(\sum_{\alpha\in S}\Big(\alpha(z_0)\|l_\alpha\|_{\left(E(\alpha)\right)^*}\Big)^{p'}\right)^{1/p'} \leq C\cdot$$

Moreover, $\|l\|_{([A]^S_{z_0,p})^*} = \inf\Big\{C > 0 :\ C \textit{ satisfies (ii)}\Big\}\cdot$

Proof. Let $l \in ([A]^S_{z_0,p})^*$. Since

$$[A]^S_{z_0,p} = \left\{a \in \mathcal{U} :\ \big(\|a\|_{E(\alpha,z_0)}\big)_\alpha \in l^p(S)\right\},$$

where $E(\alpha,z_0) = \alpha(z_0)^{-1}E(\alpha)$, we can consider the space $[A]^S_{z_0,p}$ as a subspace of the space $l^p\left((E(z_0,\alpha)_{\alpha\in S}\right)$ (identifying $a \equiv (a)_{\alpha\in S}$) and then, l can be extended to a linear operator L over this space. That is, for every $\alpha \in S$, there exists $\widetilde{l_\alpha} \in (E(\alpha,z_0))^* = \alpha(z_0)E(\alpha)^*$ such that

$$< L, (a_\alpha)_\alpha >= \sum_{\alpha\in S}\int_\Gamma < \widetilde{l_\alpha}, a_\alpha > (\gamma)\alpha(z_0)\,d\gamma\,,$$

with $a_\alpha = \int_\Gamma a_\alpha(\gamma)\,d\gamma$, $a_\alpha \in \overline{\mathcal{G}}$, and

$$(**)\qquad \left(\sum_{\alpha\in S}\|\widetilde{l_\alpha}\|^{p'}_{\left(E(\alpha)\right)^*}\right)^{1/p'} < +\infty\cdot$$

Then,

$$(l,a) =< L, (a)_{\alpha\in S} >= \sum_{\alpha\in S}\int_\Gamma \frac{< \widetilde{l_\alpha}, a_\alpha > (\gamma)}{\alpha(z_0)}\,d\gamma = \sum_{\alpha\in S}\int_\Gamma < l_\alpha, a_\alpha > (\gamma)\,d\gamma\,,$$

with $l_\alpha = \frac{\widetilde{l_\alpha}}{\alpha(z_0)}$ and $a = \int_\Gamma a_\alpha(\gamma)\,d\gamma$. Obviously, l_α satisfies (ii) by (**).

Reciprocally, by the properties of $l_\alpha \in \big(E(\alpha)\big)^*$, l is well defined by (i) and the fact that $l \in ([A]^S_{z_0,p})^*$ is trivial. □

Remark 5.4. By conditions (i) and (ii) of the previous theorem, we have that if $a \in \mathcal{A}$, then we can take $a_\alpha(\cdot)$ of the form $a\varphi_\alpha(\cdot)$ with $\int_\Gamma \varphi_\alpha(\gamma)\,d\gamma = 1$ and then we get $(l,a) = \sum_\alpha (l_\alpha, a)$; that is, we can identify, at least formally, $l = \sum_\alpha l_\alpha$. Now, since $\|l_\alpha\|_{\left(E(\alpha)\right)^*} = J_\infty(\alpha^{-1}, h^\alpha)$, we get that

$$\left(\sum_{\alpha\in S}\left(\frac{J_\infty(\alpha^{-1}, l_\alpha)}{\alpha^{-1}(z_0)}\right)^{p'}\right)^{1/p'} < +\infty\,,$$

and then, we can embed the dual space of $[A]^S_{z_0,p}$ into the space $(A^*)^{S^{-1}}_{z_0,p'}$ where $\overline{A^*} = \{A^*(\gamma) : \gamma \in \Gamma\}$ and $S^{-1} = \{\alpha^{-1} :\ \alpha \in S\}$. However this embedding is just formally since, as is well–known the family $\overline{A^*}$ need not be an i.f.

Using the same argument as in the previous theorem, one can also see that the following result holds.

Theorem 5.5

A linear functional l belongs to the dual space of $\{A\}^S_{z_0,p}$ if and only if, for every $\alpha \in S$, there exists $l_\alpha \in \left(\widetilde{E(\alpha)}\right)^*$ such that
(i) for every $a \in \mathcal{A}$, $(l,a) = \sum_{\alpha\in S}\int_\Gamma \Big(l_\alpha, a_\alpha(\gamma)\Big)\,d\gamma$ with $\|a_\alpha(\cdot)\|_\cdot \in L^1(\alpha)$, $a_\alpha(\cdot) \in \mathcal{G}$ and $a = \int_\Gamma a_\alpha(\gamma)\,d\gamma$,
(ii) there exists a constant $C > 0$ such that

$$\left(\sum_{\alpha\in S}\left(\|l_\alpha\|_{\left(\widetilde{E}(\alpha)\right)^*}\alpha(z_0)\right)^{p'}\right)^{1/p'} \le C\,\cdot$$

Moreover, $\|l\|_{(\{A\}^S_{z_0,p})^*} = \inf\Big\{C > 0 :\ C \text{ satisfies (ii)}\Big\}\cdot$

The following theorem will give us a characterization of the dual space of $(A)^S_{z_0,p,q}$.

Let us define $J_q(\alpha) = \{a \in \mathcal{A} :\ J_q(\alpha, a) < +\infty\}$ and assume that $J_q(\alpha)$ is not trivial. Then, if $\widetilde{J_q(\alpha)}$ is the completion of $J_q(\alpha)$ with respect to the norm $J_q(\alpha,\cdot)$, we get the following result.

Theorem 5.6

Let $1 \le q < +\infty$. Then $l \in \left(\widetilde{J_q(\alpha)}\right)^*$ if and only if, for every $a \in J_q(\alpha)$, there exists a measurable function $< h(\cdot), a >$ on Γ such that, for every $a, b \in J_q(\alpha)$ and every $\lambda, \mu \in \mathbb{C}$, $< h(\gamma), a + b >=< h(\gamma), a > + < h(\gamma), b >$, a.e. $\gamma \in \Gamma$ and
(i) there exists a positive constant C so that, for every $a \in J_q(\alpha)$, the function

$$H_a(\gamma) = | < h(\gamma), a > |/(\alpha(\gamma)\|a\|_\gamma) \in L^{q'}(\Gamma)$$

and $\|H_a\|_{q'} \le C$, and
(ii) for every $a \in J_q(\alpha)$, $(l, a) = \int_\Gamma < h(\gamma), a > \, d\gamma$.
Moreover, $\|l\|_{\left(\widetilde{J_q(\alpha)}\right)^*} = \inf \Big\{ C : \ C$ satisfies (i)$\Big\}$.

