

On the weak star uniformly rotund points of Orlicz spaces

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ABSTRACT

We obtain the necessary and sufficient condition of weak star uniformly rotund point in Orlicz spaces.

In this paper, X denotes a Banach space, $B(X)$, $S(X)$ denote respectively the unit ball and the unit sphere of X . $x \in S(X)$ is said to be a UR (WUR, W^* UR) point provided that $x_n \in B(X)$, $\|x_n + x\| \rightarrow 2 \Rightarrow \|x_n - x\| \rightarrow 0$ ($x_n - x \xrightarrow{w} 0$, $x_n - x \xrightarrow{w^*} 0$). Obviously, URP \Rightarrow WURP \Rightarrow W^* URP. If all points of $S(X)$ are W^* UR points, X is locally weak star uniformly rotund.

$M(\cdot)$, $N(\cdot)$ denote a pair of complemented N -functions (see [3]); “ $M \in \Delta_2$ ” (“ $M \in \nabla_2$ ”) means that $M(\cdot)$ satisfies the ∇_2 -condition (N satisfies the ∇_2 -condition). (G, Σ, μ) stands for a non-atomic finite measure space; $L_M(G, \Sigma, \mu)$ expresses the Orlicz space generated by $M(\cdot)$:

$$L_M(G, \Sigma, \mu) = \left\{ x(t) : \exists a > 0, R_M\left(\frac{x}{a}\right) = \int_G M\left(\frac{x(t)}{a}\right) d\mu < \infty \right\}$$

with the norm

$$\|x\| = \inf \left\{ a > 0 : R_M\left(\frac{x}{a}\right) \leq 1 \right\}.$$

For an Orlicz function space L_M and a sequence space l_M , a criterion of UR (WUR) point was obtained in recent years [1,2]. Here we give the criterion of W^* UR point, but the statement and method is different to WURP.

Theorem 1

Let $x \in S(L_M)$. x is a W^* URP if and only if

- (1) $\exists \tau > 0, R_M(\frac{x}{1-\tau}) < \infty$
- (2) $\mu\{t \in G : |x(t)| \in \mathbb{R} \setminus S_M\} = 0$

(3) $\mu\left\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} \{b_i\}\right\} = 0$ or $\mu\left\{t \in G : |x(t)| \in \bigcup_{i=1}^{\infty} \{a_i\}\right\} = 0$ and $M \in \nabla_2$,
where S_M is the set of all strictly convex points of $M(\cdot)$, (a_i, b_i) are affine segments
of $M(\cdot)$ for $i = 1, 2, \dots$

Proof. Without loss of generality, we assume $x(t) \geq 0$.

Necessity. Suppose that $R_M(\frac{x}{1-\varepsilon}) = \infty$ for all $\varepsilon > 0$. Take c large enough such that $\mu G_c = \mu\{t \in G : x(t) \leq c\} > 0$. Define $x' = -x\chi_{G_c} + x\chi_{G \setminus G_c}$; then $\|x'\| = \|x\| = 1$, but for all $\varepsilon > 0$

$$R_M\left(\frac{x+x'}{2(1-\varepsilon)}\right) = \int_{G \setminus G_c} M\left(\frac{x(t)}{1-\varepsilon}\right) d\mu = \infty$$

so $\|x+x'\| = 2$. From $x \neq x'$, we get a contradiction with the fact that x is a W^* UR point, which shows that (1) is true.

Since a W^* UR point is an extreme point, from [3], we get that (2) is true.

Suppose that there exist affine segments $[a, b], [c, d]$ of $M(\cdot)$ such that $A = \{t : x(t) = a\}, B = \{t : x(t) = d\}$ are sets with positive measure. Take $E \subset A, F \subset B$ satisfying $(M(b) - M(a))\mu E = (M(d) - M(c))\mu F$. Define $x' = x\chi_{G \setminus (E \cup F)} + b\chi_E + c\chi_F$, then $x' \neq x, R_M(x) = R_M(x') = 1$, and $R_M(\frac{x+x'}{2}) = \frac{R_M(x)+R_M(x')}{2} = 1$, which contradicts the fact that x is a W^* UR point.

