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On W^*UR point and UR point of Orlicz spaces with Orlicz norm*

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Abstract

For Orlicz spaces with Orlicz norm, a criterion of W^*UR point is given, and previous results about UR points and WUR points are amended.

In the sequel, X denotes a Banach space, $S(X)$ and $B(X)$ denote the unit sphere and ball of X respectively. $x \in S(X), x$ is said to be an uniformly round (UR) point, uniformly weak round (WUR) point and uniformly weak star round (W^*UR) point provided that $x_n \in S(X), \|x_n + x\| \to 2$ imply $\|x_n - x\| \to 0, x_n - x \xrightarrow{w} 0$ and $x_n - x \xrightarrow{w^*} 0$, respectively.

$M(u)$ and $N(v)$ denote a pair of complemented N-functions, $p(u)$ denotes the right-side derivative of $M(u)$. $M \in \Delta_2 (M \in \nabla_2)$ denotes that $M(u)$ satisfies Δ_2-condition (∇_2-condition) for large u. S_M denotes the set of all strictly convex points of $M(u)$. $\{a'\}$ (respectively, $\{b'\}$) denotes the sets of all left-extreme points (resp., right-extreme points) of affine segments of $M(u)$ with $p_-(a') = p(a')$ (resp., $p_-(b') = p(b)$), but for $\{a\}, \{b\}$ with $p_-(a) < p(a)$ and $p_-(b) < p(b)$.

(G, Σ, μ) denotes a non-atomic finite measure space, $x(t)$ denotes a measurable real function. We call $R_M(x) = \int_G M(x(t))d\mu$ a modular of x. By an Orlicz space we shall mean the space $L_M(G, \Sigma, \mu) = \{x(t)\colon \text{ for some } c > 0, R_M(cx) < \infty\}$ equipped with the norm

$$\|x\|^0 = \sup_{R_N(y) \le 1} \int_G x(t)y(t)d\mu.$$

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For $x \in L_M$, we denote

$$\xi(x) = \inf\left\{c > 0: R_M\left(\frac{x}{c}\right) < \infty\right\}$$
$$k_x^* = \inf\left\{k > 0: R_N(p(kx)) = \int_G N\Big(p\big(kx(t)\big)\Big)d\mu \geq 1\right\}$$
$$k_x^{**} = \sup\left\{k > 0: R_N(p(kx)) \leq 1\right\}, \mathbb{K}(x) = [k_x^*, k_x^{**}]$$

It is well known that $k \in \mathbb{K}(x)$ if and only if $\|x\|^0 = \frac{1}{k}(1 + R_M(kx))$.

For each $x \in L_M$, there is a supporting functional $f = y + \varphi$, i.e., $f(x) = \int_G x(t)y(t)d\mu + \varphi(x)$, $\|f\| = 1$ and $f(x) = \|x\|^0$ where $y \in L_N$ and φ is singular.

The criteria of UR and WUR points in L_M have been discussed in [1-3], but in [1, 2] they are restricted to $p(u)$ continuous, and the condition given in [3] is not necessary. Even without the restriction in that paper, we first give the criterion of W^*UR point and deduce easily the criteria of WUR and UR points.

Lemma 1

For any $0 < \lambda < 1, 0 < \delta < 1$ and $0 < \varepsilon < 1$ there is $0 < \delta' < 1$ such that for u, v, with $M(\lambda u + (1-\lambda)v) \leq (1-\delta)(\lambda M(u) + (1-\lambda)M(v))$ we have that for all $\lambda' \in [\varepsilon, 1-\varepsilon]$

$$M\big(\lambda' u + (1-\lambda')v\big) \leq (1-\delta')\big(\lambda' M(u) + (1-\lambda')M(v)\big).$$

Proof. The proof is easy and is left to the reader. □

Lemma 2

Let $0 \neq x \in L_M$. Then $f = y + \varphi$ is a supporting functional of $x \Leftrightarrow R_N(y) + \|\varphi\| = 1, \|\varphi\| = \varphi(kx), p_-(kx(t)) \leq y(t) \leq p(kx(t))$ μ - a.e., for $k \in \mathbb{K}(x)$.