Proof. The sufficient condition is obviously true by Hölder's inequality.

To prove the necessary condition, let $L^q(A(\cdot)\alpha(\cdot))$ be the set of all measurable functions $f : \Gamma \longrightarrow \mathcal{U}$ such that
(a) $f(\gamma) \in A(\gamma)$ a.e. $\gamma \in \Gamma$ and,
(b) $\|f(\gamma)\|_\gamma \in L^q(\alpha(\cdot))$,
endowed with the norm $\|f\|_{L^q(A(\cdot)\alpha(\cdot))} = \left(\int_\Gamma \|f(\gamma)\|_\gamma^q \alpha(\gamma)^q d\gamma\right)^{1/q}$.

Then, we can identify $J_q(\alpha)$ as a subspace of $L^q(A(\cdot)\alpha(\cdot))$ by

$$\begin{aligned} J_q(\alpha) &\longrightarrow L^q(A(\cdot)\alpha(\cdot)) \\ a &\longrightarrow [\gamma \in \Gamma \to a] \cdot \end{aligned}$$

Hence, if $l \in \left(\widetilde{J_q(\alpha)}\right)^*$, l can be extended to a functional L on $L^q(A(\cdot)\alpha(\cdot))$ such that

$$L(f) \le \|l\|_{\left(\widetilde{J_q(\alpha)}\right)^*} \left(\int_\Gamma \|f(\gamma)\|_\gamma^q \alpha(\gamma)^q d\gamma\right)^{1/q}.$$

In particular, if we fix $a \in J_q(\alpha)$, L defines a continuous functional on the space

$$\Lambda(a) = \left\{ \varphi \text{ measurable } : \ \left(\int_\Gamma |\varphi(\gamma)|^q \alpha^q(\gamma) \|a\|_\gamma^q d\gamma\right)^{1/q} < +\infty \right\} = L^q(\alpha\varphi_a)\,,$$

and therefore, there exists a measurable function $h_a \in L^{q'}(\alpha^{-1}\varphi_a^{-1})$ such that

$$L(\varphi a) = \int_\Gamma \varphi(\gamma) h_a(\gamma) d\gamma,$$

for every $\varphi \in \Lambda(a)$ and $\|h_a\|_{L^{q'}(\alpha^{-1}\varphi_a^{-1})} \le \|l\|_{\left(\widetilde{J_q(\alpha)}\right)^*}$.

Now, as α and $\varphi_a \in \mathcal{L}$ we can deduce, by standard arguments that, for every λ, $\beta \in \mathbb{C}$ and every $a, b \in J_q(\alpha)$, we get $h_{\lambda a+\beta b}(\gamma) = \lambda h_a(\gamma) + \beta h_b(\gamma)$ a.e. $\gamma \in \Gamma$. Thus, if $< h(\gamma), a >= h_a(\gamma)$ a.e. $\gamma \in \Gamma$ and every $a \in J_q(\alpha)$, we get that, for every $a \in J_q(\alpha)$,

$$(l, a) = L(a) = \int_\Gamma < h(\gamma), a > d\gamma,$$

and condition (i) holds easily. □

Using this result, we get the following one.

Theorem 5.7

Let $1 \leq p,\ q < +\infty$. A linear functional $l \in \left((A)^S_{z_0,p,q}\right)^*$ if and only if, for every $\alpha \in S$, and every $a \in J_q(\alpha)$, there exists a function $< h_\alpha(\cdot), a >$ satisfying:
(i) for every $a \in (A)^S_{z_0,p,q}$, $(l, a) = \sum_{\alpha\in S} \int_\Gamma < h_\alpha(\gamma), a_\alpha > d\gamma$, where $a = \sum_\alpha a_\alpha$ in the $\mathcal{U}$–norm and $a_\alpha \in J_q(\alpha)$, and
(ii) $\left(\alpha(z_0) \sup_{a_\alpha\in J_q(\alpha)} \| < h_\alpha(\cdot), a_\alpha > /(\alpha(\cdot)\|a_\alpha\|.) \|_{q'}\right)_{\alpha\in S} \in l^{p'}(S)$.
Moreover,

$$\|l\|_{((A)^S_{z_0,p,q})^*} = \inf\left\{\left(\sum_{\alpha\in S}\left(\alpha(z_0) \sup_{a_\alpha\in J_q(\alpha)} \| < h_\alpha(\cdot), a_\alpha > /(\alpha(\cdot)\|a_\alpha\|.) \|_{q'}\right)^p\right)^{1/p}\right\},$$

where the infimum extends over all representations of $a = \sum_\alpha a_\alpha$.

Proof. The sufficient condition is obvious. Let $l \in \left((A)^S_{z_0,p,q}\right)^*$. Since, we can identify $(A)^S_{z_0,p,q}$ with the quotient space $l^p\left(J_q(\alpha)\alpha(z_0)^{-1}\right)$ modulo the set N of all sequences $(a_\alpha)_\alpha$ such that $a_\alpha \in J_q(\alpha)$ and $\sum_\alpha a_\alpha = 0$, we get that l can be seen as an element of $l^{p'}\left((J_q(\alpha)^*\alpha(z_0)\right)$ vanishing on N. That is, there exists a sequence $(h_\alpha)_\alpha$ satisfying conditions (ii) such that l can be identified with such a sequence by

$$(l, a) = \sum_{\alpha\in S} \int_\Gamma < h_\alpha(\gamma), a_\alpha > d\gamma,$$

for $a = \sum_\alpha a_\alpha$. □

6. Reiteration

It is well–known that, for the case of a compatible pair of Banach spaces, the classic interpolated spaces $(A_0, A_1)^K_{\theta,p}$ and $(A_0, A_1)^J_{\theta,p}$ coincide algebraically and topologically. However, for $n > 2$ only the inclusion

$$(A_1, \cdots, A_n)^J_{\theta,p} \subset (A_1, \cdots, A_n)^K_{\theta,p}\,,$$

is valid in general, for the methods of Sparr, Fernández and Cobos-Peetre.

Definition 6.1. An i.f. of Banach spaces $\overline{A} = \{A(\gamma) :\ \gamma \in \Gamma;\ \mathcal{A},\ \mathcal{U}\}$ is said to satisfy the equivalence property with respect to the method S, if $[A]^S_{z_0,p,q} \equiv (A)^S_{z_0,p,q}$ for every $z_0 \in D$ and every p and q.

It turns out that when one wants to study the question whether a method of interpolation is stable under reiteration, a good condition to get a positive answer is to have the equivalence property. Without it, only some inclusions can be proved (see, for example, [10]).

