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On the Ishikawa iteration process in Hilbert spaces

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Abstract

In this paper, we shall prove that a certain sequence of points which is iteratively defined converges always to a fixed point of some contractive mappings. The results generalize corresponding theorems of Singh and Qihou.

1. Introduction

Let C be a nonempty subset of a Hilbert space X and S,T self–maps on C. An Ishikawa scheme for S and T is defined by

$$\left.\begin{aligned} &x_0 \in C\,, \\ &y_n = (1-\beta_n)x_n + \beta_n S x_n\,, \qquad n \geq 0\,, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n \geq 0\,, \end{aligned}\right\} \tag{1}$$

where the real sequence $\{\alpha_n\}$, $\{\beta_n\}$ satisfy

$$0 \leq \alpha_n \leq \beta_n \leq 1\,,\ \lim_n \alpha_n > 0\,,\ \overline{\lim_n}\, \beta_n < 1 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty\,.$$

In [2], [4], the authors studied and proved the convergence of iterates of a single mapping S to a fixed point of S under some contractive definitions in a Hilbert space.

In this paper, it is proved that for the mapping S and for two mappings S, T which satisfy condition (I) or (II) below if the sequence of Ishikawa iterates converges, then it converges to a fixed point of S and to a common fixed of S and T. These results extend the corresponding results in [2] and [4].

The contractive conditions to be used are the following (see [4]):

(I) $\|Sx - Sy\| < \max\big\{\|x-y\|,\ k\|x-Sx\|, \|y-Sy\|, \|x-Sy\|, \|y-Sx\|\big\}$ for all x, y in C, $x \neq y$ and $0 < k < 1$.

(II) $\|Sx - Ty\| \leq k \max\big\{\|x-y\|, \|x-Sx\|, \|y-Ty\|, \|x-Ty\|, \|y-Sx\|\big\}$, for all $x, y \in C$, $0 < k \leq 1$.

In order to prove our results we need the following lemma:

Lemma 1.1 [2]:

Let a real sequence $\{x_n\}_{n=1}^{\infty}$ satisfy the following condition

$$x_{n+1} \leq \alpha x_n + \beta_n \ , \tag{2}$$

where $x_n \geq 0$, $\beta_n \geq 0$ and $\lim_n \beta_n = 0$, $0 \leq \alpha < 1$. Then, $\lim_n x_n = 0$.

2. Main results

Theorem 2.1

Let H be a Hilbert space and C be a closed convex subset of H. Let $S : C \longrightarrow C$ be a mapping satisfying the condition (I) with nonempty fixed points set, $\{x_n\}$ be a sequence where x_n is defined iteratively for each integer $n \geq 0$ by

$$\begin{aligned} &x_0 \in C\,, \\ &y_n = \big(1-\beta_n\big)x_n + \beta_n S x_n,\ n \geq 0\ , \\ &x_{n+1} = \big(1-\alpha_n\big)x_n + \alpha_n S y_n,\ n \geq 0\,, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the following conditions:

$$(i)\quad 0 \leq \alpha_n \leq \beta_n \leq 1\,, \qquad (ii)\quad \lim_n \beta_n = 0$$

and

$$(iii)\quad \lim_n \alpha_n < 1 - k^2\,.$$

If $\{x_n\}$ converges, then it converges to a fixed point of S.

Proof. Ishikawa [1] has shown that for any points x, y, z in a Hilbert space and any real number λ,

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2 .$$

Let p be a fixed point of S, then we have

$$\begin{aligned}\|x_{n+1}-p\|^2 = \|\big(1-\alpha_n\big)x_n - \alpha_n Sy_n - p\|^2 = \alpha_n\|Sy_n - p\|^2 \\ + \big(1-\alpha_n\big)\|x_n-p\|^2 - \alpha_n\big(1-\alpha_n\big)\|x_n - Sy_n\|^2 . \end{aligned} \tag{3}$$