Proof. Necessity.
If f is the supporting functional of x, then

$$1 = \|f\| = \inf\left\{c > 0: R_N\left(\frac{y}{c}\right) + \frac{\|\varphi\|}{c} \leq 1\right\}$$
$$\|kx\|^0 = f(kx) = \int_G kx(t)y(t)d\mu + \varphi(kx) \leq \int_G kx(t)y(t)d\mu + \|\varphi\|$$
$$\leq R_M(kx) + R_N(y) + \|\varphi\| \leq R_M(kx) + 1 = \|kx\|^0 .$$

So we get $\int_G kx(t)y(t)d\mu = R_M(kx) + R_N(y)$, thus $p_-(kx(t)) \leq y(t) \leq p(kx(t))$. Moreover $\|\varphi\| = \varphi(kx)$ and $R_N(y) + \|\varphi\| = 1$.

The sufficiency part of the proof is clear. □

For the convenience of the reader, we split the main result into several lemmas.

Lemma 3

Let $x \in S(L_M)$ and $k \in \mathbb{K}(x)$. If x is a W^*UR point then $M \in \nabla_2$.

Proof. Take $y_n(t), R_N(y_n) = 1$ with $\int_G x(t)y_n(t)d\mu > 1 - 1/n$. Let d be such that $\mu G_d = \mu\{t \in G: |x(t)| \le d\} > 0$.

Suppose that $M \notin \nabla_2$. Then there exists a sequence (v_n) with $v_n \nearrow \infty, N(\frac{v_n}{1-\frac{1}{n}}) > 2nN(v_n)$. Take $G_n \subset G_d, N(v_n)\mu G_n = 1/n$ and define $z_n(t) = v_n\chi_{G_n}(t)$. Then $R_N(z_n) = 1/n < 1$ and $R_N(\frac{z_n}{1-\frac{1}{n}}) > 2nN(v_n)\mu G_n = 2$ so we get $1 \ge \|z_n\|_N \ge 1 - 1/n$, where $\|y\|_N = \inf\{c > 0: R_N(y/c) \le 1\}$ (it is called the Luxemburg norm of y). By [4], there are $x_n(t) = u_n\chi_{G_n}(t), \|x_n\|^0 = 1$ such that

$$u_n v_n \mu G_n = \int_{G_n} x_n(t)z_n(t)d\mu = \|z_n\|_N > 1 - \frac{1}{n}.$$

Put $g_n(t) = (1 - \frac{1}{n})(y_n(t)\chi_{G\setminus G_n}(t) + z_n(t))$. Then

$$R_N(g_n) \le \Big(1 - \frac{1}{n}\Big)(R_N(y_n) + R_N(z_n)) \le \Big(1 - \frac{1}{n}\Big)\Big(1 + \frac{1}{n}\Big) = 1 - \frac{1}{n^2}.$$

Thus

$$\begin{aligned}
\|x_n + x\|^0 &\ge \int_G (x_n(t) + x(t))g_n(t)d\mu \\
&= \Big(1 - \frac{1}{n}\Big)\Big(\int_{G\setminus G_n} x_n(t)y_n(t)d\mu + \int_{G\setminus G_n} x(t)y_n(t)d\mu \\
&\quad + \int_{G_n} x_n(t)z_n(t)d\mu + \int_{G_n} x(t)z_n(t)d\mu\Big) \\
&\ge \Big(1 - \frac{1}{n}\Big)\Big(0 + \int_G x(t)y_n(t)d\mu - \int_{G_n} x(t)y_n(t)d\mu + u_n v_n \mu G_n \\
&\quad - \Big|\int_{G_n} x(t)z_n(t)d\mu\Big|\Big) \\
&\ge \Big(1 - \frac{1}{n}\Big)\Big(1 - \frac{1}{n} - d \cdot \|\chi_{G_n}\|^0 + 1 - \frac{1}{n} - d\|\chi_{G_n}\|^0\Big) \to 2
\end{aligned}$$

