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P-convexity property in Musielak-Orlicz sequence spaces

Ye Yining and Huang Yafeng

P.O. Box 610, Math. Dept., Harbin Univ. Sci. Tech.,
Harbin, Heilongjiang, 150080, P.R. China

Abstract

We prove that in the Musielak-Orlicz sequence spaces equipped with the Luxemburg norm, P-convexity coincides with reflexivity.

In 1970, Kottman [1] introduced an important geometric property-P-convexity in order to describe a reflexive Banach space. We say that a Banach space $(X, \|\cdot\|)$ is P-convex if X is $P(n\varepsilon)$-convex for some positive integer n and a real number $\varepsilon > 0$, i.e. for any $x_1, x_2, \dots, x_n$ in the unit sphere of X, $\min_{i\neq j} \|x_i - x_j\| < 2-\varepsilon$ for some n and $\varepsilon > 0$. Moreover Kottman proved that any P-convex Banach space is reflexive. After P-convexity property was introduced, many people tried to give a distinct relation between P-convexity and reflexivity. But there are a lot of differences between them in a Banach space.

In 1978 Sastry and Naidu [2] introduced a new geometric property, O-convexity intermediate between P-convexity and reflexivity, and proved that P-convexity implies O-convexity and O-convexity implies reflexivity.

In 1984, D. Amir and C. Franhetti [3] gave two geometric properties, O-convexity and H-convexity by the preceding results and proved O-convexity implies Q-convexity, Q-convexity implies reflexivity and H-convexity implies B-convexity and these convexities do not coincide with each other.

In 1988, Yeyining, Hemiaohong and Ryszard Pluciennik [4] proved that in Orlicz spaces P-convexity coincides with reflexivity, and reflexivity coincides with $P(3,\varepsilon)$-convexity for some $\varepsilon > 0$.

In this paper we prove that in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm P-convexity coincides with reflexivity.

0. Introduction

Let X be a Banach space equipped with the norm $\|\cdot\|$ and $S(X)$ be the unit sphere of the space X, i.e. $S(X) = \{x \in X: \|x\| = 1\}$. Denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $\varphi = (\varphi_n)$ be a sequence of Young functions, i.e. for every $n \in \mathbb{N}$, $\varphi_n(\cdot): \mathbb{R} \to [0, \infty]$ is a convex, $\varphi_n(0) = 0$, $\lim\limits_{u \to \infty} \varphi_n(u) = \infty$, $\varphi_n(\cdot)$ is continuous at 0 and not identically equal to the zero function, and there exists a real number u_0, s.t. $\varphi_n(u_0) < \infty$. We define a modular on the family of all sequences $x = (x_n)$ of real numbers by the following formula

$$I_\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x_n).$$

The linear set

$$l_\varphi = \{x = (x_n): \exists a > 0, I_\varphi(ax) < \infty\}$$

equipped with so - called Luxemburg norm

$$\|x\| = \inf\{k > 0: I_\varphi(k^{-1}x) \leq 1\}$$

is said to be a Musielak-Orlicz sequence space.

We say that $\varphi = (\varphi_n)$ satisfies the δ_2-condition if there are constants a, k, and a sequence (c_n) of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad \varphi_a(2u) \leq k\varphi_n(u) + c_n$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_a(u) \leq a$.

The complementary function of Young function $\varphi = (\varphi_n)$ is defined by

$$\varphi_n^*(v) = \sup_{u \geq 0}\{u|v| - \varphi_n(u)\}, \quad \text{for all } n \in \mathbb{N}.$$

A Musielak-Orlicz sequence space l_φ is reflexive if and only if $\varphi = (\varphi_n)$ and $\varphi^* = (\varphi_n^*)$ satisfy the δ_2-condition. Let $a_n = \sup\{u > 0: \varphi_n(u) \leq 1\}$ for all $n \in \mathbb{N}$.

1. Auxiliary lemmas

Lemma 1

Let $\varphi = (\varphi_n)$ satisfy the δ_2-condition, then

(i) if $A = \inf_n \varphi_n(a_n)$, then $A > 0$,

(ii) for any $l_1 > 1, a_1 > 0$, there are $k_1 > 1$ and a sequence $(c_n^{(1)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(1)} < \infty \quad \text{and} \quad \varphi_n(lu) \leq k\varphi_n(u) + c_n^{(1)}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(l_1 u) \leq a_1$,

(iii) for any $k_1 > 1, l_2 > 1, a_2 > 0$, there are $\sigma \in (0, l_2 - 1)$ and a sequence $(c_n^{(2)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(2)} < \infty \quad \text{and} \quad \varphi_n\big((1+\delta)u\big) \leq k_2\varphi_n(u) + c_n^{(2)}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(l_2 u) \leq a_2$.

Proof. (i) Obviously $A \geq 0$, so it is enough to prove $A \neq 0$. Assume that $A = 0$. Then for any $a > 0$ there is $n_0 \in \mathbb{N}$, such that $\varphi_{n_0}(a_{n_0}) < a$. It is easy to see that $a_{n_0} \neq 0$ by the definition of $\varphi_n(u)$. We may assume without loss of generality that $a < 1$. Then $\varphi_{n_0}(a_{n_0}) < 1$ implies $\varphi_{n_0}(2a_{n_0}) = \infty$ because $\varphi_n(u)$ is a convex function and so it has the only discontinuous point u_0, such that $\varphi_{n_0}(u_0 - 0) < \infty$ and $\varphi_{n_0}(u_0+0) = \infty$. By the definition of a_{n_0} and $\varphi_{n_0}(a_{n_0}) < 1$ we may deduce that a_{n_0} is the discontinuous point of $\varphi_{n_0}(u)$, so $\varphi_{n_0}(2a_{n_0}) = \infty$. But this contradicts the δ_2-condition and so $A > 0$.

(ii) Let a positive integer α satisfy $2^{\alpha-1} < l_1 < 2^{\alpha}$.
Since $\varphi = (\varphi_n)$ satisfies the δ_2-condition, there are constants $k > 0, a > 0$ and a sequence (c_n) of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n < \infty \quad \text{and} \quad \varphi_n(2u) \leq k\varphi_n(u) + c_n$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(u) \leq a$. When $\varphi_n(l_1 u) \leq a, \varphi_n(2^{\alpha-1}u) \leq \varphi_n(l_1 u) \leq a$, then

$$\begin{aligned}\varphi_n(l_1 u) &\leq \varphi_n(2^{\alpha}u) \\ &\leq k\varphi_n(2^{\alpha-1}u) + c_n \leq \ldots \leq k^{\alpha}\varphi_n(u) + (k^{\alpha-1} + \ldots + k + 1)c_n.\end{aligned}$$

Let $c_n^{(1)} = (k^{\alpha-1} + \cdots + k + 1)c_n$. Obviously $\sum_{n=1}^{\infty} c_n^{(1)} < \infty$. Then $\varphi_n(l_1 u) \leq k^\alpha \varphi_n(u) + c_n^{(1)}$ with $\varphi_n(l_1 u) \leq a$.