In our reiteration theorems, we have to impose a "density condition" completely analogous to the one imposed in the reiteration theorem for families of Banach spaces in the complex method (see Theorem 5.2, [6]). Moreover, we are also forced to work with the method $\{A\}^S_{z_0,p}$ instead of $[A]^S_{z_0,p}$ (see Remark 5.3, [6]).

One may also ask whether reiteration works when we deal with two different kind of methods of interpolation, whenever they can be "composed". In particular, we can mix our method with the complex interpolation method for families of the St. Louis group (see Theorems 6.5, 6.6 and 6.7).

Finally, we deal with i.f. of the type $\overline{A} = \{A(\gamma) = (A_0, A_1)_{\alpha(\gamma),p(\gamma)} :\ \gamma \in \Gamma\}$, $\overline{A} = \{A(\gamma) = [A_0, A_1]_{\alpha(\gamma)} :\ \gamma \in \Gamma\}$, $\overline{A} = \{A(\gamma) = (A_1, \cdots, A_N)_{\overline{\alpha}(\gamma),p(\gamma)} :\ \gamma \in \Gamma\}$, etc. This families are very useful to obtain the interpolated spaces for families of L^p, H^p, Sobolev spaces, ...

Theorem 6.2

Let Ω be a simply connected domain contained in D such that $\Sigma = \partial\,\Omega$ is a closed rectifiable Jordan curve and $\Sigma \subset D$. For every $\sigma \in \Sigma$, let us consider the space $B(\sigma) = \{A\}^S_{\sigma,p(\sigma),q}$ with $q \geq 1$ and $p(\sigma) \geq 1$ a measurable function on Σ. Let $d\sigma$ be the normalized measure on Σ. Assume that $\mathcal{A}$ is dense in

$$\mathcal{A}^\Sigma = \left\{ a \in B(\sigma),\ a.e.\ \sigma \in \Sigma : \int_\Sigma \log^+ \|a\|_{B(\sigma)} d\sigma < +\infty \right\},$$

in the following sense: $\mathcal{A} \subset \mathcal{A}^{\Sigma}$ *and, for every* $b \in \mathcal{A}^{\Sigma}$*, there exists a sequence* $(a_n)_n \subset \mathcal{A}$ *such that*

$$\lim_{n\to\infty} \int_{\Sigma} \|a_n - b\|_{\sigma}^{q} d\sigma = 0, \tag{2}$$

where $\|a\|_{\sigma} = \|a\|_{B(\sigma)}$*. Also, assume that there exists a Banach space* $\mathcal{V}$ *containing* $B(\sigma)$ *and a function* P *measurable on* Σ *whose logarithm is integrable and, for every* $a \in \mathcal{A}^{\Sigma}$, $\|a\|_{\mathcal{V}} \le P(\sigma)\|a\|_{\sigma}$*. Then,*
(a) $\overline{B} = \{B(\sigma) : \ \sigma \in \Sigma;\ \mathcal{A}^{\Sigma},\ \mathcal{V}\}$ *is an i.f.,*
(b) there exists a constant $C > 0$ *such that, for every* $a \in \mathcal{A}$ *and* $z_0 \in \Omega$*,*

$$\|a\|_{\{A\}^{S}_{z_0,p,q}} \le C\|a\|_{\{B\}^{S_{|\Sigma}}_{z_0,p,q}},$$

where $S_{|\Sigma} = \left\{\alpha(\sigma) = \exp\left(\int_{\Gamma} \log \alpha(\gamma) P_{\sigma}(\gamma)\, d\gamma\right) \ :\ \alpha \in S\right\}$*, and, if there exists a compact set* K *in* Ω *such that condition (1) holds, then* $\mathcal{A}$ *is dense in* $\mathcal{A}^{\Sigma}$ *with respect to the norm* $\{B\}^{S_{|\Sigma}}_{z_0,p,q}$*.*

Remark 6.3. If we have that the spaces $\{A\}^{S}_{z_0,p,q}$ and $\{B\}^{S_{|\Sigma}}_{z_0,p,q}$ are embedded in a common containing space $\mathcal{W}$, then condition (1) in (b) implies

$$\left\{\{A\}^{S}_{\sigma,p(\sigma),q}\right\}^{S_{|\Sigma}}_{z_0,p,q} \subset \{A\}^{S}_{z_0,p,q}.$$

Proof.
(a) We have that, for every $b \in \mathcal{A}^{\Sigma}$, the function

$$\sigma \in \Sigma \longrightarrow \|b\|_{\sigma} = \left(\sum_{\alpha}\left(\frac{\widetilde{K}_q(\alpha, b)}{\alpha(\sigma)}\right)^{p(\sigma)}\right)^{1/p(\sigma)},$$

is measurable on Σ. The other conditions for the family to be an i.f. are imposed by hypotheses.
(b) Let $a \in \mathcal{A}$. Since $\mathcal{A} \subset \mathcal{A}^{\Sigma}$, we have that given $\varepsilon > 0$ and $\alpha \in S_{|\Sigma}$, there exists $a_{\alpha}(\cdot) \in \mathcal{G}^{\Sigma}$ (the space $\mathcal{G}$ for the i.f. $\overline{B}$), such that $a = \int_{\Sigma} a_{\alpha}(\sigma)\, d\sigma$ and

$$\left(\int_{\Sigma} \left(\alpha(\sigma)\|a_{\alpha}(\sigma)\|_{\sigma}\right)^{q} d\sigma\right)^{1/q} \le \widetilde{K}_{\Sigma,q}(\alpha, a) + \varepsilon\,,$$

where $\widetilde{K}_{\Sigma,q}$ is the $\widetilde{K}_q$–functional corresponding to the i.f. $\overline{B}$.

Let us write

$$a_\alpha(\sigma) = \sum_{n=1}^{N} a_n \chi_{E_n}(\sigma)\,,$$

with $a_n \in \mathcal{A}^\Sigma$, for every $n \in \{1, \cdots, N\}$. Now by (2), given $\varepsilon > 0$ and $a_n \in \mathcal{A}^\Sigma$, there exists $b_n \in \mathcal{A}$ such that

$$\left(\int_\Sigma (\|a_n - b_n\|_\sigma)^q \, d\sigma \right)^{1/q} \le \frac{\varepsilon}{2N\|\alpha\|_\Sigma}\,,$$

for every $n \in \{1, \cdots, N\}$, with $\|\alpha\|_\Sigma = \sup_{\sigma\in\Sigma} \alpha(\sigma)$.