as $n \to \infty$. Take $h \in E_N$, with $\int_G x(t)h(t)d\mu > 0$, then

$$\begin{aligned}
\int_G (x(t) - x_n(t))h(t)d\mu &\ge \int_G x(t)h(t)d\mu - \|x_n\|^0 \cdot \|h\chi_{G_n}\|^0 \\
&\to \int_G x(t)h(t)d\mu > 0
\end{aligned}$$

which contradicts the fact that x is a W^*UR point. □

Lemma 4

Let $x \in S(L_M)$. If x is a W^*UR point then

$$\mu\{t \in G: |kx(t)| \in \mathbb{R} \setminus S_M \cup \{a'\} \cup \{b'\}\} = 0$$

for every $k \in \mathbb{K}(x)$.

Proof. Since a W^*UR point is an extreme point, by [4], it follows

$$\mu\{t \in G: |kx(t)| \in \mathbb{R} \setminus S_M\} = 0\,.$$

Denote $G_{a'} = \{t \in G: |kx(t)| \in \{a'\}\}, G_{b'} = \{t \in G: |kx(t)| \in \{b'\}\}$. Suppose that $\mu G_{b'} > 0$. Without loss of generality, we can assume that $\mu G_0 = \mu\{t \in G: kx(t) = b'\} > 0$ for some $b' \in \{b'\}, b' > 0$. Define

$$y(t) = x(t)\chi_{G\setminus G_0}(t) + \frac{a'}{k}\chi_{G_0}(t),$$

where $a' \in \{a'\}$ is such that M is affine on the interval $[a', b']$.

For any $\eta > 0$,

$$\begin{aligned} R_N\big(p((1-\eta)ky)\big) &\le R_N\big(p((1-\eta)kx)\big) \\ &\le R_N\big(p((1-\eta)k_x^{**}x)\big) \le 1 \end{aligned}$$

and $R_N(p((1+\eta)ky)) \ge 1$. Indeed, when $R_N(p((1+\eta)kx)) = \infty$, we have $R_N(p((1+\eta)ky)) \ge R_N(p((1+\eta)kx\chi_{G\setminus G_0}(t))) = \infty$; in the other case, i.e., if $R_N(p((1+\eta)kx)) < \infty$ we have

$$\begin{aligned} R_N\big(p((1+\eta)ky)\big) &= \int_{G\setminus G_0} N\big(p((1+\eta)kx(t))\big)d\mu + N\big(p((1+\eta)a')\big)\mu G_0 \\ &\ge \lim_{s\to 0}\int_{G\setminus G_0} N\big(p((1+s)kx(t))\big)d\mu + N\big(p((1+s)a')\big)\mu G_0 \\ &= \lim_{s\to 0} R_N\big(p((1+s)kx)\big) \ge 1\,. \end{aligned}$$

Hence $k \in \mathbb{K}(y)$.

Let $f = v + \varphi$ be a supporting functional of x. By Lemma 2, we get that $R_N(v) + \|\varphi\| = 1, \|\varphi\| = \varphi(kx)$ and $p_-(kx(t)) \le v(t) \le p(kx(t))$. Clearly $\varphi(ky) = \varphi(kx)$ and $p_-(ky(t)) \le v(t) \le p(ky(t))$, so f is the supporting functional of y too, i.e., $f(y) = \|y\|^0$. Therefore

$$f\Big(x + \frac{y}{\|y\|^0}\Big) = f(x) + \frac{f(y)}{\|y\|^0} = 2$$

we get $\|x + y/\|y\|^0\|^0 = 2$. But $x \neq y/\|y\|^0$, which contradicts the fact that x is a W^*UR point.