If $a_1 \leq a$, it is enough to put $k_1 = k^\alpha$. Let $a < \varphi_n(l_1 u) \leq a_1$ and $\varphi_n(l_1' u) = a$. Then $l_1' < l$. Hence

$$\begin{aligned}\varphi_n(l_1, u) &\leq a_1 = a_1 a^{-1} a = a_1 a^{-1} \varphi_n(l_1' u) \\ &= a_1 a^{-1} \varphi_n(l_1 l_2^{-1} l_1' u) \leq a_1 a^{-1}[k^\alpha \varphi_n(l_1^{-1} l_1' u) + c_n^{(1)}] \\ &\leq a_1 a^{-1} k^\alpha \varphi_n(u) + a_1 a^{-1} c_n^{(1)}.\end{aligned}$$

Replace $a_1 a^{-1} k^\alpha$ by $k_1, a_1 a^{-1} c_n^{(1)}$ by $c_n^{(1)}$, then $\sum_{n=1}^{\infty} c_n^{(1)} < \infty$. So $\varphi_n(l_1 u) \leq k_1 \varphi_n(u) + c_n^{(1)}$ when $\varphi_n(l_1 u) \leq a_1$.

(iii) For $l_2 > 1, a_2 > 0$, by (ii) there are $k_1 > 1$ and a sequence $(c_n^{(1)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(1)} < \infty \quad \text{and} \quad \varphi_n(l_2 u) \leq k_1 \varphi_n(u) + c_n^{(1)}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(l_2 u) \leq a_2$. Take σ satisfying

$$\sigma < \min\left\{l_2 - 1, [(k_2 - 1)/(k_1 - 1)](l_2 - 1)\right\}.$$

Because $\varphi_n(u)$ is convex, when $\varphi_n(l_2 u) \leq a_2$ it follows that

$$\begin{aligned}\varphi_n((1+\sigma)u) &= \varphi_n\Big(\frac{(l_2 - 1)(l + \sigma)}{l_2 - 1} u\Big) \\ &= \varphi_n\Big(\frac{\sigma}{l_2 - 1} l_2 u + \frac{l_2 - 1 - \sigma}{l_2 - 1} u\Big) \\ &\leq \frac{\sigma}{l_2 - 1} \varphi_n(l_2 u) + \frac{l_2 - 1 - \sigma}{l_2 - 1} \varphi_n(u) \\ &\leq \frac{k_1 \sigma}{l_2 - 1} \varphi_n(u) + \frac{l_2 - 1 - \sigma}{l_2 - 1} \varphi_n(u) + \frac{\sigma}{l_2 - 1} c_n^{(1)} \\ &= \Big[\frac{(k_1 - 1)\sigma}{l_2 - 1} + 1\Big] \varphi_n(u) + \frac{\sigma}{l_2 - 1} c_n^{(1)} \\ &\leq \Big[\frac{(k_1 - 1)(k_2 - 1)}{(l_2 - 1)(k_1 - 1)}(l_2 - 1) + 1\Big] \varphi_n(u) \\ &\quad + \frac{c_n^{(1)}(k_2 - 1)}{(l_2 - 1)(k_1 - 1)}(l_2 - 1) \\ &= k_2 \varphi_n(u) + \frac{k_2 - 1}{k_1 - 1} c_n^{(1)}.\end{aligned}$$

Let $c_n^{(2)} = [(k_2 - 1)/(k_1 - 1)]c_n^{(1)}$, which completes the proof of (iii). □

Lemma 2

If $\varphi = (\varphi_n)$ and $\varphi^* = (\varphi_n^*)$ satisfy the δ_2-condition, then for any $l_3 > 1$, $b > 1$ there are $k_3 > 1$ and a sequence $(c_n^{(3)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(3)} < \infty \quad \text{and} \quad \varphi_n^*(v) < \frac{1}{l_3 k_3}\varphi_n^*(l_3 v) + c_n^{(3)},$$

for all $n \in \mathbb{N}$ and $v \in \mathbb{R}$ with $\varphi_n^*(v) \le b$.

Proof. First we prove when $\varphi_n^*(v) \le b$, there is $a > 0$ such that $\varphi_n(u) \le a$ for all $n \in \mathbb{N}$ where $v = p_n(u)$.

Otherwise, there is a sequence $\{u_k\}_{k=1}^{\infty}$ of real numbers such that $\varphi_{n_k}(u_k) \to \infty$ as $k \to \infty$, while $\varphi_{n_k}^*(v) \le b$.

Notice that for some $l_3 > 1$, there is $b' > 0$, such that $\varphi_n^*(l_3 v) \le b'$ for all $n \in \mathbb{N}$. It is enough to put $b' = 2l_3 b$. If $\varphi_n^*(l_3 v) > 2l_3 b$, Lemma 2 obviously holds.

By Lemma 1, there is $\sigma \in (0, l_3 - 1)$ such that $\varphi_{n_k}^*((1+\sigma)v_k) \le k_2 \varphi_{n_k}^*(v_k) + c_k$ for all $n \in \mathbb{N}$ with

$$\varphi_{n_k}^*(l_3 v) \le b', \quad \text{where} \quad k_2 > 1, \sum_{n=1}^{\infty} c_k < \infty.$$

Let $b_1 = k_2 b + \max_k c_k$. Then $\varphi_{n_k}^*((1+\sigma)v_k) \le b_1$ for all $k \in \mathbb{N}$.

On the other hand, when $v_k = p_{n_k}(u_k), \varphi_{n_k}^*(v_k) = |u_k v_k| - \varphi_{n_k}(u_k) > 0$, and $\varphi_{n_k}(u_k) \to \infty$ as $k \to \infty$, i.e. there is $k_0 \in \mathbb{N}$ such that $\varphi_{n_k}(u_k) > b_1\sigma^{-1}$ with $k > k_0$. So, when $k > k_0$, we have

$$\begin{aligned}\varphi_{n_k}^*\big((1+\sigma)v_k\big) &= \sup_{u \ge 0}\big\{(1+\sigma)|v_k|u - \varphi_{n_k}(u)\big\} \\ &\ge (1+\sigma)|v_k u_k| - \varphi_{n_k}(u_k) \\ &\ge (1+\sigma)\varphi_{n_k}(u_k) - \varphi_{n_k}(u_k) = \sigma\varphi_{n_k}(u_k) > b_1.\end{aligned}$$

This contradicts the inequality $\varphi_{n_k}^*((1+\sigma)v_k) \le b_1$.

Therefore, there is $a > 0$ such that $\varphi_n(u) \le a$ for all $n \in \mathbb{N}$ with $\varphi_n^*(v) \le b$. Hence by $\varphi_n^*(l_3 v) \le b'$ there is $a' > 0$ such that $\varphi_n(l_3 u) \le a'$ for all $n \in \mathbb{N}$.

