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Bounded variation functions of order k on sequence spaces

ZHAO LINSHENG

Dept. of Basic Sciences, Heilongjiang Commercial College,
Dao Li District, 150076 Harbin, China

ABSTRACT

In this paper, we generalize some results concerning bounded variation functions on sequence spaces.

Some properties of bounded variation functions on sequence spaces were investigated by Wu Congxin [1], [2], [3]. Later on, he and Zhao Linsheng in [4], [5], [6] introduced and discussed bounded variation functions of order 2 on sequence spaces. In this paper, we generalize these results to bounded variation functions of order k on sequence spaces.

Let λ be a real linear sequence space. The Köthe dual λ^* of λ is the real linear sequence space consisting of all real sequences $U = (u_1, u_2, \dots)$ satisfying $\sum_{k=1}^{\infty} |u_k x_k| < \infty$ for all $X = (x_1, x_2, \dots) \in \lambda$. When $\lambda = \lambda^{**}$, we say that λ is a perfect space.

For a real function $x(t)$ defined on $[a,b]$ and $k+1$ different points $t_0, t_1, \dots, t_k \in [a,b]$, we denote

$$Q_k(x; t_0, t_1, \dots, t_k) = \sum_{i=0}^{k} \frac{x(t_i)}{\prod\limits_{\substack{j=0\\ i\neq j}}^{k} (t_i - t_j)}$$

DEFINITION 1 [7, p. 87]. The variation of order k of a function $x(t)$ defined on $[a,b]$ is

$$\mathop{V}_{a\ k}^{b}(x) \triangleq \sup_{\pi} \sum_{i=0}^{n-k} \left| Q_{k-1}(x; t_i, \dots, t_{i+k-1}) - Q_{k-1}(x; t_{i+1}, \dots, t_{i+k}) \right|.$$

Where the "sup" is taken over all partitions $\pi: a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$. When $\overset{b}{\underset{a\ k}{V}}(x) < \infty$, we say that $x(t)$ is a bounded variation function of order k and denote by $x(t) \in V_k[a,b]$.

Lemma 1 [7, p. 88].

For any $x \in V_k[a,b]$ and k different points $a_0, a_1, \ldots, a_{k-1}$ in $[a,b]$, we have

$$|Q_{k-1}(x; t_0, 1_1, \ldots, t_{k-1})| \le |Q_{k-1}(x; a_0, \ldots, a_{k-1})| + 2 \overset{b}{\underset{a\ k}{V}}(x).$$

Lemma 2 [7, p. 79].

$$Q_{r-1}(x; t_1, t_2, \ldots, t_r) - Q_{r-1}(x; t_0, t_1, \ldots, t_{r-1}) = (t_r - t_0) Q_r(x; t_0, \ldots, t_r).$$

Lemma 3

For any $k \ge 3$, we have $(k-1)! \overset{b}{\underset{a\ k}{V}}(x) = \overset{b}{\underset{a\ 2}{V}}(x^{(k-2)})$.

Lemma 4 [11, p. 179].

Let $x: [a,b] \to \mathbb{R}$, $y: [a,b] \to \mathbb{R}$, then

$$\begin{aligned} Q_k(xy; t_0, t_1, \ldots, t_k) = {} & y(x_0) Q_k(x; t_0, t_1, \ldots, t_k) \\ & + Q_1(y; t_0, t_1) Q_{k-1}(x; t_1, \ldots, t_k) \\ & + Q_2(y; t_0, t_1, t_2) Q_{k-2}(x; t_2, t_3, \ldots, t_k) \\ & + \ldots + Q_k(y; t_0, t_1, \ldots, t_k) x(t_k). \end{aligned}$$

Lemma 5

Suppose $x: [a,b] \to \mathbb{R}$ and $a_0, a_1, \ldots, a_r \in [a,b] (a_i \ne a_j$ when $i \ne j)$, then

$$|Q_r(x; a_0, a_1, \ldots, a_r)| \le \frac{1}{\min\limits_{j \ne i} |a_i - a_j|^r} \sum_{i=0}^{r} |x(a_i)| \quad r = 1, 2, \ldots$$