Consider $b(\sigma) = \sum_{n=1}^N (\chi_{E_n}(\sigma) - |E_n|) b_n + a$. Then $b(\cdot) \in \mathcal{G}^\Sigma$ with $b(\sigma) \in \mathcal{A}$ and $a = \int_\Sigma b(\sigma) d\sigma$. Moreover, since

$$b(\sigma) = \sum_{n=1}^{N} (\chi_{E_n}(\sigma) - |E_n|)(b_n - a_n) + a_\alpha(\sigma)\,,$$

we get

$$\begin{aligned}
&\left(\int_\Sigma (\alpha(\sigma)\|b(\sigma)\|_\sigma)^q \, d\sigma \right)^{1/q} \\
&\le \left(\int_\Sigma (\alpha(\sigma)\|a_\alpha(\sigma)\|_\sigma)^q \, d\sigma \right)^{1/q} + 2\|\alpha\|_\Sigma \sum_{n=1}^{N} \left(\int_\Sigma (\|b_n - a_n\|_\sigma)^q \, d\sigma \right)^{1/q} \\
&\le \left(\int_\Sigma (\alpha(\sigma)\|a_\alpha(\sigma)\|_\sigma)^q \, d\sigma \right)^{1/q} + \varepsilon \le \widetilde{K}_{\Sigma,q}(\alpha, a) + 2\varepsilon\,\cdot
\end{aligned}$$

Hence, we can assume $a = \int_\Sigma b(\sigma) d\sigma$ with $b(\cdot) \in \mathcal{G}^\Sigma$ and $b(\sigma) \in \mathcal{A}$.

Now, $b(\sigma)$ can be written as $\sum_n c_n \chi_{A_n}$, with A_n pairwise disjoint and $c_n \in \mathcal{A}$. Given c_n, let $\sigma_0 \in \Sigma$ be such that, for every $\sigma \in \Sigma$, $\alpha(\sigma_0)\|c_n\|_{\sigma_0} \le (1+\varepsilon)\alpha(\sigma)\|c_n\|_\sigma$. Then, for $\varepsilon > 0$, set $F_n \in \mathcal{G}$ so that $\int_\Gamma F_n(\gamma) d\gamma = c_n$ and

$$\left(\int_\Gamma \big(\alpha(\gamma)\|F_n(\gamma)\|_\gamma\big)^q d\gamma \right)^{1/q} \le (1+\varepsilon)\alpha(\sigma_0)\|c_n\|_{\sigma_0}\,\cdot$$

Let us consider $F(\sigma, \gamma) = \sum_n F_n(\gamma)\chi_{A_n}(\sigma)$. We have that $a = \int_\Gamma \left(\int_\Sigma F(\sigma,\gamma) d\sigma \right) d\gamma$

and $\int_\Sigma F(\sigma,\cdot)\,d\sigma \in \mathcal{G}$. Therefore,

$$\begin{aligned}
\widetilde{K}_q(\alpha,a) &\le \left(\int_\Gamma \left(\alpha(\gamma)\left\|\int_\Sigma F(\sigma,\gamma)d\sigma\right\|_\gamma\right)^q d\gamma\right)^{1/q} \\
&\le \left(\int_\Gamma \left(\int_\Sigma \alpha(\gamma)\|F(\sigma,\gamma)\|_\gamma d\sigma\right)^q d\gamma\right)^{1/q} \\
&\le \int_\Sigma \left(\int_\Gamma \left(\sum_n \|F_n(\gamma)\|_\gamma \chi_{A_n}(\sigma)\alpha(\gamma)\right)^q d\gamma\right)^{1/q} d\sigma \\
&= \sum_n \int_\Sigma \chi_{A_n}(\sigma)\left(\int_\Gamma \left(\|F_n(\gamma)\|_\gamma \alpha(\gamma)\right)^q d\gamma\right)^{1/q} d\sigma \\
&\le (1+\varepsilon)^2 \sum_n \int_{A_n} \alpha(\sigma)\|c_n\|_\sigma\,d\sigma = (1+\varepsilon)^2\int_\Sigma \alpha(\sigma)\|b(\sigma)\|_\sigma\,d\sigma \\
&\le (1+\varepsilon)^2\left(\int_\Sigma (\alpha(\sigma)\|b(\sigma)\|_\sigma)^q\,d\sigma\right)^{1/q} \le (1+\varepsilon)^2\left(\widetilde{K}_{\Sigma,q}(\alpha,a)+2\varepsilon\right)\cdot
\end{aligned}$$

Thus, since $\alpha(z_0)=\alpha_\Sigma(z_0)=\exp\left(\int_\Sigma \log\alpha(\sigma)\mu_{z_0}(\sigma)\,d\sigma\right)$ with μ_{z_0} the harmonic measure with respect to Ω and z_0, and letting ε go to zero, we get

$$\frac{\widetilde{K}_q(\alpha,a)}{\alpha(z_0)} \le \frac{\widetilde{K}_{\Sigma,q}(\alpha,a)}{\alpha_\Sigma(z_0)}\,.$$

That is, for every $\alpha\in\mathcal{A}$, $\|a\|_{\{A\}^S_{z_0,p,q}} \le \|a\|_{\{B\}^{S|\Sigma}_{z_0,p,q}}$.

We need to show now that, under condition (1), $\mathcal{A}$ is dense in $\mathcal{A}^\Sigma$ with respect to the norm $\{B\}^{S|\Sigma}_{z_0,p,q}$. Now, given $a\in\mathcal{A}^\Sigma$ and $\varepsilon>0$, let us consider $b\in\mathcal{A}$ such that $\left(\int_\Sigma \|a-b\|^q_\sigma d\sigma\right)^{1/q}\le\varepsilon$. Then, given $K\subset\Omega$ satisfying the hypothesis, there exists $C>0$ such that

$$\begin{aligned}
\|a-b\|_{\{B\}^{S|\Sigma}_{z_0,p,q}} &= \left(\sum_{\alpha\in S}\left(\frac{\widetilde{K}_{\Sigma,q}(\alpha,a-b)}{\alpha(z_0)}\right)^p\right)^{1/p} \\
&\le C\left(\sum_{\alpha\in S}\left(\frac{\inf_{z\in K}(\alpha(z))}{\alpha(z_0)}\right)^p\right)^{1/p}\left(\int_\Sigma \|a-b\|^q_\sigma d\sigma\right)^{1/q} \le C'\varepsilon\,\cdot\,\Box
\end{aligned}$$

Theorem 6.4

Let Ω be a simply connected domain contained in D such that $\Sigma=\partial\Omega$ is a closed rectifiable Jordan curve and $\Sigma\subset D$. For every $\sigma\in\Sigma$, let us consider the *space $B(\sigma) = (A)^S_{\sigma,p(\sigma),q}$ with $q \geq 1$ and $p(\sigma) \geq 1$ a measurable function on Σ. Let us assume that $\mathcal{A} \subset B(\sigma)$ a.e. $\sigma \in \Sigma$ and that $\overline{B} = \Big\{B(\sigma) : \sigma \in \Sigma\Big\}$ is an i.f. Then, $(A)^S_{z_0,p,q}$ is embedded in the space $(B)^{S_{|\Sigma}}_{z_0,p,q}$, for every $z_0 \in \Omega$.*