By Lemma 1 (iii) for $k_2 = l_3, l_2 = l_3, a_2 = a'$, there are $\varepsilon \in (0, l_3 - 1)$ and a sequence $(c_n^{(2)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(2)} < \infty \quad \text{and} \quad \varphi_n\big((1+\varepsilon)u\big) \le l_3\varphi_n(u) + c_n^{(2)}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(l_3 u) \le a'$. Then

$$\begin{aligned}
\varphi_n^*(v) &= \sup\left\{u|v| - \varphi_n(u) : u \ge 0, \varphi_n^*(l_3 v) \le b'\right\} \\
&\le \sup\left\{u|v| - \varphi_n(u) : \varphi_n(l_3 u) \le a'\right\} \\
&\le \sup_{u \ge 0}\left\{u|v| - \frac{\varphi_n(u+\varepsilon)u) - c_n^{(2)}}{l_3}\right\} \\
&= \frac{1}{l_3}\sup_{u\ge 0}\left\{\frac{l_3|v|}{1+\varepsilon}(1+\varepsilon)u - \varphi_n\big((1+\varepsilon)u\big)\right\} + \frac{c_n^{(2)}}{l_3} \\
&= \frac{1}{l_3}\varphi_n^*\left(\frac{l_3 v}{1+\varepsilon}\right) + \frac{c_n^{(2)}}{l_3} \\
&< \frac{1}{l_3(1+\varepsilon)}\varphi_n^*(l_3 v) + \frac{c_n^{(2)}}{l_3}.
\end{aligned}$$

Let $k_3 = 1+\varepsilon, c_n^{(3)} = c_n^{(2)}/l_3$, which completes the proof of Lemma 2. □

Lemma 3

If $\varphi = (\varphi_n)$ and $\varphi^* = (\varphi_n^*)$ satisfy the δ_2-condition, then there is a sequence (c_n) of non-negative real numbers such that $\sum\limits_{n=1}^{\infty} \varphi_n(c_n) < \infty$, and if

$$d_n = \sup\left\{\alpha(u,n) : \varphi_n\left(\frac{u}{\alpha(u,n)}\right) \ge \frac{1}{2}\varphi_n(u), c_n \le |u| \le a_n\right\},\ n = 1, 2, \dots$$

$$d_1 = \lim_{m\to\infty}\sup_{n>m} d_n,$$

then $d_1 < 2$.

Proof. Let $l_3 = 2, b = 1$ in Lemma 2. Then there are $k_3 > 1$ and a sequence $(c_n^{(3)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(3)} < \infty \quad \text{and} \quad \varphi_n(u) \le \frac{1}{2k_3}\varphi_n(2u) + c_n^{(3)} \tag{1}$$

for all n and u with $\varphi_n(u) \le 1$.

In Lemma 1 (iii) let $k_2 = (k_3+1)/2, l_2 = 2, a_2 = 1$. There are $\varepsilon \in (0,1)$ and a sequence (β_n) of positive numbers such that $\sum\limits_{n=1}^{\infty} \beta_n < \infty$, and when $\varphi_n(2u) \le 1$,

$$\varphi_n\big((1+\varepsilon)u\big) < \frac{1}{2}(k_3+1)\varphi_n(u) + \beta_n. \tag{2}$$

Let

$$c'_n = \frac{2k_3(k_3+1)}{k_3-1}c_n^{(3)} + \frac{4k_3}{k_3-1}\beta_n.$$

Obviously $\sum_{n=1}^{\infty} c'_n < \infty$.

Since $A = \inf_n \varphi_n(a_n) > 0$ is true by Lemma 1 (i), so there is $n_0 \in \mathbb{N}$ such that $c'_n < A$ for $n > n_0$. We define a sequence (c_n) by

$$c_n = \begin{cases} 0 & \text{when } n \le n_0 \\ \varphi_n^{-1}(c'_n) & \text{when } n > n_0. \end{cases}$$

Then $\sum_{n=1}^{\infty} \varphi_n(c_n) \le \sum_{n=1}^{\infty} c'_n < \infty$.

We will show the sequence (c_n) satisfies Lemma 3.

Obviously $d_1 \le 2$. If $d_1 = 2$, for $n > n_0$ there are subsequence $\{u_n\}_{n>n_0}$ and $\{\alpha(u_n,n)\}_{n>n_0}$ (let the subsequence be $\{u_n\}$ and $\{\alpha(u_n,n)\}$) such that

$$\varphi_n\Big(\frac{u_0}{\alpha(u_n,n)}\Big) \ge \frac{1}{2}\varphi_n(u_n), \quad c_n \le |u_n| < a_n \tag{3}$$

and $\alpha(u_n,n) \to 2$ as $n \to \infty$.

So there is $n_1 \in \mathbb{N}$, such that $2/\alpha(u_n,n) < 1+\varepsilon$ for $n > n_1$.
Let $\alpha_n = \alpha(u_n,n)$. By formula (2) it follows that

$$\varphi_n\Big(\frac{u_n}{\alpha_n}\Big) \le \varphi_n\Big((1+\varepsilon)\frac{u_n}{2}\Big) < \frac{k_3+1}{2}\varphi_n\Big(\frac{u_n}{2}\Big) + \beta_n.$$

By (1), we get

$$\varphi_n\Big(\frac{u_n}{\alpha_n}\Big) < \frac{k_3+1}{2}\Big[\frac{1}{2k_3}\varphi_n(u_n) + c_n^{(3)}\Big] + \beta_n = \frac{k_3+1}{4k_3}\varphi_n(u_n) + \frac{k_3+1}{2}c_n^{(3)} + \beta_n.$$

By (3), we have

$$\frac{1}{2}\varphi_n(u_n) < \frac{k_3+1}{4k_3}\varphi_n(u_n) + \frac{k_3+1}{2}c_n^{(3)} + \beta_n,$$

i.e.

$$\varphi_n(u_n) < \frac{2k_3(k_3+1)}{k_3-1}c_n^{(3)} + \frac{3k_3}{k_3-1}\beta_n\,. \tag{4}$$

But when $n > \max(n_0,n_1)$, we have

$$\varphi_n(u_n) \ge \varphi_n(c_n) = c'_n = \frac{2k_3(k_3+1)}{k_3-1}c_n^{(3)} + \frac{4k_3}{k_3-1}\beta_n.$$

This contradicts (4), so Lemma 3 is true. □

2. Result

Theorem

A Musielak-Orlicz sequence space l_φ is P-convex if and only if l_φ is reflexive.

Proof. We may obtain necessity according to paper [1], so it is enough to prove sufficiency.

Assume sufficiency is false. Let l_φ be reflexive i.e. $\varphi = (\varphi_n)$ and $\varphi^* = (\varphi_n^*)$ satisfy the δ_2-condition but l_φ is not P-convex. Then for any $\varepsilon > 0$ and positive integer N_1, there is a set $X = \{x^i\}$ having N_1 elements in $S(l_\varphi)$ such that

$$\|x^i - x^j\| \geq 2(1-\varepsilon); \quad i \neq j, \ i, j = 1, 2, \ldots, N_1.$$

We will complete the proof of theorem in two steps.