DEFINITION 2 Let $X(t) = (x_1(t), x_2(t), \ldots)$ be an abstract function from $[a,b]$ to a sequence space λ. If for each $U = (u_1, u_2, \ldots) \in \lambda^*$, we have

$$\begin{aligned} \overset{b}{\underset{a\ k}{V}}(X, U) \triangleq \sup_{\pi} \sum_{i=0}^{n-k} \sum_{m=1}^{\infty} \big| u_m [& Q_{k-1}(x; t_i, \ldots, t_{i+k-1}) \\ & - Q_{k-1}(x; t_{i+1}, \ldots, t_{i+k})] \big| < \infty \end{aligned}$$

then $X(t)$ is called a bounded variation function of order k and denoted by $X(t) \in V_k([a,b], \lambda)$.

Theorem 1

$X(t) \in V_k([a,b],\lambda)$ *iff*

1^0 $x_m(t) \in V_k[a,b]$, $m = 1,2,\ldots$ and

2^0 $\sum_{m=1}^{\infty} \{\overset{b}{\underset{a}{V}}{}_k (x_m)\} \in \lambda^{**}$.

Proof. Necessity. 1^0 Pick $U = (\overbrace{0,0,\ldots,0}^{i-1},1,0,0,\ldots) \in \lambda^*$, then from

$$\sup_{\pi} \sum_{i=0}^{n-1} \sum_{m=1}^{\infty} |u_m||t_{i+k} - t_i|\ |Q_k(x_m; t_i, \ldots, t_{i+k})| =$$

$$= \sup_{\pi} \sum_{i=0}^{n-k} |t_{i+k} - t_i|\ |Q_k(x_m; t_i, \ldots, t_{i+k})| < \infty$$

we see $x_s \in V_k[a,b], s = 1,2,\ldots$

Next we turn to 2^0. If 2^0 is not true, then there exist $U^{(0)} = (u_1^{(0)}, u_2^{(0)}, \ldots) \in \lambda^*, u_m^{(0)} \neq 0$, $m = 1,2,\ldots$ and $N_n \geq 1, \varepsilon_n > 0$ such that

$$\sum_{m=1}^{N_n} |u_m^{(0)}| \overset{b}{\underset{a}{V}}{}_k (x_m) = n + \varepsilon_n.$$

Since $x_m \in V_k[a,b], m = 1,2,\ldots,N_n$, there exists a partition π_m: $a = t_0^{(m)} < t_1^{(m)} < \ldots < t_{n_m}^{(m)} = b$ such that

$$\overset{b}{\underset{a}{V}}{}_k (x_m) \leq \sum_{i=0}^{n_m-k} |t_{i+k} - t_i|\ |Q_k(x_m; t_i^{(m)}, \ldots, t_{i+k}^{(m)})| + \frac{\varepsilon_n}{2^{m+1}|u_m^{(0)}|}.$$

Let π be the partition consisting of all points $\{t_i^{(m)}\ i \leq n_m, m \leq N_n\}$: π: $a = s_0^{(N_n)} < s_1^{(N_n)} < \cdots < s_{l(N_n)}^{(N_n)} = b$ then by Theorem 3 in [8], we have

$$\overset{b}{\underset{a}{V}}{}_k (x_m) \leq \sum_{i=1}^{l(N_n)-k} |s_{i+k}^{(N_n)} - s_i^{(N_n)}|\ |Q_k(x_m; s_i^{(N_n)}, \ldots, s_{i+k}^{(N_n)}| + \frac{\varepsilon_n}{2^{m+1}|u_m^{(0)}|}.$$

Hence

$$\sum_{i=0}^{l(N_n)-k} \sum_{m=1}^{\infty} |u_m^{(0)}|\, |s_{i+k}^{(N_n)} - s_i^{(N_n)}|\, |Q_k(x_m; s_i^{(N_n)}, \dots, s_{i+k}^{(N_n)})|$$

$$\geq \sum_{i=0}^{l(N_n)-k} \sum_{m=1}^{N_n} |u_m^{(0)}|\, |s_{i+k}^{(N_n)} - s_i^{(N_n)}|\, |Q_k(x_m; s_i^{(N_n)}, \dots, s_{i+k}^{(N_n)})|$$

$$\sum_{m=1}^{N_n} \sum_{i=0}^{l(N_n)-k} |u_m^{(0)}|\, |s_{i+k}^{(N_n)} - s_i^{(N_n)}|\, |Q_k(x_m; s_i^{(N_n)}, \dots, s_{i+k}^{(N_n)})|$$

$$\geq \sum_{m=1}^{N_n} |u_m^{(0)}| \overset{b}{\underset{a}{V}} (x_m) - \sum_{m=1}^{N_n} \frac{\varepsilon_n}{2^{m+1}} \geq n$$

contradicting that $X(t) \in V_k([a,b], \lambda)$.