Proof. Let $a \in (A)^S_{z_0,p,q}$. Then given $\varepsilon > 0$, $a = \sum_{\alpha \in S} a_\alpha$, with $(a_\alpha)_\alpha$ in $\mathcal{A}$ and

$$\left(\sum_\alpha \left(\frac{J_q(\alpha, a_\alpha)}{\alpha(z_0)}\right)^p\right)^{1/p} \leq (1+\varepsilon)\|a\|_{(A)^S_{z_0,p,q}} \cdot$$

Now, since $a_\alpha \in \mathcal{A} \cap B(\sigma)$, a.e. $\sigma \in \Sigma$, we get

$$\begin{aligned} J_{\Sigma,q}(\alpha_{|\Sigma}, a_\alpha) &= \left(\int_\Sigma \big(\alpha(\sigma)\|a_\alpha\|_\sigma\big)^q d\sigma\right)^{1/q} \\ &\leq \left(\int_\Sigma \Big(\alpha(\sigma)\frac{J_q(\alpha, a_\alpha)}{\alpha(\sigma)}\Big)^q d\sigma\right)^{1/q} = J_q(\alpha, a_\alpha) \cdot \end{aligned}$$

Therefore,

$$\|a\|_{(B)^{S_{|\Sigma}}_{z_0,p,q}} \leq \left(\sum_\alpha \left(\frac{J_{\Sigma,q}(\alpha_{|\Sigma}, a_\alpha)}{\alpha_\Sigma(z_0)}\right)^p\right)^{1/p} \leq (1+\varepsilon)\|a\|_{(A)^S_{z_0,p,q}} \,. \;\square$$

The following theorem will be related to the reiteration between the real and complex (in this order) methods. We get as an application the corresponding reiteration result of Hernández-Soria (see [8]).

Let $B\{z_0\}$ and $B[z_0]$ be the interpolated spaces of $\{B(\sigma) : \sigma \in \Sigma\}$ with the complex method (see [6]).

Theorem 6.5

If we are under the hypotheses of the reiteration Theorem 6.2 with $B(\sigma) = \{A\}^S_{\sigma,p,q}$ and there exists a positive constant M such that

$$\sup_{\sigma \in \Sigma} \left(\sum_\alpha \left(\frac{\inf_{z\in\Omega} \alpha(z)}{\alpha(\sigma)}\right)^p\right)^{1/p} \leq M, \tag{3}$$

then, if $z_0 \in \Omega$,
(a) there exists a constant $C > 0$ such that, for every $a \in \mathcal{A}$,

$$\|a\|_{\{A\}^S_{z_0,p,q}} \leq C\|a\|_{B\{z_0\}} \,,$$

$\mathcal{A}$ is dense in $\mathcal{A}^\Sigma$ with respect to the norm $B\{z_0\}$, and
(b) $B[z_0]$ is embedded in $[A]^S_{z_0,p,q}$.

Proof.
(a) We already know that $\overline{B} = \Big\{B(\sigma) :\ \sigma \in \Sigma\Big\}$ is an i.f. Since $\mathcal{A} \subset \mathcal{A}^{\Sigma}$, we have that, for every $a \in \mathcal{A}$ and every $\varepsilon > 0$, there exists a function $G(z) = \sum_j a_j \varphi_j(z) \in \mathcal{G}^1(B(\cdot), \Sigma)$ such that $G(z_0) = a$ and $\int_\Sigma \|G(\sigma)\|_\sigma \, d\sigma \le (1+\varepsilon)\|a\|_{B\{z_0\}}$ (see [6]).

Let us consider, for every $j \in \{1, \cdots, N\}$, elements $b_j \in \mathcal{A}$ such that

$$\Big(\int_\Sigma \|a_j - b_j\|_\sigma^q d\sigma\Big)^{1/q} \le \frac{\varepsilon}{2N\|\varphi_j\|_\infty}\cdot$$

Set $\tilde{G}(\sigma) = \sum_{j=1}^N \Big(\varphi_j(\sigma) - \varphi_j(z_0)\Big) b_j + a$. Then, $\int_\Sigma \tilde{G}(\sigma)\mu_{z_0}(\sigma)\, d\sigma = a$, $\tilde{G}(\sigma) \in \mathcal{A}$ and since $\tilde{G}$ can be written as

$$\tilde{G}(\sigma) = \sum_{j=1}^N \Big(\varphi_j(\sigma) - \varphi_j(z_0)\Big)(b_j - a_j) + G(\sigma)\,,$$

we get that

$$\int_\Sigma \|\tilde{G}(\sigma) - G(\sigma)\|_\sigma \, d\sigma \le \sum_{j=1}^N 2\|\varphi_j\|_\infty \int_\Sigma \|a_j - b_j\|_\sigma \, d\sigma \le \varepsilon\cdot$$

Now, by (3), $\sup_{\sigma\in\Sigma} \|b_j\|_\sigma = C_j < +\infty$ for every $j \in \{1, \cdots, N\}$ and, hence, we can consider, for each j, a simple function s_j such that $\int_\Sigma \frac{\tilde{\alpha}(z_0)}{\tilde{\alpha}(\sigma)} s_j(\sigma)\mu_{z_0}(\sigma)\, d\sigma = \varphi_j(z_0)$ and

$$\int_\Sigma |s_j(\sigma) - \varphi_j(\sigma)|\, d\sigma \le \frac{\varepsilon}{NC_j}\cdot$$

Set

$$F(\sigma) = \sum_{j=1}^N \Big(s_j(\sigma) - \varphi_j(z_0)\Big) b_j + a\cdot$$

We have that F is a simple function such that $F(\sigma) \in \mathcal{A}$, $\int_\Sigma \|F(\sigma) - \tilde{G}(\sigma)\|_\sigma \, d\sigma \le \varepsilon$ and

$$a = \int_\Sigma \frac{\tilde{\alpha}(z_0)}{\tilde{\alpha}(\sigma)} F(\sigma)\mu_{z_0}(\sigma)\, d\sigma\cdot$$

Assume $F = \sum' a_n \chi_{E_n}$, with $a_n \in \mathcal{A}$ and E_n measurable pairwise disjoint sets. Let $(C_\alpha)_\alpha$ be a sequence of positive numbers such that $\sum_\alpha \Big(C_\alpha \int_\Sigma \frac{\mu_{z_0}(\sigma)}{\alpha(\sigma)}\, d\sigma\Big)^p < 1$ and let $a_n(\cdot) \in \mathcal{G}$ be such that $a_n = \int_\Gamma a_n(\gamma)\, d\gamma$, and

$$\Big(\int_\Gamma \Big(\alpha(\gamma)\|a_n(\gamma)\|_\gamma\Big)^q d\gamma\Big)^{1/q} \le \widetilde{K}_q(\alpha, a_n) + \varepsilon\, C_\alpha\cdot$$