Step 1. There is $\varepsilon_0 > 0$ such that $\|x_n\| < (1-\varepsilon_0)a_n$ for any $x = (x_n) \in X$ and all $n \in \mathbb{N}$.

(1a) We define some constants.
By Lemma 3, there are a sequence (c_n) of non-negative real numbers, $N' \in N, d > 0$ such that $\sum_{n=1}^{\infty} c_n < \infty, d_1 < d < 2$ and $d_n < d$ with $n > N'$. Let $\beta = \varepsilon_0/4$, then $\beta < 1$.

By Lemma 1 (ii), for $l_1 = 1/\beta$ and $a_1 = 1$, there are $k_1 > 1$ and a sequence $(c_n^{(2)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(2)} < \infty \quad \text{and} \quad \varphi_n(u/\beta) \leq k_1\varphi_n(u) + c_n^{(2)} \tag{1}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(u/\beta) \leq 1$. Let $\lambda_1 = (2-d)/(24k_1)$, $\lambda_2 = (2-d)/2d$. By Lemma 1 (iii), for $k_2 = 1 + \min(\lambda_1, \lambda_2), l_2 > 1$ and $a = 1$, there are $a \in (0, l-1)$ and a sequence $(c_n^{(3)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(3)} < \infty \quad \text{and} \quad \varphi_n\big((1+\delta)u\big) \leq k_2\varphi_n(u) + c_n^{(3)} \tag{2}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(l_2u) \leq 1$.

By Lemma 1 (ii), for $l_1 = 2$, and $a_1 = 1$, there are $k > 1$ and a sequence $(c_n^{(1)})$ of non-negative real numbers such that

$$\sum_{n=1}^{\infty} c_n^{(1)} < \infty \quad \text{and} \quad \varphi_n(2u) \leq k\varphi_n(u) + c_n^{(1)} \tag{3}$$

for all $n \in \mathbb{N}$ and $u \in \mathbb{R}$ with $\varphi_n(2u) \leq 1$. Let h_1 be, such that $0 < h_1 < 1$. Let

$$h_2 = \min\left\{\frac{2-d}{8k}, \frac{2-d}{4}\right\}$$
$$r_1 = \min\left\{\frac{1-h_1}{4(1+k_1)}, \frac{h_2(1-h_1)}{12kk_1}\right\}$$
$$r_2 = \frac{h_2(1-h_1)}{12(3k+1)}.$$

By $\sum_{n=1}^{\infty} \varphi_n(c_n) < \infty$ and (1), (2), (3), there is $N_0 > N'$, such that

$$\sum_{n=1}^{\infty} \varphi_n(c_n) < r_1, \quad \sum_{n=N_0}^{\infty} c_n^{(i)} < r, \quad i = 1, 2, 3. \tag{4}$$

(1b) Now we will prove that for any h_1, $0 < h_1 < 1$, there do not exist three elements x^1, x^2 and x^3 in X, such that

$$\sum_{n=1}^{\infty} \varphi_n(x_n^i) \geq I_\varphi(x^i) - h_1 = 1 - h_1, \quad i = 1, 2, 3. \tag{5}$$

Assume (1b) is false:

(i) If $0 < \varepsilon < \varepsilon_0/4$, then $\varphi_n((x_n^i - x_n^j)/2(1-\varepsilon)) < \infty$ for all $n \in N, i \neq j,\ i, j = 1, 2, 3$.

Let $u_n = \max\{|x_n^1|, |x_n^2|, |x_n^3|\}, w_n = \min\{|x_n^1|, |x_n^2|, |x_n^3|\}, v_n$ be the arithmetic mean of u_n and w_n. Since $u_n v_n \geq 0$, or $u_n w_n \geq 0$, or $v_n w_n \geq 0$ is true, we first consider $v_n, w_n \geq 0$.

Divide positive integers $n \geq N_0$ into the following sets:

$$I_1 = \left\{n: \left|\frac{v_n}{u_n}\right| \geq \beta \quad \text{and} \quad |v_n| \geq c_n\right\}$$
$$I_2 = \left\{n: \left|\frac{v_n}{u_n}\right| \geq \beta \quad \text{and} \quad |v_n| < c_n\right\}$$
$$I_3 = \left\{n: \left|\frac{v_n}{u_n}\right| < \beta \quad \text{and} \quad |u_n| \geq c_n\right\}$$
$$I_4 = \left\{n: \left|\frac{v_n}{u_n}\right| < \beta \quad \text{and} \quad |u_n| < c_n\right\}.$$

When $n \in I_1$, by formula (2) for $l_2 = (1-\varepsilon_0/2)/[(1-\varepsilon_0)(1-\varepsilon)]$, if $\sigma = 1/(1-\varepsilon) - 1$, then $\sigma < l_2 - 1$. Since

$$\begin{aligned}\varphi_n\Big(l_2\frac{u_n - v_n}{2}\Big) &= \varphi_n\Big(\frac{1-\varepsilon_0/2}{1-\varepsilon_0}\cdot\frac{u_n - v_n}{2(1-\varepsilon)}\Big)\\ &\le \varphi_n\Big(\frac{1-\varepsilon_0/2}{1-\varepsilon_0/4}\cdot\frac{2u_n}{2(1-\varepsilon_0)}\Big) \le \varphi_n(a_n) \le 1\end{aligned}$$

by (2) and $k_2 = 1 + \min(\lambda_1, \lambda_2)$, it follows that

$$\begin{aligned}\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) &= \varphi_n\Big((1+\sigma)\frac{u_n - v_n}{2}\Big) \le k_2\varphi_n\Big(\frac{u_n - v_n}{2}\Big) + c_n^{(3)}\\ &\le (1+\lambda_1)\varphi_n\Big(\frac{u_n - v_n}{2}\Big) + c_n^{(3)}\\ &\le (1+\lambda_1)\frac{\varphi_n(u_n) + \varphi_n(v_n)}{2} + c_n^{(3)}\\ &< \frac{1}{2}\varphi_n(u_n) + \frac{1}{2}\varphi_n(v_n) + \lambda_1\varphi_n(u_n) + c_n^{(3)}.\end{aligned} \tag{6}$$

By the same argumentation, we get

$$\varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big) \le \frac{1}{2}\varphi_n(u_n) + \frac{1}{2}\varphi_n(w_n) + \lambda_1\varphi_n(u_n) + c_n^{(3)}. \tag{7}$$

By $v_n, w_n \ge 0$ and $|v_n| \ge |w_n|$, it follows that

$$\varphi\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) \le \varphi_n\Big(\frac{v_n}{2(1-\varepsilon)}\Big) \le (1+\lambda_1)\varphi_n\Big(\frac{v_n}{2}\Big) + c_n^{(3)}.$$

By $|v_n| \ge c_n$ and the definition of d, we get

$$\varphi_n\Big(\frac{v_n}{2}\Big) = \varphi_n\Big(\frac{d}{2}\cdot\frac{v_n}{d}\Big) \le \frac{d}{2}\varphi_n\Big(\frac{v_n}{d}\Big) \le \frac{d}{4}\varphi_4(v_n),$$

so

$$\varphi_n\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) \le \frac{d}{4}(1+\lambda_1)\varphi_n(v_n) + c_n^{(3)}. \tag{8}$$