Sufficiency. Notice that

$$\sup_{\pi} \sum_{i=0}^{n-k} \sum_{m=1}^{\infty} |u_m|\, |t_{i+k} - t_i|\, |Q_k(x_m; t_i, \dots, t_{i+k})|$$

$$\leq \sum_{m=1}^{\infty} \sup_{\pi} \sum_{i=0}^{n-k} |u_m|\, |t_{i+k} - t_i|\, |Q_k(x_m; t_i, \dots, t_{i+k})|$$

$$= \sum_{m=1}^{\infty} |u_n| \overset{b}{\underset{a\ k}{V}} (x_m) < \infty$$

we find that $X(t)$ is bounded variation of order k. □

Theorem 2

$V_k([a,b], \lambda) \subset V_r([a,b], \lambda)$ for all $1 \leq r < k$.

Proof. It is sufficient to consider the case $r = k - 1$. By Theorem 1, $X(t) = \{x_m(t)\}_{m=1}^{\infty} \in V_k([a,b], \lambda)$ implies

$$x_m(t) \in V_k[a,b] \quad \text{and} \quad \{\overset{b}{\underset{a\ k}{V}} (x_m)\} \in \lambda^{**}.$$

For k different points $a_0 < a_1 < \cdots < a_{k-1}$ in $[a,b]$, by Lemma 1, we have

$$|Q_{k-1}(x_m; t_i, \dots, t_{i+k-1})| \leq |Q_{k-1}(x_m; a_0, \dots, a_{k-1})| + 2 \overset{b}{\underset{a\ k}{V}} (x_m).$$

Hence

$$\sup_{\pi} \sum_{i=0}^{n-k+1} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k-1} - t_i|\ |Q_{k-1}(x_m; t_i, \ldots, t_{i+k-1})|$$

$$\leq \sup_{\pi} \sum_{i=0}^{n-k+1} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k-1} - t_i|\ (|Q_{k-1}(x_m; a_0, a_1, \ldots, a_{k-1})| + 2 \mathop{V}_{a\ k}^{b} (x_m))$$

$$\leq \sup_{\pi} \sum_{i=0}^{n-k+1} |t_{i+k-1} - t_i| \sum_{m=1}^{\infty} |u_m|\ \ (|Q_{k-1}(x_m; a_0, a_1, \ldots, a_{k-1})| + 2 \mathop{V}_{a\ k}^{b} (x_m))$$

$$\leq k(b-a)\Big(\frac{1}{\min\limits_{i \neq j, i,j=0,1,r,\ldots,k-1} |a_i - a_j|^{k-1}} \sum_{m=1}^{\infty} |u_m| \sum_{i=0}^{k-1} |x_m(a_i)| + 2 \sum_{m=1}^{\infty} |u_m \mathop{V}_{a\ k}^{b} (x_m)|\Big) < \infty .$$

It follows that $X(t) \in V_{k-1}([a, b], \lambda)$. □

Corollary 1

If $\{\mathop{V}_{a\ k}^{b} (x_m)\} \in \lambda^{**}$, *then* $\{\mathop{V}_{a\ r}^{b} (x_m)\} \in \lambda^{**}$ *for all* $1 \leq r < k$.

Theorem 3

Suppose $k \geq 3$, *then* $X(t) \in V_k([a, b], \lambda)$ *iff* $X'(t) \equiv \{x'_m(t)\} \in V_{k-1}([a, b], \lambda)$.