Suppose now that there is an affine segment $[a, b]$ with positive measure set $D = \{t : x(t) = b\}$ and $M \notin \nabla_2$, i.e. there are $u_n \nearrow \infty$ such that $u_1 > b$ and $M(\frac{u_n}{2}) > (1 - \frac{1}{n})\frac{M(u_n)}{2}$ ($n = 1, 2, \dots$). Take subsets $E_n, D \supset E_1 \supset E_2 \supset \dots$ satisfying

$$M(u_n - b)\mu E_n + M(a)\mu(D \setminus E_n) = M(b)\mu D$$

then $\mu E_n \rightarrow 0$. Define $x_n = x\chi_{G \setminus D} + a\chi_{D \setminus E_n} + (u_n - b)\chi_{E_n}$ so $R_M(x_n) = R_M(x) = 1$ ($n = 1, 2, \dots$) and

$$\begin{aligned}
R_M\left(\frac{x_n+x}{2}\right) &= R_M(x\chi_{G \setminus D}) + M\left(\frac{a+b}{2}\right)\mu(D \setminus E_n) + M\left(\frac{u_n}{2}\right)\mu E_n \\
&\geq R_M(x\chi_{G \setminus D}) + \frac{M(a) + M(b)}{2}\mu(D \setminus E_n) + \frac{1}{2}\left(1 - \frac{1}{n}\right)M(u_n)\mu E_n \\
&\geq \frac{R_M(x\chi_{G \setminus D}) + M(b)\mu(D \setminus E_n)}{2} \\
&\quad + \frac{R_M(x\chi_{G \setminus D}) + M(a)\mu(D \setminus E_n) + (1 - \frac{1}{n})M(u_n - b)\mu E_n}{2} \\
&\longrightarrow R_M(x) = 1.
\end{aligned}$$

From $\langle x - x_n, \chi_{D \setminus E_1} \rangle = (b - a)\mu(D \setminus E_1) > 0$, we get a contradiction since x is a W*UR point.

Sufficiency. Suppose that $x_n \in B(L_M)$, $\|x_n + x\| \rightarrow 2$. We will first show that $R_M(x_n) \rightarrow 1$ and $R_M\left(\frac{x_n+x}{2}\right) \rightarrow 1$.

Suppose that $R_M(x_n) \leq 1 - \delta$ for some $\delta > 0$ ($n = 1, 2, \dots$). Take ε small enough such that $\frac{1+\varepsilon}{1-\varepsilon} < \frac{1}{1-\delta}$. For such ε , while n is large enough we have $\|(1+\varepsilon)\frac{x_n+x}{2}\| > 1$, and so $R_M((1+\varepsilon)\frac{x_n+x}{2}) > 1$. Hence applying assumption (1) we get

$$\begin{aligned}
1 &< R_M\left((1+\varepsilon)\frac{x_n+x}{2}\right) = R_M\left(\frac{1+\varepsilon}{2}x_n + \frac{1-\varepsilon}{2}\frac{1+\varepsilon}{1-\varepsilon}x\right) \\
&\leq \frac{1+\varepsilon}{2}R_M(x_n) + \frac{1-\varepsilon}{2}R_M\left(\frac{1+\varepsilon}{1-\varepsilon}x\right) \leq \frac{1+\varepsilon}{2}(1-\delta) \\
&\quad + \frac{1-\varepsilon}{2}(1+o(\varepsilon))R_M(x).
\end{aligned}$$

If $\varepsilon \rightarrow 0$, $1 \leq 1 - \delta/2$, so we get $R_M(x_n) \rightarrow 1$.

Clearly $\|\frac{x_n+x}{2} + x\| \rightarrow 2$, so we get similarly $R_M\left(\frac{x_n+x}{2}\right) \rightarrow 1$.

In the following, we show $x_n - x \xrightarrow{\text{w}^*} 0$, and so it is enough to show $x_n - x \xrightarrow{\mu} 0$. Denote $E = \{t \in G : x(t) \in S_M \setminus (\{a_i\} \cup \{b_i\})\}$. We first show $x_n - x \xrightarrow{\mu} 0$ on E . Suppose for a contrary that there exist $\varepsilon, \sigma > 0$ with $\mu\{t \in E : |x_n(t) - x(t)| \geq \varepsilon\} \geq \sigma$. Since

$$1 \geq R_M(x_n) \geq \int_{\{t : |x_n(t)| \geq d\}} M(x_n(t))d\mu \geq M(d)\mu\{t : |x_n(t)| > d\},$$

we have that for d large enough $\mu\{t : |x_n(t)| > d\} < \sigma/3$ ($n = 1, 2, \dots$) and $\mu\{t : |x(t)| > d\} < \sigma/3$. Thus $\mu E_n \geq \sigma/3$, where

$$E_n = \{t \in E : |x_n(t) - x(t)| \geq \varepsilon, \quad |x_n(t)| \leq d, \quad |x(t)| \leq d\}.$$