Let

$$\begin{aligned}f(n) = \varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) + \varphi_n\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) + \varphi\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big)\\ - \varphi_n(u_n) - \varphi_n(v_n) - \varphi_n(w_n).\end{aligned}$$

By (1) we get $\varphi_n(u_n) \le k_1\varphi_n(\beta u_n) + c_n^{(2)}$. By (6), (7) and (8) it follows

$$\begin{aligned}
\sum_{n\in I_1} f(n) &\le \sum_{n\in I_1} \Big[2\lambda_1\varphi_n(u_n) + \frac{d}{4}(1+\lambda_1)\varphi_n(v_n) + 3c_n^{(3)} - \frac{1}{2}\varphi_n(v_n)\Big] \\
&\le \sum_{n\in I_1} \Big[3\lambda_1\varphi_n(u_n) - \frac{2-d}{4}\varphi_n(v_n)\Big] + 3\sum_{n\in I_1} c_n^{(3)} \\
&\le \sum_{n\in I_1} \Big[3\lambda_1\varphi_n(u_n) - \frac{2-d}{4}\varphi(\beta u_n)\Big] + 3\sum_{n\in I_1} c_n^{(3)} \\
&\le \sum_{n\in I_1} \Big[3\lambda_1\varphi_n(u_n) - \frac{2-d}{4k_1}\varphi_n(u_n)\Big] + \frac{2-d}{4k_1}\sum_{n\in I_1} c_n^{(3)} \\
&\quad + 3\sum_{n\in I_1} c_n^{(3)} \\
&= \frac{2-d}{8k_1}\sum_{n\in I_1}\varphi_n(u_n) + \frac{2-d}{4k_1}\sum_{n\in I_1} c_n^{(3)} + 3\sum_{n\in I_1} c_n^{(3)} .
\end{aligned} \tag{9}$$

When $n \in I_2, |\frac{v_n}{u_n}| \ge \beta, |v_n| < c_n$. Since

$$\varphi_n\Big(\frac{2u_n}{2(1-\varepsilon)}\Big) \le \varphi_n\Big(\frac{u_n}{1-\varepsilon_0}\Big) \le \varphi_n(a_n) \le 1,$$

by (3) we get

$$\varphi_n\Big(\frac{2u_n}{2(1+\varepsilon)}\Big) \le k\varphi_n\Big(\frac{u_n}{2(1-\varepsilon)}\Big) + c_n^{(1)} \le k\varphi_n(u_n) + c_n^{(1)},$$

so

$$\begin{aligned}
\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) &\le \varphi_n\Big(\frac{2u_n}{2(1-\varepsilon)}\Big) \le k\varphi_n(u_n) + c_n^{(1)} \\
&\le kk_1\ \varphi_n(\beta u_n) + kc_n^{(2)} + c_n^{(1)} \le kk_1\varphi_n(c_n) + kc_n^{(2)} + c_n^{(1)}.
\end{aligned}$$

We have also

$$\begin{aligned}
\varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big) &\le kk_1\varphi_n(c_n) + c_n^{(1)} + kc_n^{(2)} \\
\varphi_n\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) &\le kk_1\varphi_n(c_n) + c_n^{(1)} + kc_n^{(2)},
\end{aligned}$$

so we get

$$\begin{aligned}
\sum_{n\in I_2} f(n) &\le \sum_{n\in I_2} \Big[\varphi_n\Big(\frac{u_n - v_n)}{2(1-\varepsilon)}\Big) + \varphi_n\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) + \varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big)\Big] \\
&\le 3kk_1\sum_{n\in I_2}\varphi_n(c_n) + 3\sum_{n\in I_2} c_n^{(1)} + 3k\sum_{n\in I_3} c_n^{(3)} .
\end{aligned} \tag{10}$$

When $n \in I_3, |\frac{v_n}{u_n}| < \beta, |u_n| \geq c_n$, by

$$\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) \leq \varphi_n\Big(\frac{(1+\varepsilon_0/4)u_0}{2(1-\varepsilon)}\Big),$$

denoting $(1+\varepsilon_0/4)/(1-\varepsilon) = 1/(1-\varepsilon'), \sigma' = 1/(1-\varepsilon') - 1$, we get as in (6),

$$\begin{aligned}\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) &\leq \varphi_n\Big((1+\sigma')\frac{u_n}{2}\Big) \leq (1+\lambda_2)\varphi_n\Big(\frac{u_n}{2}\Big) + c_n^{(3)} \\ &\leq \frac{d}{4}(1+\lambda_2)\varphi_n(u_n) + c_n^{(3)}\end{aligned}$$

and

$$\varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big) \leq \frac{d}{4}(1+\lambda_2)\varphi_n(u_n) + c_n^{(3)}.$$

By $\varphi_n(\frac{u_n - w_n}{2(1-\varepsilon)}) \leq \varphi_n(\frac{v_n}{2(1-\varepsilon)}) \leq \varphi_n(v_n)$ we get

$$\begin{aligned}\sum_{n\in I_1} f(n) &\leq \sum_{n\in I_3}\Big[\frac{d}{2}\varphi_n(u_n) + \frac{d}{2}\lambda_2\varphi_n(u_n) + 2c_n^{(3)} - \varphi_n(u_4)\Big] \\ &\leq \sum_{n\in I_2}\Big[-\frac{2-d}{2}\varphi_n(u_n) + \frac{2-d}{4}\varphi_n(u_n)\Big] + 2\sum_{n\in I_3} c_n^{(3)} \\ &= -\frac{2-d}{4}\sum_{n\in I_3}\varphi_n(u_n) + 2\sum_{n\in I_3} c_n^{(3)}.\end{aligned} \tag{11}$$

When $n \in I_4, |u_n| < c_n$, as in the case of $n \in I_2$, we get

$$\begin{aligned}\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) &\leq k\varphi_n(u_n) + c_n^{(1)} \leq k\varphi_n(c_n) + c_n^{(1)} \\ \varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) &\leq k\varphi_n(c_n) + c_n^{(1)} \\ \varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big) &\leq k\varphi_n(c_n) + c_m^{(1)}.\end{aligned}$$

Then

$$\sum_{n\in I_4} f(n) \leq 3k\sum_{n\in I_4}\varphi_n(c_n) + 3\sum_{n\in I_4} c_n^{(1)}. \tag{12}$$

By (9), (10), (11) and (12), we get

$$\sum_{n=N_0}^{\infty} f(n) \le -h_2 \sum_{n=N_0}^{\infty} \varphi_n(u_n) + h_2 \sum_{n\in I_2\cup I_4} \varphi_n(u_n) + 3\sum_{n=N_0}^{\infty} (c_n^{(1)} + c_n^{(3)}) + 3kk_1 \sum_{n=N_0} \varphi_n(c_n) + \Big(3k + \frac{2-d}{4k_1}\Big) \sum_{n=N_0}^{\infty} c_n^{(2)} . \tag{13}$$