Proof. Necessity. By Theorem 1, $x'_m(t) \in V_{k-1}[a, b]$. Moreover, by Lemma 3, when $k = 3$, from $\{\mathop{V}_{a\ 3}^{b} (x_m)\} \in \lambda^{**}$, we have $\{\mathop{V}_{a\ 2}^{b} (x'_m)\} \in \lambda^{**}$, and when $k > 3$, we have

$$(k-2)| \mathop{V}_{a\ k-1}^{b} (x'_m) = \mathop{V}_{a\ 2}^{b} (x_m^{(k-2)}).$$

Hence, $\{\mathop{V}_{a\ k-1}^{b} (x'_m)\} \in \lambda^{**}$ and the conclusion follows from Theorem 1.

Sufficiency. By Theorem 1 in [9], we have $x_m(t) \in V_k[a,b]$. Observing that $k = 3$ implies $2! \overset{b}{\underset{a\ 3}{V}} (x_m) = \overset{b}{\underset{a\ 2}{V}} (x'_m)$ and $k > 3$ implies

$$(k-2)! \overset{b}{\underset{a\ k-1}{V}} (x'_m) = \overset{b}{\underset{a\ 2}{V}} (x_m^{(k-2)}) = (k-1)! \overset{b}{\underset{a\ k}{V}} (x_m)$$

we find that $\{\overset{b}{\underset{a\ k}{V}} (x_m)\} \in \lambda^*(k \geq 3)$ and that Theorem 1 implies $X(t) \in V_k[a,b],\lambda)$. □

Theorem 1 and Theorem 3 imply

Corollary 2

Let $k \geq 3$ then the following are equivalent

1^0 $X(t) \in V_k([a,b],\lambda)$;

2^0 $\forall 2 \leq r < k$, $X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a,b],\lambda)$;

3^0 $\exists 2 \leq r < k$, *such that* $X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a,b],\lambda)$;

4^0 $\forall 2 \leq r < k$, *we have*

(i) $x_m^{(k-r)}(t) \in V_r[a,b], m = 1,2,\ldots$

(ii) $\{\overset{b}{\underset{a\ r}{V}} (x_m^{(k-r)})\} \in \lambda^{**}$

5^0 $\exists 2 \leq r < k$ *such that*

(i) $x_m^{(k-r)}(t) \in V_r[a,b], m = 1,2,\ldots$

(ii) $\{\overset{b}{\underset{a\ r}{V}} (x_m^{(k-r)}\} \in \lambda^{**}$.

Theorem 4

Assume that λ is a perfect space, then $X(t) \in V_k([a,b],\lambda)$ iff there exist convex functions $X^{(i)}(t) \in V_k([a,b],\lambda)$ $(i = 1,2)$ of order k such that

$$X(t) = X^{(1)}(t) - X^{(2)}(t) \quad (t \in [a,b])$$

($X(t)$ is called a convex function of order k, if for each natural number $m, x_m(t)$ is a usual convex function of order k and $x(t)$ is called a usual convex function of order k, if for any partition $\pi: a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, we have $Q_k(x;t_0,t_1,\ldots,t_k) \geq 0$).

Proof. The sufficiency is obvious. Now we prove the necessity. The necessity is already known for $k = 1$ and $k = 2$. Suppose that the condition is necessary for $k = m - 1$, we investigate the case $k = m$. Since by Theorem 3, $X'(t) \in V_{m-1}([a,b],\lambda)$ by the assumption, there exist convex functions $Y^{(i)}(t) \in V_{m-1}([a,b],\lambda)$ of order $m-1$ $(i = 1,2)$ such that $X'(t) = Y^{(1)}(t) - Y^{(2)}(t)$. For any $c \in (a,b)$, we have

$$X(t) = \int_0^t X'(s)ds - X(c) = \int_0^t Y^{(1)}(s)ds - \int_0^t Y^{(2)}(s)ds - X(c).$$