If we suppose that $\mu G_{a'} > 0$, without loss of generality, we can assume that for some $a' \in \{a'\}, a' > 0$ we have $\mu G_0 = \mu\{t \in G: kx(t) = a'\} > 0$. Define

$$y(t) = x(t)\chi_{G\setminus G_0}(t) + \frac{a' + b'}{2k}\chi_{G_0}(t),$$

where $b' \in \{b'\}$ is such that M is affine on the interval $[a', b']$. For any $\eta > 0$ sufficiently small, $R_N(p((1+\eta)ky)) \geq R_N(p((1+\eta)kx)) \geq 1$, and $R_N(p((1-\eta)ky)) \leq 1$, which shows that $k \in \mathbb{K}(y)$. Indeed if suppose that

$$\begin{aligned} 1 < R_N\big(p(1-\eta)ky)\big) &= \int_{G\setminus G_0} N\big(p((1-\eta)kx(t))\big)d\mu + N\big(p(a')\big)\mu G_0 \\ &= \int_{G\setminus G_0} N\big(p((1-\eta)kx(t))\big)d\mu + N\big(p_-(a')\big)\mu G_0 \,. \end{aligned}$$

By $p_-(a') = p(a')$, it follows that there is $0 < \xi < \eta$ such that

$$R_N\big(p((1-\xi)kx)\big) \geq \int_{G\setminus G_0} N\big(p((1-\eta)kx(t))\big)d\mu + N\big(p((1-\xi)a')\big)\mu G_0 > 1$$

which is a contradiction with $k \in \mathbb{K}(x)$.

Finally one can reach a contradiction analogously as in the case $\mu G_{b'} > 0$. □

Lemma 5

Let $x \in S(L_M)$ and $k \in \mathbb{K}(x)$. If x is a W^*UR point, then
(i) $\mu G_a > 0$ implies $R_N(p_-(kx)) = 1$
(ii) $\mu G_b > 0$ implies $R_N(p(kx)) = 1$ and, for some $0 < \tau < 1$,

$$R_N\left(p\Big(\frac{kx}{1-\tau}\Big)\right) < \infty$$

where $G_a = \{t \in G: |kx(t)| \in \{a\}\}, G_b = \{t \in G: |kx(t)| \in \{b\}\}$.

Proof. (i) If $\mu G_a > 0$, then $\mu G_0 = \mu\{t \in G: |kx(t)| = a\} > 0$ for some $a \in \{a\}$. We first show $R_N(p(kx)) \geq 1$. Indeed, if $R_N(p(kx)) < 1$, then, for the supporting functional $f = v' + \varphi$ of x, we have $\varphi \neq 0$. Take $\lambda > 0$ with

$$R_N\big(p(kx)\big) + \|\lambda\varphi\| = 1\,.$$

Put $y(t) = x(t)\chi_{G\setminus G_0}(t)+(a+b)/(2k)\chi_{G_0}(t)$. Then $R_N(p(ky)) = R_N(p(kx)) < 1$ and $R_N(p((1+\eta)ky)) \geq R_N(p((1+\eta)kx)) \geq 1$, so $k \in \mathbb{K}(y)$.

Put $v(t) = p(kx)\chi_{G\setminus G_0}(t) + p(a)\chi_{G_0}(t)$. Then $f = v + \lambda\varphi$ is the supporting functional of y, since $R_N(v) + \lambda\|\varphi\| = 1$, $\|\lambda\varphi\| = \lambda\|\varphi\| = \lambda\varphi(kx) = \lambda\varphi(ky)$, and $v(t) = p(ky(t))$. Further, f is the supporting functional of x too. Therefore $f(x + y/\|y\|^0) = 2$, $\|x + y/\|y\|^0\|^0 = 2$. But $x \neq y/\|y\|^0$, which contradicts the fact that x is a W^*UR point.

Now suppose that $R_N(p_-(kx)) \neq 1$. Then $R_N(p_-(kx)) < 1$. We shall consider two cases.