When $n \in I_2$, since (1) implies $\varphi_n(u_n) \le k_1\varphi_n(c_n) + c_n^{(2)}$, then

$$\begin{aligned} h_2 \sum_{n\in I_2\cup I_4} \varphi_n(u_n) &= h_2 \sum_{n\in I_2} \varphi_n(u_n) + h_2 \sum_{n\in I_4} \varphi_n(u_n) \\ &\le h_2 \sum_{n\in I_2} [k_1\varphi_n(c_n) + c_n^{(2)}] + h_2 \sum_{n\in I_4} \varphi_n(c_n) \\ &\le h_2(k_1+1) \sum_{n=N_0}^{\infty} \varphi_n(c_n) + h_2 \sum_{n\in I_2} c_n^{(2)} . \end{aligned} \tag{14}$$

It we put (14) into (13), by (4) and (5), we get

$$\begin{aligned} \sum_{n=N_0}^{\infty} f(n) &\le -h_2 \sum_{n=N_0}^{\infty} \varphi_n(u_n) + h_2(k_1+1) \sum_{n=N_0}^{\infty} \varphi_n(c_n) \\ &\quad + 3kk_1 \sum_{n=N_0}^{\infty} \varphi_n(c_n) \\ &\quad + 3\sum_{n=N_0}^{\infty} (c_n^{(1)} + c_n^{(2)}) + (3k+1) \sum_{n=N_0}^{\infty} c_n^{(2)} \\ &< -h_2(1-h_1) + h_2(k_1+1)r_1 + 3kk_1r_1 + 3(3k+1)r_2 \\ &< -\frac{h_2(1-h_1)}{4} . \end{aligned} \tag{15}$$

(ii) Formula (5) implies $\sum_{n=1}^{N_0-1} \varphi_n(x_n^i) < h$, $i = 1, 2, 3$. We deduce that $|2x_n^i| < a_n$ for all $n < \mathbb{N}$, and $i = 1, 2, 3$. Let

$$\alpha' = \min_{n<N_0} \varphi_n^{-1}\Big(\frac{h_2}{48N_0}\Big) .$$

Then $k' = \max_{n<N_0} \max_{\alpha' \le u \le a_n} \varphi_n(u)/\varphi_n(\frac{u}{2}) < \infty$.

So when $|2u_n| \in [\alpha', a_n], \varphi_n(2u_n) \le k'\varphi_n(u_n)$; when $|2u_n| < \alpha', \varphi_n(2u_n) \le \varphi_n(\alpha')$. Hence

$$\sum_{n=1}^{N_0-1} f(n) < \sum_{n=1}^{N_0-1} \left[\varphi_n\Big(\frac{u_n - v_n}{2(1-\varepsilon)}\Big) + \varphi_n\Big(\frac{v_n - w_n}{2(1-\varepsilon)}\Big) + \varphi_n\Big(\frac{u_n - w_n}{2(1-\varepsilon)}\Big)\right]$$
$$\le 3\sum_{n=1}^{N_0-1} \varphi_n(2u_n) \le 3k\sum_{n=1}^{N_0-1} \varphi_n(u_n) + 3\sum_{n=1}^{N_0-1} \varphi_n(\alpha')$$

and when $h_1 < \frac{1}{3k_1} \cdot \frac{h_2}{16} \cdot h_1 < \frac{1}{2}$, then

$$\sum_{n=1}^{N_0-1} f(n) < 3k'h_1 + 3N_0\frac{h_2}{48N_0} \le \frac{h_2}{16} + \frac{h_2}{16} = \frac{h_2}{8} < \frac{h_2(1-h_1)}{4}. \tag{16}$$

By (15) and (16), we get $\sum_{n=1}^{\infty} f(n) < 0$, i.e.

$$I_\varphi\Big(\frac{x^1 - x^2}{2(1-\varepsilon)}\Big) + I_\varphi\Big(\frac{x^2 - x^3}{2(1-\varphi)}\Big) + I_\varphi\Big(\frac{x^1 - x^3}{2(1-\varepsilon)}\Big) - I_\varphi(x^1) - I_\varphi(x^2) - I_\varphi(x^3) < 0.$$

Since $I_\varphi(x^i) = 1, i = 1,2,3$, so $I_\varphi(\frac{x^1-x^2}{2(1-\varepsilon)}) < 1$, or $I_\varphi(\frac{x^2-x^3}{2(1-\varepsilon)}) < 1$, or $I_\varphi(\frac{x^1-x^3}{2(1-\varepsilon)}) < 1$, and this implies $\|x^1 - x^2\| < 2(1-\varepsilon)$ or $\|x^2 - x^3\| < 2(1-\varepsilon)$, or $\|x^1 - x^3\| < 2(1-\varepsilon)$. This contradicts the assumption in the theorem, so result (1b) is true.

Repeating the same argumentation, we may prove result (1b) in case of $uw > 0$ and $uv > 0$.

(1c) Let $N_1 = 2N_0 + 1, N_1$ is the number of elements of X. Result (1b) implies that there are at least $2N_0 - 1$ elements in X such that

$$\sum_{n=1}^{N_0-1} \varphi_n(x_n) > h_1. \tag{17}$$

Let

$$\alpha_1 = \frac{h_1}{N_0 - 1}, \quad u_0 = \min_{n<N_0} \frac{1}{4}\varphi_n^{-1}\Big(\frac{\alpha_1}{4(N_0-1)}\Big).$$

The fact that a continuous function is uniformly continuous in a closed interval implies that there is $\delta_n' > 0$ such that

$$\varphi\Big(\frac{u}{1-\delta}\Big) \le \varphi_n(u) + \frac{\alpha_1}{4(N_0-1)}, \quad n =, 1, 2, \dots, N_0 - 1 \tag{18}$$

for all $\delta < \delta_n'$ and $u \in [u_0, a_n]$.

Let $\delta' = \min_{n<N_0} \delta'_n$. Take $\varepsilon < \varepsilon_0/4$ and $0 < \varepsilon < \delta'$. Among the elements satisfying (17), there are three ones x^1, x^2, x^3 and $n_0 < N_0$ such that

$$\varphi_{n_0}(x^i_{n_0}) > \frac{h_1}{N_0 - 1}, \quad i = 1, 2, 3$$

this is because $2N_0 - 1$ elements satisfy (17) in the former $N_0 - 1$ components, then there are three elements satisfying the above formula in the same component.