Then

$$a = \int_\Sigma \frac{\tilde\alpha(z_0)}{\tilde\alpha(\sigma)} F(\sigma)\mu_{z_0}(\sigma)d\sigma = \sum_n \int_{E_n} \frac{\tilde\alpha(z_0)}{\tilde\alpha(\sigma)}\mu_{z_0}(\sigma)\left(\int_\Gamma a_n(\gamma)d\gamma\right)d\sigma$$
$$= \int_\Gamma \left(\sum_n a_n(\gamma)\int_{E_n}\frac{\tilde\alpha(z_0)}{\tilde\alpha(\sigma)}\mu_{z_0}(\sigma)d\sigma\right)d\gamma,$$

and

$$\widetilde{K}_q(\alpha,a) \le \sum_n \int_{E_n} \frac{\alpha(z_0)}{\alpha(\sigma)}\Big(\widetilde{K}_q(\alpha,a_n) + \varepsilon\, C_\alpha\Big)\mu_{z_0}(\sigma)d\sigma$$
$$\le \int_\Sigma \frac{\alpha(z_0)}{\alpha(\sigma)}\widetilde{K}_q(\alpha,F(\sigma))\mu_{z_0}(\sigma)d\sigma + \varepsilon\, C_\alpha \int_\Sigma \frac{\alpha(z_0)}{\alpha(\sigma)}\mu_{z_0}(\sigma)d\sigma\,\cdot$$

That is,

$$\frac{\widetilde{K}_q(\alpha,a)}{\alpha(z_0)} \le \int_\Sigma \frac{\widetilde{K}_q(\alpha,F(\sigma))}{\alpha(\sigma)}\mu_{z_0}(\sigma)d\sigma + \varepsilon\, C_\alpha \int_\Sigma \frac{1}{\alpha(\sigma)}\mu_{z_0}(\sigma)d\sigma\,,$$

and thus,

$$\left(\sum_{\alpha\in S}\left(\frac{\widetilde{K}_q(\alpha,a)}{\alpha(z_0)}\right)^p\right)^{1/p}$$
$$\le \int_\Sigma \left(\sum_{\alpha\in S}\left(\frac{\widetilde{K}_q(\alpha,F(\sigma))}{\alpha(\sigma)}\right)^p\right)^{1/p}\mu_{z_0}(\sigma)d\sigma + \varepsilon\left(\sum_{\alpha\in S}\left(C_\alpha\int_\Sigma \frac{\mu_{z_0}(\sigma)}{\alpha(\sigma)}d\sigma\right)^p\right)^{1/p}$$
$$\le C\int_\Sigma \|F(\sigma)\|_\sigma\, d\sigma + \varepsilon \le C\Big(\varepsilon + \int_\Sigma \|\tilde G(\sigma)\|_\sigma d\sigma\Big) + \varepsilon$$
$$\le C\Big(2\varepsilon + \int_\Sigma \|G(\sigma)\|_\sigma\, d\sigma\Big) + \varepsilon$$
$$\le C\Big(2\varepsilon + (1+\varepsilon)\|a\|_{B\{z_0\}}\Big) + \varepsilon\,\cdot$$

Letting ε go to zero we are done.

The density of $\mathcal{A}$ in $\mathcal{A}^\Sigma$ is trivial by (2), since

$$\|a-b\|_{B\{z_0\}} \le \exp\left(\int_\Sigma \log\|a-b\|_{B(\sigma)}\mu_{z_0}(\sigma)d\sigma\right) \le C\int_\Sigma \|a-b\|_\sigma d\sigma \le C\varepsilon\,\cdot$$

(b) The proof of this part is completely analogous to (a). Let $a = F(z_0)$ for some $F \in \mathcal{F}^1(B(\cdot),\Sigma)$. Let $(G_N)_N$ be in $\mathcal{G}^1(B(\cdot),\Sigma)$ such that G_N converges to F. Then, as in (a), we can assume G_N to be a simple function and, hence, F is a suitable function for the definition of $[A]^S_{z_0,p}$. Following the computations in (a) one can show that

$$\|G_N(z_0) - G_M(z_0)\|_{[A]^S_{z_0,p}} \leq C\|G_N - G_M\|_{\mathcal{G}^1}\,,$$

and thus, $G_N(z_0)$ converges to an element b in $[A]^S_{z_0,p}$ and obviously $b = a$. □

Theorem 6.6

Under the hypotheses of Theorem 6.2 with $B(\sigma) = (A)^S_{\sigma,p(\sigma),q}$, $p(\sigma) < +\infty$ a.e. $\sigma \in \Sigma$ and $p = p(z_0)$, where $\frac{1}{p(z)} = \int_\Sigma \frac{1}{p(\sigma)}\mu_z(\sigma)\,d\sigma$, we have that,
(a) if $B\{z_0\}$ is embedded in $\mathcal{U}$, $(A)^S_{z_0,p,q} \subset B\{z_0\}$, and
(b) $(A)^S_{z_0,p,q} \subset B[z_0]$.

Proof.
(a) Given $\varepsilon > 0$, let $a \in (A)^S_{z_0,p,q}$ be such that $\|a\|_{(A)^S_{z_0,p,q}} = 1$ and let $a = \sum_n a_n$ be such that $\left(\sum_n \left(J_q(\alpha_n, a_n)/\alpha_n(z_0)\right)^p\right)^{1/p} \leq (1+\varepsilon)$, with $\{\alpha_n\}_n \subset S$.