Set

$$X^{(1)}(t) = \int_0^t Y^{(1)}(s)ds, \quad X^{(2)}(t) = \int_0^t Y^{(2)}(s)ds + X(c)$$

then by Theorem 13 in [8], $X^{(i)}(t)$ is convex of order $m, i = 1,2$. But $(X^{(i)}(t))' \in V_{m-1}([a,b],\lambda)$, by Theorem 3, $X^{(i)}(t) \in V_m([a,b],\lambda)$ $i = 1,2$. Clearly, $X(t) = X^{(1)}(t) - X^{(2)}(t)$, and $X^{(1)}(t) \in \lambda$ (and thus $X^{(2)}(t) \in \lambda$) can be deduced as follows

$$\begin{aligned}|x_m^{(1)}(t)| \le \int_a^b |y_m^{(1)}(s)|ds \le \int_a^b |y_m^{(1)}(a)| + \overset{b}{\underset{a}{V}}(y_m^{(1)})ds \\ + (b-a)\big(|y_m^{(1)}(a)| + \overset{b}{\underset{a}{V}}(y_m^{(1)})\big)\end{aligned}$$

therefore, $\{X_m^{(1)}(t)\} \in \lambda^{**} = \lambda$. □

Theorem 5

Let λ be perfect, then $X(t), Y(t) \in V_k([a,b],\lambda)$ implies $X(t)Y(t) \in V_k([a,b],\lambda)$ iff for any $Z(t) \in V_k([a,b]),\lambda), U \in \lambda^*$ and $c \in [a,b]$, we have

$$\Big\{|u_m|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}}(Z_m) + \overset{b}{\underset{a\ 2}{V}}(Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}}(Z_m)\big)\Big\} \in \lambda^*.$$

Proof. Sufficiency. Let $X(t) \in V_k([a,b],\lambda), Y(t) \in V_k([a,b],\lambda)$, by Lemma 4,

$$\sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i|\ |Q_k(x_m y_m; t_i, t_{i+1}, \dots, t_{i+k})|$$
$$= \sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i|\ |y_m(t_i) Q_k(x_m; t_i, \dots, t_{i+k})$$
$$+ Q_1(y_m; t_i, t_{i+1}) Q_{i-1}(x_m; t_{i+1}, \dots, t_{i+k})$$
$$+ \dots + Q_k(y_m; t_i, \dots, t_{i+k}) x_m(t_{i+k})|$$
$$\le \sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| \big[|y_m(t_i) Q_k(x_m; t_i, \dots, t_{i+k})$$
$$+ Q_1(y_m; t_i, t_{i+1}) Q_{i-1}(x_m; t_{i+1}, \dots, t_{i+k})$$
$$+ \dots + Q_k(y_m; t_i, \dots, t_{i+k}) x_m(t_{i+k})| \big]$$
$$\le \sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| \big(|y_m(a_0)|$$
$$+ 2 \overset{b}{\underset{a}{V}} (y_n) \big) |Q_k(x_m; t_i, \dots, t_{i+k})|$$
$$+ \sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| (|Q_1(y_m; a_0, a_1)|$$
$$+ \overset{b}{\underset{a\ 2}{V}} (y_m)(|Q_{k-1}(x_m; a_0, \dots, a_{t-1})| + \overset{b}{\underset{a\ k}{V}} (x_m)$$
$$+ \dots + \sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i|\ |Q_k(y_m; t_i, \dots, t_{i+k})| (|x_m(a_0)| + 2 \overset{b}{\underset{a}{V}} (x_m)),$$

where $\{a_i\}_{i=0}^{k-1}$ are different points in (a, b). For the first term, we have

$$\sup_\pi \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| \big(|y_m(a_0)| + 2 \overset{b}{\underset{a}{V}} (y_m) \big) |Q_k(x_m; t_i, \dots, t_{i+k})|$$
$$\le \sum_{m=1}^{\infty} |u_m| c |y_m(a_0)| + 2 \overset{b}{\underset{a}{V}} (y_m)) \sup_\pi \sum_{i=0}^{n-k} |t_{i+k} - t_i|\ |Q_k(x_m; t_1, \dots, t_{i+k})|$$
$$= \sum_{m=1}^{\infty} |u_m| \big(|y_m(a_0)| + 2 \overset{b}{\underset{a}{V}} (y_m) \big) \overset{b}{\underset{a\ k}{V}} (x_m) < \infty$$

(note that $\{|u_m|(|y_m(a_0)| + 2\, V_a^b\ (y_m))\} \in \lambda^*, \{V_{a\ k}^b\ (x_m)\} \in \lambda^{**}$).