Clearly there is $\delta > 0$ such that for $t \in E_n$,

$$M\left(\frac{x_n(t) + x(t)}{2}\right) \leq (1 - \delta)\frac{M(x_n(t)) + M(x(t))}{2}.$$

Hence

$$\begin{aligned} 0 &\leftarrow \frac{R_M(x_n) + R_M(x)}{2} - R_M\left(\frac{x_n + x}{2}\right) \\ &= \int_G \left(\frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t) + x(t)}{2}\right) \right) d\mu \\ &\geq \int_{E_n} \left(\frac{M(x_n(t)) + M(x(t))}{2} - M\left(\frac{x_n(t) + x(t)}{2}\right) \right) d\mu \\ &\geq \frac{\delta}{2} \int_{E_n} (M(x_n(t)) + M(x(t))) d\mu \geq \frac{\delta}{2} M\left(\frac{\varepsilon}{2}\right) \frac{\sigma}{3}. \end{aligned}$$

This contradiction shows that $x_n - x \xrightarrow{\mu} 0$ on E . Denote

$$\begin{aligned} F_a^+(n) &= \{t \in G: x(t) \in \{a_i\}, x_n(t) \geq x(t)\} \\ F_a^-(n) &= \{t \in G: x(t) \in \{a_i\}, x_n(t) < x(t)\} \\ F_b^+(n) &= \{t \in G: x(t) \in \{b_i\}, x_n(t) \geq x(t) \text{ or } x_n(t) < 0\} \\ F_b^-(n) &= \{t \in G: x(t) \in \{b_i\}, x_n(t) < x(t) \text{ and } x_n(t) \geq 0\}. \end{aligned}$$

Analogously as above we have that for any $\varepsilon > 0$

$$\begin{aligned} \mu\{t \in F_a^-(n): x_n(t) \leq x(t) - \varepsilon\} &\rightarrow 0 \quad (n \rightarrow \infty) \\ \mu\{t \in F_b^+(n): x_n(t) \geq x(t) + \varepsilon\} &\rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

whence

$$\limsup_{n \rightarrow \infty} \int_{E \cup F_a^-(n) \cup F_b^+(n)} M(x_n(t)) d\mu \geq \limsup_{n \rightarrow \infty} \int_{E \cup F_a^-(n) \cup F_b^+(n)} M(x(t)) d\mu.$$

Consider the case $\mu(F_b^+(n) \cup F_b^-(n)) = 0$ ($n = 1, 2, \dots$). From

$$\limsup_{n \rightarrow \infty} \int_{E \cup F_a^-(n)} M(x_n(t)) d\mu \geq \limsup_{n \rightarrow \infty} \int_{E \cup F_a^-(n)} M(x(t)) d\mu$$

and $R_M(x_n) \rightarrow 1 = R_M(x)$, it follows that

$$\liminf_{n \rightarrow \infty} \int_{F_a^+(n)} M(x_n(t)) d\mu \leq \liminf_{n \rightarrow \infty} \int_{F_a^+(n)} M(x(t)) d\mu$$

thus

$$\liminf_{n \rightarrow \infty} \int_{F_a^+(n)} (M(x_n(t)) - M(x(t))) d\mu = 0$$

so for any $\varepsilon > 0$

$$\mu\{t \in F_a^+(n) : x_n(t) \geq x(t) + \varepsilon\} \rightarrow 0 \quad (n \rightarrow \infty).$$

Combining the above, we have $x_n - x \xrightarrow{\mu} 0$ on $E \cup F_a^-(n) \cup F_a^+(n) = G$.

Consider the case of $\mu(F_a^+(n) \cup F_a^-(n)) = 0$ ($n = 1, 2, \dots$) and $M \in \nabla_2$. We first prove that $\limsup_{\mu e \rightarrow 0} \int_M(x_n(t)) d\mu = 0$. Suppose that there exist $\varepsilon > 0$ and $e_n \subset G$ with $\mu e_n \searrow 0$ such that $\int_{e_n} M(x_n(t)) d\mu \geq \varepsilon$.