Let $a^N = \sum_{n=-N}^N a_n \in \mathcal{A}$. We have that $(a^N)_N$ is a Cauchy sequence in $(A)^S_{z_0,p,q}$. Let us consider, for $z \in \Omega$,

$$G^N(z) = \sum_{n=-N}^{N} \frac{\tilde{\alpha}_n(z)}{\alpha_n(z_0)} \left(\frac{J_q(\alpha_n, a_n)}{\alpha_n(z_0)}\right)^{p(z_0)(\widetilde{\frac{1}{p(z)}})-1} a_n\,,$$

where $(\widetilde{1/p(z)})$ is the analytic extension of $1/p(z) = \int_\Sigma 1/p(\sigma)\ \mu_z(\sigma)d\sigma$ such that $(\widetilde{1/p(z_0)}) = 1/p(z_0)$. Then $G^N \in \mathcal{G}^1(B(\cdot),\Sigma)$, $G^N(z_0) = a^N$ and, given $\varepsilon > 0$ there exist N and M such that

$$\|a^N - a^M\|_{B\{z_0\}} \leq \|G^N - G^M\|_{\mathcal{G}^1} = \int_\Sigma \|G^N(\sigma) - G^M(\sigma)\|_{B(\sigma)}d\sigma$$

$$\leq \int_\Sigma \left(\sum_{N\leq|n|\leq M} \left(\frac{\alpha_n(\sigma)}{\alpha_n(z_0)}\right)^{p(\sigma)} \left(\frac{J_q(\alpha_n,a_n)}{\alpha_n(z_0)}\right)^{p(z_0)-p(\sigma)} \left(\frac{J_q(\alpha_n,a_n)}{\alpha_n(\sigma)}\right)^{p(\sigma)}\right)^{1/p(\sigma)} d\sigma$$

$$= \int_\Sigma \left(\sum_{N\leq|n|\leq M} \left(\frac{J_q(\alpha_n,a_n)}{\alpha_n(z_0)}\right)^{p(z_0)}\right)^{1/p(\sigma)} d\sigma \leq \varepsilon\cdot$$

That is, $(a^N)_N$ is a Cauchy sequence in $B\{z_0\}$ and, therefore, since we are assuming that $B\{z_0\}$ is embedded in $\mathcal{U}$, we have that a^N converges to a in $B\{z_0\}$ and, hence, $a \in B\{z_0\}$ and $\|a\|_{B\{z_0\}} \leq \|a\|_{(A)^S_{z_0,p}}$.
(b) The proof of this part is completely analogous to (a). □

Now, we deal with reiteration with the complex method first and then real.

Theorem 6.7

Under the hypotheses of the Theorem 6.2 with $B(\sigma) = A\{\sigma\}$, we have that
(a) $\overline{B} = \{B(\sigma): \ \sigma \in \Sigma, \ \mathcal{A}^{\Sigma}, \mathcal{V}\}$ *is an i.f.,*
(b) if we assume that, for every $a \in \mathcal{A}$, the function $\gamma \to \|a\|_\gamma$ is in L^p, for some $p > q$, then, for every $a \in \mathcal{A}$,

$$\|a\|_{\{A\}^S_{z_0,p,q}} \leq C\|a\|_{\{B\}^{S_{|\Sigma}}_{z_0,p,q}},$$

(c) if $\mathcal{U} = \mathcal{V}$, then $(A)^S_{z_0,p,q}$ is embedded in $(B)^{S_{|\Sigma}}_{z_0,p,q}$.

Remark 6.8. If we have that the spaces $\{A\}^S_{z_0,p,q}$ and $\{B\}^{S_{|\Sigma}}_{z_0,p,q}$ are embedded in a common containing space $\mathcal{W}$ and if there exists a compact set $K \subset \Omega$ satisfying (3), then $\mathcal{A}$ is dense in $\big\{A\{\sigma\}\big\}^{S_{|\Sigma}}_{z_0,p,q}$ and therefore, (b) implies

$$\big\{A\{\sigma\}\big\}^{S_{|\Sigma}}_{z_0,p,q} \subset \{A\}^S_{z_0,p,q}\,.$$

Proof. The proof of (a) can be found in [6].
(b) Let $a \in \mathcal{A}$. Since $\mathcal{A} \subset \mathcal{A}^{\Sigma}$ we have that given $\varepsilon > 0$ and $\alpha \in S_{|\Sigma}$, there exists $a_\alpha(\cdot) \in \mathcal{G}^{\Sigma}$ (the space $\mathcal{G}$ for the i.f. $\overline{B}$), such that $a = \int_\Sigma a_\alpha(\sigma)\, d\sigma$ and

$$\left(\int_\Sigma \left(\alpha(\sigma)\|a_\alpha(\sigma)\|_\sigma\right)^q d\sigma\right)^{1/q} \leq \widetilde{K}_{\Sigma,q}(\alpha, a) + \varepsilon,$$

where $\widetilde{K}_{\Sigma,q}$ is the $\widetilde{K}_q$–functional corresponding to the i.f. $\overline{B}$.

Let us write

$$a_\alpha(\sigma) = \sum_{n=1}^{N} a_n \chi_{E_n}(\sigma)\,,$$

with $a_n \in \mathcal{A}^{\Sigma}$, for every $n \in \{1, \cdots, N\}$. Now, we can assume (as we did in Theorem 6.2) that $a_\alpha(\sigma) \in \mathcal{A}$.

Given a_n, let $\sigma_0 \in \Sigma$ be such that $\alpha(\sigma_0)\|a_n\|_{\sigma_0} \leq (1+\varepsilon)\alpha(\sigma)\|a_n\|_\sigma$, for every $\sigma \in \Sigma$. Then, for $\varepsilon > 0$, set $F_n \in \mathcal{G}(A(\cdot), \Gamma)$ so that $F_n(\sigma_0) = a_n$ and $\|F_n\|_{\mathcal{G}} \leq (1+\varepsilon)\|a_n\|_{\sigma_0}$. Let us consider the function $\tilde{F}_n(\cdot) = F_n(\cdot)\alpha(\sigma_0)/\tilde{\alpha}(\cdot)$. We can assume (see the proof of Theorem 4.1) that $\tilde{F}_n(\cdot)P_{\sigma_0}(\cdot) \in \mathcal{G}$. Let us consider $F(\sigma, \gamma) = \sum_n \tilde{F}_n(\gamma)\chi_{A_n}(\sigma)$. We have that $a = \int_\Gamma \left(\int_\Sigma F(\sigma,\gamma) d\sigma\right) P_{\sigma_0}(\gamma) d\gamma$ and $\int_\Sigma F(\sigma, \cdot)P_{\sigma_0}(\cdot) d\sigma \in \mathcal{G}$. The proof now follows as in Theorem 6.2 (b).