Assume that $\int_{G\setminus G_0} N(p_-(kx(t)))d\mu + N(p(a))\mu G_0 < 1$.
Since $R_N(p(kx)) \geq 1$, there exists $v(t)$, such that $v(t) = p(a) \quad (t \in G_0)$; $p_-(kx(t)) \leq v(t) \leq p(kx(t)) \quad (t \in G \setminus G_0)$ and $R_N(v) = 1$. Hence v is the supporting functional of x. Define $y(t) = x(t)\chi_{G\setminus G_0}(t) + (a+b)/(2k)\chi_{G_0}(t)$. Since $R_N(p(ky)) \geq R_N(p(kx)) \geq 1$ and

$$R_N(p((1-\eta)ky)) \leq \int_{G\setminus G_0} N(p_-(kx(t)))d\mu + N(p(a))\mu G_0 < 1\,,$$

$k \in \mathbb{K}(y)$ and v is the supporting functional of y too.
From $\int_G (x(t) + y(t)/\|y\|^0)v(t)d\mu = 2$, we get $\|x + y/\|y\|^0\|^0 = 2$. But $x \neq y/\|y\|^0$ which contradicts the fact that x is a W^*UR point.

Now, assume that $\int_{G\setminus G_0} N(p_-(kx(t)))d\mu + N(p(a))\mu G_0 \geq 1$.

Take $\widetilde{G}_0 \subset G_0$ with $\int_{G\setminus \widetilde{G}_0} N(p_-(kx(t)))d\mu + N(p(a))\mu\widetilde{G}_0 = 1$. Define $v(t) = p_-(kx(t))\chi_{G\setminus\widetilde{G}_0}(t) + p(a)\chi_{\widetilde{G}_0}(t)$, then $R_N(v) = 1$ and v is the supporting functional of x. Put $y(t) = x(t)\chi_{G\setminus\widetilde{G}_0}(t) + (a+b)/(2k)\chi_{\widetilde{G}_0}(t)$. Then $R_N(p(ky)) \geq R_N(p(kx)) \geq 1$ and $R_N(p((1-\eta)ky)) \leq \int_{G\setminus\widetilde{G}_0} N(p_-(kx(t)))d\mu + N(p(a))\mu\widetilde{G}_0 = 1$, so $k \in \mathbb{K}(y)$. Clearly v is the supporting functional of y too. Hence $\int_G (x(t) + y(t)/\|y\|^0)v(t)d\mu = 2$, so $\|x + y/\|y\|^0\|^0 = 2$. But $x \neq y/\|y\|^0$, a contradiction.

(ii) If $\mu G_b > 0$, then $\mu G = \mu\{t \in G: |kx(t)| = b\} > 0$, for some $b \in \{b\}$. Supposing that $R_N(p(kx)) > 1$ and applying the fact that $R_N(p_-(kx)) \leq 1$, analogously as in (i) we obtain a contradiction. Thus $R_N(p(kx)) \leq 1$.

Now we show that $R_N(p(\frac{kx}{1-\tau})) < \infty$ for some $\tau > 0$. In fact if we suppose that $R_N(p((1+\varepsilon)kx)) = \infty$ for any $\varepsilon > 0$, then $M \notin \Delta_2$, i. e., there exists a sequence $u_n \nearrow \infty, M(u_n)/u_n p(u_n) \to 0$. Since, by Lemma 3, $M \in \nabla_2$, there is $d > 0$ such that $up(u) \leq dN(p(u))$ for every large u, so $M(u_n)/N(p(u_n)) \to 0$. Take $G_n \subset G_0$ satisfying

$$\int_{G\setminus G_0} N(p(kx(t)))d\mu + N(p(a))\mu(G_0 \setminus G_n) + N(p(u_n))\mu G_n = 1\,.$$

Clearly $\mu G_n \to 0$ $(n \to \infty)$. Define

$$y_n(t) = x(t)\chi_{G\setminus G_0}(t) + \frac{a}{k}\chi_{G_0\setminus G_n}(t) + \frac{u_n}{k}\chi_{G_n}(t)\,.$$