From (I) we have

$$\begin{aligned}&\|Sy_n - p\|^2 = \|Sy_n - Sp\|^2 < \max\big\{\|y_n-p\|^2,\ k^2\|y_n - Sy_n\|^2, \|p-Sp\|^2 , \\ &\|p-Sy_n\|^2, \|y_n-p\|^2\big\} \leq \max\big\{\|y_n-p\|^2,\ k^2\|y_n-Sy_n\|^2\big\} ,\end{aligned}$$

since for real nonnegative numbers a, b we have

$$\max\big\{a, b\big\} \leq a+b .$$

Hence, we get

$$\|Sy_n - p\|^2 \leq \|y_n-p\|^2 + h\|y_n - Sy_n\|^2 , \quad \text{where} \quad h = k^2 . \tag{4}$$

On the other hand

$$\begin{aligned}\|y_n-p\|^2 = \|\big(1-\beta_n\big)x_n + \beta_n Sx_n - p\|^2 = \beta_n\|Sx_n-p\|^2 + \big(1-\beta_n\big)\|x_n-p\|^2 \\ - \beta_n\big(1-\beta_n\big)\|x_n - Sx_n\|^2 , \end{aligned} \tag{5}$$

and

$$\begin{aligned}\|y_n - Sy_n\|^2 = \beta_n\|Sx_n - Sy_n\|^2 + \big(1-\beta_n\big)\|x_n - Sy_n\|^2 \\ - \beta_n\big(1-\beta_n\big)\|x_n - Sx_n\|^2 . \end{aligned} \tag{6}$$

However,

$$\|Sx_n - p\|^2 \leq \|x_n - p\|^2 + h\|x_n - Sx_n\|^2 . \tag{7}$$

Introducing (7), (6) and (5) into (4), we obtain

$$\begin{aligned}\|Sy_n - p\|^2 \leq \beta_n\|x_n-p\|^2 + h\beta_n\|x_n - Sx_n\|^2 + \big(1-\beta_n\big)\|x_n-p\|^2 \\ - \beta_n\big(1-\beta_n\big)\|x_n - Sx_n\|^2 \\ + h\beta_n\|Sx_n - Sy_n\|^2 + h\big(1-\beta_n\big)\|x_n - Sy_n\|^2 \\ - h\beta_n\big(1-\beta_n\big)\|x_n - Sx_n\|^2 .\end{aligned}$$

Hence,

$$\|Sy_n - p\|^2 \le \|x_n - p\|^2 - \beta_n\big(1-\beta_n - h\beta_n\big)\|x_n - Sx_n\|^2 + h\beta_n\|Sx_n - Sy_n\|^2 + h\big(1-\beta_n\big)\|x_n - Sy_n\|^2 \,. \tag{8}$$

Substituting (8) in (3), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 &+ h\alpha_n\beta_n\|Sx_n - Sy_n\|^2 \\ &- \alpha_n\beta_n(1-\beta_n - h\beta_n)\|x_n - Sx_n\|^2 \\ &- \alpha_n\beta_n(1-\beta_n - h\beta_n)\|x_n - Sx_n\|^2 \\ &- \alpha_n(1-\alpha_n - h + h\beta_n)\|x_n - Sy_n\|^2.\end{aligned}$$

This shows that $\big\{\|x_n - p\|^2\big\}$ is decreasing for all sufficiently large n. Since conditions (ii) and (iii) are satisfied, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_k \|x_{n_k} - Sx_{n_k}\| = 0 \,.$$

Now, we show that $\{Sx_{n_k}\}$ is a Cauchy sequence.

Indeed,

$$\begin{aligned}&\|Sx_{n_k} - Sx_{n_\ell}\| < \max\big\{\|x_{n_k} - x_{n_\ell}\|,\, k\|x_{n_k} - Sx_{n_k}\|, \\ &\|x_{n_\ell} - Sx_{n_\ell}\|,\, \|x_{n_\ell} - Sx_{n_k}\|,\, \|x_{n_k} - Sx_{n_\ell}\|\big\} \\ &\le \max\big\{\|x_{n_k} - Sx_{n_k}\| + \|Sx_{n_k} - Sx_{n_\ell}\| + \|Sx_{n_\ell} - x_{n_\ell}\|\big\}\,.\end{aligned}$$

Taking limit as $k, \ell \longrightarrow \infty$, we have

$$\|Sx_{n_k} - Sx_{n_\ell}\| \longrightarrow 0 \,.$$

Thus $\{Sx_{n_k}\}$ is a Cauchy sequence, hence convergent. Call the limit q. Then $\lim_k Sx_{n_k} = x_{n_k} = q$.