(c) Let $a = \sum_n a_n$ and $\varepsilon > 0$ such that

$$\left(\sum_n \left(\frac{J_q(\alpha_n, a_n)}{\alpha_n(z_0)} \right)^p \right)^{1/p} \leq (1+\varepsilon) \|a\|_{(A)^S_{z_0,p,q}} .$$

Then, since $\|a\|_\sigma \leq J_q(\alpha, a)/\alpha(\sigma)$ for every $a \in \mathcal{A}$ and every $\alpha \in S$, we get

$$\begin{aligned} \|a\|_{(B)^{S_{|\Sigma}}_{z_0,p,q}} &\leq \left(\sum_n \left(\frac{J_{\Sigma,q}(\alpha_n, a_n)}{\alpha_n(z_0)} \right)^p \right)^{1/p} \\ &= \left(\sum_n \left(\frac{\left(\int_\Sigma \left(\alpha_n(\sigma) \|a_n\|_\sigma \right)^q d\sigma \right)^{1/q}}{\alpha_n(z_0)} \right)^p \right)^{1/p} \\ &\leq \left(\sum_n \left(\frac{J_q(\alpha_n, a_n)}{\alpha_n(z_0)} \right)^p \right)^{1/p} \leq (1+\varepsilon) \|a\|_{(A)^S_{z_0,p,q}} . \quad \square \end{aligned}$$

Reiteration for a compatible pair of Banach spaces

Theorem 6.9

Let (A_0, A_1) be a compatible pair of Banach spaces. Let $0 \leq \alpha(\cdot) \leq 1$ and $p(\cdot) \geq 1$ be two measurable functions on Γ. Let us define, for each $\gamma \in \Gamma$, the space $A(\gamma) = (A_0, A_1)_{\alpha(\gamma), p(\gamma)}$. Set $S = \{\alpha_n(\cdot) = 2^{n\alpha(\cdot)} : n \in \mathbb{Z}\}$. Then, for every $1 \leq p < +\infty$ and $q \geq 1$,

$$[A]^S_{z_0,p,q} \equiv (A_0, A_1)_{\alpha(z_0),p} \equiv (A)^S_{z_0,p,q} ,$$

where $\alpha(z_0) = \int_\Gamma \alpha(\gamma) P_{z_0}(\gamma) d\gamma$.

Proof. Let us start with the first embedding. Let $a \in [A]^S_{z_0,p,q}$. Given $\varepsilon > 0$, we have that, for every $n \in \mathbb{Z}$, there exists $a_n(\cdot)$ such that $a = \int_\Gamma a_n(\gamma) d\gamma$, $a_n(\cdot) \in \overline{\mathcal{G}}$, $a_n(\gamma) \in (A_0, A_1)_{\alpha(\gamma), p(\gamma)}$ and

$$\int_\Gamma 2^{n\alpha(\gamma)} \|a_n(\gamma)\|_\gamma d\gamma \leq \widetilde{K}_q(\alpha_n, a_n) + \varepsilon ,$$

for every $n \in \mathbb{Z}$. We now observe that we can always assume that $a \in \mathcal{A}$ and that $a_n \in G$. Hence, for each $n \in \mathbb{Z}$, let $a_n^j(\cdot) : \Gamma \longrightarrow A_j$ be a simple function $(j = 0, 1)$ so that $a_n(\gamma) = a_n^0(\gamma) + a_n^1(\gamma)$ and $\|a_n^0(\gamma)\|_{A_0} + 2^n \|a_n^1(\gamma)\|_{A_1} \le (1+\varepsilon) 2^{n\alpha(\gamma)} \|a_n(\gamma)\|_\gamma$. Let us define $a_n^j = \int_\Gamma a_n^j(\gamma) d\gamma \in A_j$. Hence, $a_n^0 + a_n^1 = \int_\Gamma a_n(\gamma) d\gamma = a$ and

$$\begin{aligned} K(2^n, a) &\le \|a_n^0\|_{A_0} + 2^n \|a_n^1\|_{A_1} \le (1+\varepsilon) \int_\Gamma 2^{n\alpha(\gamma)} \|a_n(\gamma)\|_\gamma d\gamma \\ &\le (1+\varepsilon) \big(\widetilde{K}_q(\alpha_n, a) + \varepsilon \big) \cdot \end{aligned}$$

Dividing by $2^{n\alpha(z_0)}$ and taking the l^p–norm, we get $\|a\|_{(A_0,A_1)_{\alpha(z_0),p}} \le \|a\|_{\{A\}^S_{z_0,p,1}}$.

To see the embedding with the J–method, let $a \in (A_0, A_1)_{\alpha(z_0),p}$. Using the equivalence of this space with the corresponding space for the J–method, we get that, given $\varepsilon > 0$, $a = \sum_n a_n$ in the $(A_0 + A_1)$–norm with $(a_n)_n$ in $A_0 \cap A_1$ and

$$\left(\sum_n \left(\frac{\max(\|a_n\|_0, 2^n \|a_n\|_1)}{2^{n\alpha(z_0)}} \right)^p \right)^{1/p} \le (1+\varepsilon) \|a\|_{(A_0,A_1)_{\alpha(z_0),p}} .$$

Now, $a_n \in \mathcal{A}$ and

$$\begin{aligned} J_q(2^{n\alpha(\cdot)}, a_n) &= \left(\int_\Gamma \Big(2^{n\alpha(\gamma)} \|a_n\|_\gamma \Big)^q d\gamma \right)^{1/q} \\ &\le \left(\int_\Gamma \Big(2^{n\alpha(\gamma)} \frac{\max(\|a_n\|_0, 2^n \|a_n\|_1)}{2^{n\alpha(\gamma)}} \Big)^q d\gamma \right)^{1/q} \\ &\le \max(\|a_n\|_0, 2^n \|a_n\|_1) = J(2^n, a_n) \cdot \end{aligned}$$

Hence, $a \in (A)^S_{z_0,p,q}$ and $\|a\|_{(A)^S_{z_0,p,q}} \le \|a\|_{(A_0,A_1)_{\alpha(z_0),p}}$. Therefore,

$$(A_0, A_1)_{\alpha(z_0),p} \subset (A)^S_{z_0,p,q} \subset [A]^S_{z_0,p,q} \subset (A_0, A_1)_{\alpha(z_0),p} ,$$

and, therefore, all the equivalences are proved. □

By Corollary 4.12 in [2], we get the following result.

Corollary 6.10

If S' contains the set $S = \{\alpha_n(\cdot) = 2^{n\alpha(\cdot)} : \ n \in \mathbb{Z}\}$, we have that, for every $q \ge 1$,

$$[A]^{S'}_{z_0,p,q} \equiv (A_0, A_1)_{\alpha(z_0),p} \equiv (A)^{S'}_{z_0,p,q} .$$

Using the relationship between the real and complex methods, one can also prove the following:

Corollary 6.11

If $A(\gamma) = [A_0, A_1]_{\alpha(\gamma)}$, a.e $\gamma \in \Gamma$ *and* S *as in Theorem 6.9, then*

$$[A]^S_{z_0,p,q} \equiv (A_0, A_1)_{\alpha(z_0),p} \equiv (A)^S_{z_0,p,q} .$$

Using the same kind of proof as in the case of a compatible pair, we can get similar results in the case of Sparr, Fernández and Cobos-Peetre. However, since for these methods the equivalence between the K and J–methods does not hold, we can only get the corresponding embeddings, unless the family satisfies the equivalence property.

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