From $R_N(p(ky_n)) = 1$, it follows that $k \in \mathbb{K}(y_n), k\|y_n\|^0 \in \mathbb{K}(y_n/\|y_n\|^0)$. Hence

$$\begin{aligned} k'_n = k\|y_n\|^0 &= 1 + R_M(ky_n) \\ &= 1 + \int_{G\setminus G_0} M\big(kx(t)\big)d\mu + M(a)\mu(G_0 \setminus G_n) + M(u_n)\mu G_n \\ &\longrightarrow 1 + \int_{G\setminus G_0} M\big(kx(t)\big)d\mu + M(a)\mu G_0 = k' < k\,. \end{aligned}$$

Since

$$\begin{aligned} \frac{k\cdot k'_n}{k+k'_n}\Big(x(t) + \frac{y_n(t)}{\|y_n\|^0}\Big) &= \frac{k'_n}{k+k'_n}kx(t) + \frac{k}{k+k'_n}ky_n(t) \\ &= \begin{cases} kx(t) & (t \in G \setminus G_0) \\ \dfrac{k'_n}{k+k'_n}b + \dfrac{k}{k+k'_n}a & (t \in G_0 \setminus G_n) \\ \dfrac{k'_n}{k+k'_n}b + \dfrac{k}{k+k'_n}u_n & (t \in G_n), \end{cases} \end{aligned}$$

we have

$$\begin{aligned} R_N\left(p\Big(\frac{k\cdot k'_n}{k+k'_n}(x + \frac{y_n}{\|y_n\|^0})\Big)\right) < \int_{G\setminus G_0} & N(p(kx(t)))d\mu \\ & + N\big(p(a)\big)\mu(G_0 \setminus G_n) + N\big(p(u_n)\big)\mu G_n = 1 \end{aligned}$$

and

$$R_n\Big(\frac{k\cdot k'_n}{k+k'_n}(1+\eta)\Big(x + \frac{y_n}{\|y_n\|^0}\Big)\Big) > \int_{G\setminus G_0} N\big(p((1+\eta)kx(t))\big)d\mu = \infty.$$

Hence $\frac{k\cdot k'_n}{k+k'_n} \in \mathbb{K}(x + y/\|y\|^0)$. Therefore

$$\begin{aligned}
\Big\|x+\frac{y_n}{\|y_n\|^0}\Big\|^0 &= \frac{k+k_n'}{k\cdot k_n'}\Big(1+R_M\Big(\frac{k\cdot k_n'}{k+k_n'}\Big(x+\frac{y_n}{\|y_n\|^0}\Big)\Big)\Big)\\
&\geq \frac{k+k_n'}{k\cdot k_n'}\Big[1+\int_{G\setminus G_0} M(kx(t))d\mu + M\Big(\frac{k_n'}{k_n'+k}b+\frac{k}{k_n'+k}a\Big)\mu(G_0\setminus G_n)\Big]\\
&= \frac{1}{k}\Big[1+\int_{G\setminus G_0} M(kx(t))d\mu + M(b)\mu(G_0\setminus G_n)\Big]\\
&\quad + \frac{1}{k_n'}\Big[1+\int_{G\setminus G_0} M(kx(t))d\mu + M(a)\mu(G_0\setminus G_n)\Big]\\
&\longrightarrow \frac{1}{k}(1+R_M(kx)) + \frac{1}{k'}\Big(1+\int_{G\setminus G_0} M(kx(t))d\mu + M(a)\mu G_0\Big) = 2\,.
\end{aligned}$$