Since there are at least two elements having same sign among $x^1_{n_0}, x^2_{n_0}, x^3_{n_0}$ and without loss of generality we have

$$x^1_{n_0} x^2_{n_0} \geq 0 \quad \text{and} \quad |x^1_{n_0}| \geq |x^2_{n_0}|.$$

By analogy of the former proof we get

$$\sum_{n=N_0}^{\infty} \varphi_n\Big(\frac{x^1_n - x^2_n}{2(1-\varepsilon)}\Big) < \frac{1}{2}\sum_{n=N_0}^{\infty} \varphi_n(x^1_n) + \frac{1}{2}\sum_{n=N_0}^{\infty} \varphi_n(x^2_n) + \frac{\alpha_1}{4}. \tag{19}$$

Divide the positive integers of $n < N_0 (n \neq n_0)$ into three sets:

$$\begin{aligned} I_5 &= \big\{n\colon \max(|x^1_n|, |x^2_n|) \geq 2u_0 \quad \text{and} \quad x^1_n x^2_n < 0\big\} \\ I_6 &= \big\{n\colon \max(x^1_n|, |x^2_n|) \geq 2u_0 \quad \text{and} \quad x^1_n x^2_n \geq 0\big\} \\ I_7 &= \big\{n\colon \max(|x^1_n|, |x^2_n|) < 2u_0\big\}. \end{aligned}$$

When $n \in I_5$, $|\frac{x^1_n - x^2_n}{2}| \geq \frac{1}{2}\max(|x^1_n|, |x^2_n|) \geq u_0$, we get by $\varepsilon \leq \delta_n$ and (18)

$$\begin{aligned} \varphi_n\Big(\frac{x^1_n - x^2_n}{2(1-\varepsilon)}\Big) &\leq \varphi_n\Big(\frac{x^1_n - x^2_n}{2}\Big) + \frac{\alpha_1}{4(N_0 - 1)} \\ &\leq \frac{1}{2}\varphi_n(x^1_n) + \frac{1}{2}\varphi_n(x^2_n) + \frac{\alpha_1}{4(N_0 - 1)}. \end{aligned} \tag{20}$$

When $n \in I_6$,

$$\begin{aligned} \varphi_n\Big(\frac{x^1_n - x^2_n}{2(1-\varepsilon)}\Big) &\leq \max\Big\{\Big(\frac{x^1_n}{2(1-\varepsilon)}\Big), \varphi_n\Big(\frac{x^2_n}{2(1-\varepsilon)}\Big)\Big\} \\ &\leq \frac{1}{2}\varphi_n(x^1_n) + \frac{1}{2}\varphi_n(x^2_n) + \frac{\alpha_1}{4(N_0 - 1)}. \end{aligned} \tag{21}$$

When $n \in I_7$,

$$\varphi_n\Big(\frac{x^1_n - x^2_n}{2(1-\varepsilon)}\Big) \leq \varphi_n\Big(\frac{4u_0}{2(1-\varepsilon)}\Big) \leq \varphi_n(4u_0) \leq \frac{\alpha_1}{4(N_0 - 1)} \tag{22}$$

since

$$\begin{aligned}\varphi_{n_0}\Big(\frac{x_n^1 - x_n^2}{2(1-\varepsilon)}\Big) < \varphi_{n_0}\Big(\frac{x_{n_0}^1}{2(1-\varepsilon)}\Big) \le \varphi_{n_0}\Big(\frac{x_{n_0}^1}{2}\Big) + \frac{\alpha_1}{4(N_0-1)} \\ \le \frac{1}{2}\varphi_{n_0}(x_{n_0}^1) + \frac{\alpha_1}{4(N_0-1)}\end{aligned} \tag{23}$$

notice $\varphi_{n_0}(x_{n_0}^2) > \frac{h_1}{N_0-1} = \alpha_1$, by (19) and (23)

$$\begin{aligned}I_\varphi\Big(\frac{x_n^1 - x_n^2}{2(1-\varepsilon)}\Big) &= \varphi_{n_0}\Big(\frac{x_{n_0}^1 - x_{n_0}^2}{2(1-\varepsilon)}\Big) + \sum_{\substack{n=1\\ n\ne n_0}}^{N_0-1} \varphi_n\Big(\frac{x_n^1 - x_n^2}{2(1-\varepsilon)}\Big) \\ &\quad + \sum_{n=N_0}^{\infty} \varphi_n\Big(\frac{x_n^1 - x_n^2}{2(1-\varepsilon)}\Big) \\ &< \frac{1}{2}\varphi_{n_0}(x_{n_0}^1) + \frac{\alpha_1}{4(N_0-1)} \\ &\quad + \sum_{\substack{n<N_0\\ n\ne n_0}} \Big[\frac{1}{2}\varphi_n(x_n^1) + \frac{1}{2}\varphi_n(x_n^2) + \frac{\alpha_1}{4(N_0-1)}\Big] \\ &\quad + \sum_{n=N_0}^{\infty} \Big[\frac{1}{2}\varphi_n(x_n^1) + \frac{1}{2}\varphi_n(x_n^2)\Big] + \frac{\alpha_1}{4} \\ &= \frac{1}{2}I_\varphi(x^1) + \frac{1}{2}I_\varphi(x^2) - \frac{1}{2}\varphi_{n_0}(x_{n_0}^2) + \frac{\alpha_1}{4} + \frac{\alpha_1}{4} \\ &< \frac{1}{2}I_\varphi(x^1) + \frac{1}{2}I_\varphi(x^2) = 1\end{aligned}$$

so $\|x^1 - x^2\| < 2(1-\varepsilon)$, and we get a contradiction again.

Steps (1b) and (1c) complete the proof of theorem.

Step 2. We discuss the general case without the restriction of step 1. For any $\varepsilon \le 1/4$, let $A = \inf_n \varphi_n((1-\varepsilon)a_n)$. By the proof of Lemma 1 (i) we get $A > 0$. Let $N_2 = [1/A]$, i.e. N_2 be the integer part of $1/A$. If l_φ is reflexive but not P-convex, then for any $\varepsilon' : 0 < \varepsilon' < \varepsilon/4$, there is a set X consisted of any finite elements in $S(l_\varphi)$ such that

$$\|x^i - x^j\| \ge 2(1-\varepsilon'), \quad i \ne j.$$

Let the number of X be $(2N_0 + 1)2^{(N_2+1)N_2/2}$ where N_0 is the positive integer satisfying (4).

Take any element x^0 in X. The definition of A implies that x^0 has at most N_2 numbers of components, such that $|x_n^0| \geq (1-\varepsilon)a_n$; hence

$$I_\varphi(x^0) = \sum_{n=1}^{\infty} \varphi_n(x_n^0) \geq (N_2+1)A > \frac{1}{A} \cdot A = 1,$$

this leads to contradiction. Without loss of generality we have $|x_n^0| \geq (1-\varepsilon)a_n$ for $n \leq N_2$. For any $x \in X$, we define a map: $x \rightarrow (r_1^x, r_2^x, \ldots, r_{N_2}^x)$, i.e. for $n = 1, 2, \ldots, N_2$

$$r_n^x = \begin{cases} 1, & \text{when } x_n^0 x_n < 0 \text{ and } |x_n| \geq (1-\varepsilon)a_n \\ 0, & \text{otherwise.} \end{cases}$$

This makes us classify the elements of X into 2^{N_1} categories, we say that the category mapping the vector $(0, 0, \ldots, 0)$ is 0-category.