Similarly, for the last term, we have

$$\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\;|t_{i+k}-t_i|\;|Q_k(y_m;t_i,\ldots,t_{i+k})|\big(|x_m(a_0)|+2\mathop{V}_a^b(x_m)\big)<\infty.$$

Now, we show that the other terms are also bounded. Without loss of generality, we only consider the term

$$\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\;|t_{i+k}-t_i|(|Q_1((y_m;a_0,a_1)| + 2\mathop{V}_{a\ 2}^{b}(y_m))(|Q_{k-1}(x_m;a_0,\ldots,a_{i-1})|+2\mathop{V}_{a\ k}^{b}(x_m)).$$

By Lemma 5,

$$\begin{aligned}&\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\;|t_{i+k}-t_i|(|Q_1((y_m;a_0,a_1)|\\&\quad+2\mathop{V}_{a\ 2}^{b}(y_m))(|Q_{k-1}(x_m;a_0,\ldots,a_{i-1})|+2\mathop{V}_{a\ k}^{b}(x_m))\\&\le k(b-a)\sum_{m=1}^{\infty}|u_n|(|Q_1(y_m;a_0,a_1)|\\&\quad+2\mathop{V}_{a\ 2}^{b}(y_m))(|Q_{k-1}(x_m;a_0,\ldots,a_{k-1})|+2\mathop{V}_{a\ k}^{b}(x_m))<\infty.\end{aligned}$$

Thus, $X(t)Y(t)\in V_k([a,b],\lambda)$.

Necessity. By Theorem 2.6 in [7], the condition is necessary for $k=1$. Now, suppose $k\ge 2$. Define

$$\left.\begin{aligned}x_m(t)&=\big(|Z_m(c)|+\mathop{V}_a^b(Z_m)+\ldots+\mathop{V}_{a\ k}^{b}(Z_m)\big)t^{k-1}\\y_m(t)&=|x_m^{(0)}|t\end{aligned}\right.\qquad(a\le t\le b)$$

then from

$$x_m^{(k-1)}(t)=(k-1)!\big(|Z_m(c)|+\mathop{V}_a^b(Z_m)+\cdots+\mathop{V}_{a\ k}^{b}(Z_m)\big)$$

$$y_m^{(k-1)}(t)=\begin{cases}|x_m^{(0)}|, & k=2\\ 0, & k>2\end{cases}$$

we have

$$x_m^{(k-2)} \in V_2[a,b], \quad y_m^{(k-2)} \in V_2[a,b].$$

Hence, Theorem 1 in [9] implies $x_m, y_m \in V_k[a,b]$, and Proposition 3.4 in [7] claims $\overset{b}{\underset{a\ k}{V}}(x_m) = \overset{b}{\underset{a\ k}{V}}(y_m) = 0$. Thus, $\{\overset{b}{\underset{a\ k}{V}}(x_m)\} \in \lambda^{**}$ and $\{\overset{b}{\underset{a\ k}{V}}(y_m)\} \in \lambda^{**}$, and so, by Theorem 1,

$$X(t) = \{x_m(t)\} \in V_k([a,b],\lambda)$$

$$y(t) = \{y_m(t)\} \in V_k([a,b],\lambda).$$

Therefore $X(t)y(t) \in V_k([a,b]),\lambda)$.

For any $t_0 \neq t_1 \neq \cdots \neq t_k$ in (a,b), by Proposition 3.5 in [7] p. 82,

$$Q_k(x_m y_m; t_0, \ldots, t_k) = |x_m^{(0)}|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}}(Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}}(Z_m)|\big)$$

and from

$$\infty > \overset{b}{\underset{a\ k}{V}}(XY;U) \geq \sum_{m=1}^{\infty} |u_m|\ |t_k - t_0|\ |Q_k(x_m y_m; t_0, \ldots, t_k)|$$

$$= |t_k - t_0| \sum_{m=1}^{\infty} |u_m|\big(|Z_m(c) + \overset{b}{\underset{a}{V}}(Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}}(Z_k)|\big)\ |x_m^{(0)}|$$

we find

$$\big\{|u_m|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}}(Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}}(Z_m)\big)\big\} \in \lambda^*. \ \square$$

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