But

$$\begin{aligned}
\int_G \Big(\frac{y_n(t)}{\|y_n\|^0}-x(t)\Big)\operatorname{sign} x(t)\chi_{G\setminus G_0}(t)d\mu &= \int_{G\setminus G_0}\Big(\frac{k}{k_n'}-1\Big)|x(t)|d\mu\\
&\longrightarrow \Big(\frac{k}{k'}-1\Big)\int_{G\setminus G_0}|x(t)|d\mu > 0
\end{aligned}$$

which contradicts the fact that x is a W^*UR point. Thus there is $0 < \tau < 1$, such that $R_N(p(\frac{kx}{1-\tau})) < \infty$. Applying the right-hand-side continuity of $p(u)$, we get $R_M(kx) \geq 1$. Since we have verified that $R_M(p(kx)) \leq 1$, we deduce $R_M(p(kx)) = 1$. □

Lemma 6

Let us assume that $x \in S(L_M), k \in \mathbb{K}(x)$ and x is a W^*UR point with $R_N(p(kx)) < 1$. Then, for any $\varepsilon, \varepsilon' > 0$, there is $\delta > 0$ such that, for any measurable function $u(t)$ and $e \subset G$ satisfying $\mu e < \delta$ and $\varepsilon\varepsilon' \leq \varepsilon' u(t) \leq kx(t) \leq u(t)$ for all $t \in e$, if $M(u(t)) \geq \varepsilon N(p_-(u(t)))$ and

$$M\Big(\frac{u(t)+kx(t)}{2}\Big) > (1-\delta)\frac{M(u(t))+M(kx(t))}{2},$$

then $R_M(u\chi_e) < \varepsilon$.

Proof. Suppose on the contrary, that for some $\varepsilon > 0$, for all n, there exist $u_n(t)$ and e_n with $\mu e_n < 1/n$, such that $\varepsilon^2 \leq \varepsilon u_n(t) \leq kx(t) \leq u_n(t)$ for $t \in e_n$, $M(u_n(t)) \geq \varepsilon N(p_-(u_n(t)))$, $M(\frac{u_n(t)+kx(t)}{2}) > (1-\frac{1}{n})\frac{M(u_n(t)+M(kx(t))}{2}$ but $R_M(u_n\chi_{e_n}) \geq \varepsilon$.

If $R_N(p_-(kx)) + R_N(p_-(u_n\chi_{e_n})) \leq 1$, put $E_n = e_n$. Then $R_M(u_n\chi_{E_n}) \geq \varepsilon$. If $R_N(p_-(kx)) + R_N(p_-(u_n\chi_{e_n})) > 1$, take $E_n \subset e_n$, such that

$$R_N\big(p_-(kx)\big) + R_N\big(p_-(u_n\chi_{E_n})\big) = 1\,.$$

Then

$$R_M(u_n\chi_{E_n}) \geq \varepsilon R_N\big(p_-(u_n\chi_{E_n})\big) \geq \varepsilon\big(1 - R_N(p_-(kx))\big) = (1-\theta)\varepsilon.$$

Define

$$x_n(t) = x(t)\chi_{G\setminus E_n}(t) + \frac{1}{k}u_n(t)\chi_{E_n}(t) \quad (n = 1, 2, \ldots)\,.$$

Since $R_N(p((1-\eta)kx_n)) \leq \int_{G\setminus E_n} N(p_-(kx(t)))d\mu + \int_{E_n} N(p_-(u_n(t)))d\mu \leq 1$ and $R_N(p((1+\eta)kx_n)) \geq R_N(p((1+\eta)kx)) \geq 1$, from $kx(t) \leq u_n(t)$, we get that $k \in \mathbb{K}(x_n)$ $(n = 1, 2, \ldots)$, $k\|x_n\|^0 \in \mathbb{K}(x_n/\|x_n\|^0)$. Since

$$k_n = k\|x_n\|^0 = 1 + R_M(kx\chi_{G\setminus E_n}) + R_M(u_n\chi_{E_n}) \to k' \geq k + (1-\theta)\varepsilon$$