Using (I) we have

$$\|Sq - Sx_{n_k}\| \le \max\big\{\|q - x_{n_k}\| + \|x_{n_k} - Sx_{n_k}\| + \|Sq - Sx_{n_k}\|\big\}\,.$$

Taking limit as $k \longrightarrow \infty$ we have

$$\lim_k \|Sq - Sx_{n_k}\| = 0 \,.$$

Hence, we have

$$\|q - Sq\| \le \|q - x_{n_k}\| + \|x_{n_k} - Sx_{n_k}\| + \|Sx_{n_k} - Sq\| \,.$$

Taking limit as $k \longrightarrow \infty$, we have $\|q - Sq\| = 0$, i.e., $q = Sq$. □

Corollary 2.1 [4].

Let H be a Hilbert space and C be a closed convex subset of H. Let $S : C \longrightarrow C$ be a mapping (I) with nonempty fixed points set, $\{x_n\}$ be a sequence defined by:

$$x_{n+1} = \big(1 - d_n\big)x_n + d_n S x_n,\ n \geq 0\,, \tag{$*$}$$

where $\{d_n\}$ satisfies the following conditions

$$d_0 = 1,\ 0 < d_n \leq 1,\ \sum_{k=0}^{\infty} d_k = \infty \quad \text{and} \quad \overline{\lim}\ d_n < 1 - k^2\,.$$

Then the iteration scheme (*) converges to a fixed point of S.

Theorem 2.2

Let S and T be self–mappings in a bounded closed convex subset C of a Hilbert space H such that (II) holds. Suppose $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the following conditions

$$(a) \quad 0 \leq \beta_n \leq 1\ , \qquad (b) \quad \lim_n \beta_n = 0$$

and

$$(c) \quad \frac{1-k^2}{2} \leq \alpha_n \leq 1 - k^2\,.$$

Then, for each $x_0 \in C$, the sequence of Ishikawa iterates (1) converges to a unique common fixed point of S and T.

Proof. It follows from Rashwan [3] that, S and T have a unique common fixed point $p \in C$. From a known equality (Ishikawa [1]), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 = \|\big(1-\alpha_n\big)x_n + \alpha_n T y_n - p\|^2 = \alpha_n \|T y_n - p\|^2 \\ + \big(1-\alpha_n\big)\|x_n - p\|^2 - \alpha_n\big(1-\alpha_n\big)\|T y_n - p\|^2\,. \end{aligned} \tag{9}$$

From (II), we have

$$\begin{aligned} \|T y_n - p\|^2 = \|Sp - T y_n\|^2 \leq k^2 \max\big\{\|Y_n - p\|^2\,,\ \|Sp - p\|^2\,,\ \|Y_n - T y_n\|^2\,, \\ \|p - T y_n\|^2\,,\ \|y_n - Sp\|^2\big\} \end{aligned}$$

and

$$\|T y_n - p\|^2 \leq k^2\ \max\big\{\|y_n - p\|^2\,,\ \|y_n - T y_n\|^2\,,\ \|T y_n - p\|^2\big\}\,.$$

It is easily seen that

$$\|Ty_n - p\|^2 \le k^2\|y_n - p\|^2 + k^2\|y_n - Ty_n\|^2 .$$

Suppose that $k^2 = h$, hence

$$\|Ty_n - p\|^2 \le h\|y_n - p\|^2 + h\|y_n - Ty_n\|^2. \tag{10}$$

On the other hand

$$\begin{aligned}\|y_n - p\|^2 = \|\beta_n Sx_n + (1-\beta_n)x_n - p\|^2 = \beta_n\|Sx_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\ - \beta_n(1-\beta_n)\|Sx_n - x_n\|^2\end{aligned} \tag{11}$$

and

$$\begin{aligned}\|y_n - Ty_n\|^2 = \|\beta_n Sx_n + (1-\beta_n)x_n - Ty_n\|^2 = \beta_n\|Sx_n - Ty_n\|^2 \\ + (1-\beta_n)\|x_n - Ty_n\|^2 - \beta_n(1-\beta_n)\|Sx_n - x_n\|^2 .\end{aligned} \tag{12}$$