First we assume: apart from 0-category, the number of elements in other category is less than $(2N_0+1)2^{(N+1+1)N_1/2}/2^{N_2} = (2N_0+1)2^{N_2(N_2-1)/2}$. Take another element from 0-category and let it be x^0, then classify X again by the former program.

After we classify each time, if the number of the elements in category, except 0-category, is less than $(2N_0+1)2^{N_1(N_1-1)/2}$, when we classify $(2N_0+1)$–times we get a set X_0 having $(2N_0+1)$ elements such that

$$x_n^i x_n^j > 0 \quad \text{or} \quad |x_n^i| \geq (1-\varepsilon)a_n \quad \text{and} \quad |x_n^j| \geq (1-\varepsilon)a_n \tag{24}$$

for any $x^i, x^j \in X_0 (i \neq j)$ and $n \in \mathbb{N}$, then

$$\left|\frac{x_n^1 - x_n^2}{2(1-\varepsilon)}\right| < \left|\frac{a_n + (1-\varepsilon)a_n}{2(1-\varepsilon/4)}\right| = \frac{2-\varepsilon}{2-\varepsilon/2}a_n < a_n,$$

i.e. $|x_n^i| < (1-\varepsilon')a_n$ for all $n \leq N_2$, and this is the case of section 1. But in section 1, we proved that there is no set X having $(2N_0+1)$ elements such that

$$\|x^i - x^j\| \geq 2(1-\varepsilon), \quad i \neq j, \; x^i, x^j \in X,$$

so we deduce that apart from 0-category there is a category X_1 such that the number of elements in X is $(2N_0+1)2^{N_1(N_2-1)/2}$ and the element x of x_1 satisfies $r_{n_1}^x = 1$ for some $n_1 \leq N_2$.

Apart from n_1-th component, any $x = (x_n)$ in X_1 has at most $(N_2 - 1)$ numbers of components such that $|x_n| \geq (1-\varepsilon)a_n$. Let $|x_n| \geq (1-\varepsilon)a_n$ for $n = N_2+1, N_2+2, \ldots, 2N_2-1$.

For any $x \in X_1$, define a map: $x \to (r_1^x, r_2^x, \ldots, r_{N_1-1}^x)$, i.e. for $n = N_2+1, N_2+2, \ldots, 2N_2-1$

$$r_n^x = \begin{cases} 1, & \text{when } x_n^0 x_n < 0 \text{ and } |x_n| \geq (1-\varepsilon)a_n \\ 0, & \text{otherwise} \end{cases}$$

then we may classify X_1 into 2^{N_2-1} categories.

If the number of elements in category except 0-category is less than $(2N_0 + 1)2^{(N_1-1)(N_2-2)/2}$, we take one element from those mapping 0-category and let it be x^0, and then classify X_1 by the former program. When we classify $(2N_0+1)$ times, the number of elements in the category except 0-category is less than $(2N_0 + 1)2^{(N_2-1)(N_2-2)/2}$, then we get a set having $(2N_0+1)$ elements such that (24), which leads a contradiction again.

We assume there a category X_2 having $(2N_0+1)2^{(N_1-1)(N_2-2)/2}$ elements except 0-category. Repeating the same discussion, when we classify N_2-times we get a category X_{N_2} having $(2N_0+1)$ elements such that

$$x_n^i x_n^j > 0 \quad \text{and} \quad |x_n^i| \geq (1-\varepsilon)a_n, \ |x_n^j| \geq (1-\varepsilon)a_n$$

for any $x^i, x^j \in X_{N_2}, i \neq j.n = n_1, n_2, \ldots, n_{N_2}$. Then for any $x \in X_{N_2}$

$$\begin{aligned} I = I_\varphi(x) &= \sum_{j \leq N_2} \varphi_{n_j}(x_{n_j}) + \sum_{n \neq n_j} \varphi_n(x_n) \\ &\geq \sum_{j \leq N_0} \varphi_{n_j}\big((1-\varepsilon)a_{n_j}\big) + \sum_{n \neq n_j} \varphi_n(x_n) \geq N_2 A + \sum_{n \neq n_j} \varphi_n(x_n) \end{aligned}$$

i.e.

$$\sum_{n \neq n_j} \varphi_n(x_n) \leq 1 - N_1 A = \frac{A}{A} - \Big[\frac{I}{A}\Big] A < A = \inf_n \varphi_n\big((1-\varepsilon)a_n\big)$$

so $|x_n| < (1-\varepsilon)a_n$ with $n \neq n_j$, but when $n = n_j$ $x_n^i x_n^j > 0 (i \neq j)$. This shows that (24) is true for any $x \in X_{N_2}$ and all $n \in \mathbb{N}$, which leads to a contradiction again.

Section 1 and section 2 complete the proof of theorem. □

Now we give an example of a Musielak-Orlicz sequence space which is P-convex but not $P(3,\varepsilon)$-convex.

Let a Young function $\varphi = (\varphi_n)$ and $\varphi^* = (\varphi_n^*)$ satisfy the δ_2-condition, and such that there are two positive integers n_1 and n_2 $(n_1 < n_2)$

$$\varphi_{n_1}(a_{n_1}) + \varphi_{n_2}(a_{n_2}) \leq 1 \quad \text{and} \quad \varphi_{n_1}(a_{n_1}) > 0,\ \varphi_{n_2}(a_{n_2}) > 0.$$

By Theorem we know that the l_φ generated by φ is P-convex but not $P(3,\varepsilon)$-convex. Let

$$x_1 = (0,\ldots,0,a_{n_1},0,\ldots,0,a_{n_2},0,\ldots)$$
$$x_2 = (0,\ldots,0,a_{n_1},0,\ldots,0,-a_{n_2},0,\ldots)$$
$$x_3 = (0,\ldots,0,-a_{n_1},0,\ldots,0,a_{n_1},0,\ldots).$$

Then $x_1, x_2, x_3 \in S(l_\varphi)$. But for any $\varepsilon > 0$

$$I_\varphi\Big(\frac{x_1 - x_2}{2(1-\varepsilon)}\Big) = \varphi_{n_2}\Big(\frac{2a_{n_2}}{2(1-\varepsilon)}\Big) > 1$$
$$I_\varphi\Big(\frac{x_1 - x_i}{2(l-\varepsilon)}\Big) = \varphi_{n_1}\Big(\frac{2a_{n_1}}{2(1-\varepsilon)}\Big) > 1$$
$$I_\varphi\Big(\frac{x_2 i x_j}{2(1-\varepsilon)}\Big) = \varphi_{n_1}\Big(\frac{2a_{n_1}}{2(1-\varepsilon)}\Big) + \varphi_{n_2}\Big(\frac{2a_{n_1}}{2(1-\varepsilon)}\Big) > 1$$

so $\|x_1 - x_2\| \geq 2(1-\varepsilon)$, $\|x_2 - x_3\| \geq 2(1-\varepsilon)$, $\|x_1 - x_4\| \geq 2(1-\varepsilon)$, hence l_φ is not $P(3,\varepsilon)$-convex.

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