Take $c > 0$ small enough, $M(c)\mu G < \varepsilon/2$, denote $e'_n = \{t \in e_n : |x_n(t)| \geq c\}$. Then

$$\begin{aligned} \int_{e'_n} M(x_n(t)) d\mu &= \int_{e_n} M(x_n(t)) d\mu - \int_{e_n \setminus e'_n} M(x_n(t)) d\mu \\ &\geq \varepsilon - M(c)\mu G \geq \frac{\varepsilon}{2}. \end{aligned}$$

Combining $M \in \nabla_2$, for $\tau, \delta > 0$, while $u \geq c$, $M(\frac{u}{1+\tau}) \leq (1-\delta)\frac{M(u)}{1+\tau}$ whence

$$\begin{aligned} 1 &\leftarrow R_M\left(\frac{x_n+x}{2}\right) = R_M\left(\frac{x_n+x}{2}\chi_{G \setminus e'_n}\right) + R_M\left(\frac{x_n+x}{2}\chi_{e'_n}\right) \\ &\leq \frac{R_M(x_n\chi_{G \setminus e'_n}) + R_M(x\chi_{G \setminus e'_n})}{2} + \int_{e'_n} M\left(\frac{1+\tau}{2}\frac{x_n(t)}{1+\tau} + \frac{1-\tau}{2}\frac{x(t)}{1-\tau}\right) d\mu \\ &\leq \frac{R_M(x_n\chi_{G \setminus e'_n}) + R_M(x\chi_{G \setminus e'_n})}{2} + \frac{1+\tau}{2} \int_{e'_n} M\left(\frac{x_n(t)}{1+\tau}\right) d\mu \\ &\quad + \frac{1-\tau}{2} \int_{e'_n} M\left(\frac{x(t)}{1-\tau}\right) d\mu \\ &\leq \frac{R_M(x_n\chi_{G \setminus e'_n}) + R_M(x\chi_{G \setminus e'_n})}{2} + \frac{1+\tau}{2} \frac{1-\delta}{1+\tau} \int_{e'_n} M(x_n(t)) d\mu \\ &\quad + \frac{1-\tau}{2} R_M\left(\frac{x}{1-\tau}\chi_{e'_n}\right) \\ &\leq \frac{R_M(x_n\chi_{G \setminus e'_n}) + R_M(x\chi_{G \setminus e'_n})}{2} + \frac{1-\tau}{2} o(\mu e'_n) + \frac{R_M(x_n\chi_{e'_n})}{2} - \frac{\delta\varepsilon}{4} \\ &\leq \frac{R_M(x_n) + R_M(x)}{2} - \frac{\delta\varepsilon}{4} + o(\mu e'). \end{aligned}$$

If $n \rightarrow \infty$, it gives a contradiction: $1 \leq 1 - \delta\varepsilon/4$. So we get

$$\limsup_{n \rightarrow \infty} \int_{E \cup F_b^+(n)} M(x_n(t)) d\mu = \limsup_{n \rightarrow \infty} \int_{E \cup F_b^-(n)} M(x(t)) d\mu$$

hence

$$\liminf_{n \rightarrow \infty} \int_{F_b^-(n)} M(x_n(t)) d\mu = \liminf_{n \rightarrow \infty} \int_{F_b^-(n)} M(x(t)) d\mu.$$

From $\liminf_{n \rightarrow \infty} \int_{F_b^-(n)} (M(x(t)) - M(x_n(t))) d\mu = 0$, it follows that $x_n - x \xrightarrow{\mu} 0$ on $F_b^-(n)$, and consequently $x_n - x \xrightarrow{\mu} 0$ on G . \square

By an analogous argumentation, we get the same result for Orlicz sequence spaces.

Theorem 2

- Let $x \in S(l_M)$. Then x is a W^* URP point if and only if
- (1) there is $\tau > 0$, $R(\frac{x}{1-\tau}) < \infty$
 - (2) if there is “ i ”, $|x(i)| \in (a, b]$, then $M \in \nabla_2$ and there is no “ j ”, $j \neq i$ with $|x(j)| \in [c, d]$, where $[a, b]$ and $[c, d]$ are affine segments of $M(\cdot)$.

References

1. T. Wang, Z. Ren and Y. Zhang, On UR point and WUR point of Orlicz spaces, *J. Math.* **13** (1993), 443–452.
2. Y. Li, Lec Pengyee, T. Wang, On UR point and WUR point of Orlicz sequence spaces, *Ann. Math. Res.* **27** (1994).
3. M.A. Krasnoselskii and Ya. B. Rutiskii, *Convex function and Orlicz spaces*, Groningen 1961.