Hence (10) can be written as follows

$$\begin{aligned}\|Ty_n - p\|^2 \le{}& h\beta_n\|Sx_n - p\|^2 + h(1-\beta_n)\|x_n - p\|^2 - h\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 \\ &+ h\beta_n\|Sx_n - Ty_n\|^2 + h(1-\beta_n)\|x_n - Ty_n\|^2 \\ &- h\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 .\end{aligned} \tag{13}$$

However,

$$\|Sx_n - p\|^2 \le h\|x_n - p\|^2 + h\|x_n - Sx_n\|^2 . \tag{14}$$

Substituting (14) in (13), we get

$$\begin{aligned}\|Ty_n - p\|^2 \le{}& h^2\beta_n\|x_n - p\|^2 + h^2\beta_n\|x_n - Sx_n\|^2 + h(1-\beta_n)\|x_n - p\|^2 \\ &- h\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 + h\beta_n\|Sx_n - Ty_n\|^2 \\ &+ h(1-\beta_n)\|x_n - Ty_n\|^2 - h\beta_n(1-\beta_n)\|Sx_n - x_n\|^2 \\ \le{}& h(1-\beta_n + h\beta_n)\|x_n - p\|^2 - h\beta_n(2 - 2\beta_n h)\|Sx_n - x_n\|^2 \\ &+ h(1-\beta_n)\|x_n - Ty_n\|^2 + h\beta_n\|Sx_n - Ty_n\|^2 \\ \le{}& h\|x_n - p\|^2 - h\beta_n(2 - h - 2\beta_n)\ \|Sx_n - x_n\|^2 \\ &+ h(1-\beta_n)\|x_n - Ty_n\|^2 + h\beta_n\|Sx_n - Ty_n\|^2 .\end{aligned} \tag{15}$$

Introducing (15) into (9), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \alpha_n h\|x_n-p\|^2-\alpha_n\beta_n h\big(2-h-2\beta_n\big)\|Sx_n-x_n\|^2\\ &\quad+\alpha_n\big(1-\beta_n\big)h\|x_n-Ty_n\|^2+\alpha_n\beta_n h\|Sx_n-Ty_n\|^2\\ &\quad+\big(1-\alpha_n\big)\|x_n-p\|^2-\alpha_n\big(1-\alpha_n\big)\|x_n-Ty_n\|^2\\ &\le \big[1-\alpha_n(1-h)\big]\|x_n-p\|^2-\alpha_n\beta_n h\big(2-h-2\beta_n\big)\|Sx_n-x_n\|^2\\ &\quad+\alpha_n\beta_n h\|Sx_n-Ty_n\|^2-\alpha_n\big(1-\alpha_n-h+h\beta_n\big)\|x_n-Ty_n\|^2. \quad (16)\end{aligned}$$

Since $\frac{1-h}{2}\le \alpha_n \le 1-h$, $0<h<1$, $\beta_n\ge 0$ and $\lim_n \beta_n=0$, there exists a natural number N, such that for $n>N$,

$$2-h-2\beta_n\ge 0 \quad \text{and} \quad 1-\alpha_n-h+h\beta_n\ge 0\,.$$

Then, for $n\ge N$, we have

$$\|x_{n+1}-p\|^2\le \tilde{h}\|x_n-p\|^2+\alpha_n\beta_n h\|Sx_n-Ty_n\|^2\ ,$$

where $0<\tilde{h}=1-\left(\frac{(1-h)^2}{2}\right)$.

It follows from the boundedness of C that $\|Sx_n-Ty_n\|^2$ is bounded. Thus

$$\lim_n \alpha_n\beta_n h\|Sx_n-Ty_n\|^2=0\,.$$

It follows from Lemma 1.1 that $\lim_n x_n=p$. This completes the proof of the theorem. □

If we put $S=T$ in Theorem 2.2, we get the following corollary.

Corollary 2.2 [2].

Let S be a self–mapping in a bounded closed convex subset C of a Hilbert space such that

$$\|Sx-Sy\|\le k\ \max\big\{\|x-y\|\,,\ \|x-Sx\|\,,\ \|y-Sy\|\,,\ \|x-Sy\|\,,\|y-Sx\|\big\}\,,$$

for all x,y in C, where $0<k\le 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions (a), (b), (c) from Theorem 2.2. Then for each $x_0\in C$, the sequence of Ishikawa iterates converges to a unique point of S.

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