and

$$\begin{aligned}\frac{k\cdot k_n}{k+k_n}\Big(x(t) + \frac{x_n(t)}{\|x_n\|^0}\Big) &= \frac{k_n}{k+k_n}kx(t) + \frac{k}{k+k_n}kx_n(t)\\ &= \begin{cases} kx(t) & t \in G\setminus E_n \\ \dfrac{k_n}{k+k_n}kx(t) + \dfrac{k}{k+k_n}u_n(t) & t \in E_n \end{cases}\end{aligned}$$

we have that $R_N(p((1+\eta)\frac{k\cdot k_n}{k+k_n}(x + x_n/\|x_n\|^0))) \geq R_N(p((1+\eta)kx)) \geq 1$ and $R_N(p((1-\eta)\frac{k\cdot k_n}{k+k_n}(x + x_n/\|x_n\|^0))) \leq R_N(p_-(kx)\chi_{G\setminus E_n}) + R_N(p_-(u_n\chi_{E_n})) \leq 1$, so $k\cdot k_n/(k+k_n) \in \mathbb{K}(x + x_n/\|x_n\|^0)$. By Lemma 1, it follows that there exists $\delta_n \downarrow 0$ such that if $M(\frac{u+v}{2}) \geq (1-\frac{1}{n})\frac{M(u)+M(v)}{2}$, then $M(\lambda_n u + (1-\lambda_n)v) \geq (1-\delta_n)(\lambda_n M(u) + (1-\lambda_n)M(v))$ for $\lambda_n \in [\frac{1}{1+\overline{k}}, \frac{\overline{k}}{1+\overline{k}}]$, where $\overline{k} = \sup k_n < \infty$

(because of $M \in \nabla_2$). Thus we have

$$\begin{aligned}\Big\|x+\frac{x_n}{\|x_n\|^0}\Big\|^0 &= \frac{k+k_n}{k\cdot k_n}\bigg(1+\int_{G\setminus E_n} M\big(kx(t)\big)d\mu \\ &\quad + \int_{E_n} M\Big(\frac{k_n}{k+k_n}kx(t)+\frac{k}{k+k_n}u_n(t)\Big)d\mu\bigg) \\ &\geq \frac{k+k_n}{k\cdot k_n}\bigg(1+\int_{G\setminus E_n} M\big(kx(t)\big)d\mu \\ &\quad +(1-\delta_n)\int_{E_n}\Big[\frac{k_n}{k+k_n}M\big(kx(t)\big)+\frac{k}{k+k_n}M\big(u_n(t)\big)\Big]d\mu\bigg) \\ &= \frac{1}{k}\bigg(1+\int_{G\setminus E_n} M\big(kx(t)\big)d\mu+(1-\delta_n)\int_{E_n} M\big(kx(t)\big)d\mu\bigg) \\ &\quad +\frac{1}{k_n}\bigg(1+\int_{G\setminus E_n} M\big(kx(t)\big)d\mu+(1-\delta_n)\int_{E_n} M\big(u_n(t)\big)d\mu\bigg)\to 2\,.\end{aligned}$$

But $x_n/\|x_n\|^0 \xrightarrow{\mu} kx/k' \neq x$, which contradicts the fact that x is a W^*UR point. □

Lemma 7

If $M \in \nabla_2, 1=\|x\|^0=\frac{1}{k}(1+R_M(kx))$,

$$1=\|x_n\|^0=\frac{1}{k_n}\big(1+R_M(k_nx_n)\big), \qquad \|x_n+x\|^0\to 2$$

and

$$\mu\big\{t\in G: |kx(t)|\in \mathbb{R}\setminus S_M\cup\{a'\}\cup\{b'\}\cup\{a\}\cup\{b\}\big\}=0,$$

then $k_nx_n-kx \xrightarrow{\mu} 0$.

Proof. Suppose that $k_nx_n-kx \overset{\mu}{\not\to} 0$. Then there are $\varepsilon,\delta>0$ such that

$$\mu\big\{t\in G: |k_nx_n(t)-kx(t)|\geq\varepsilon\big\}\geq\sigma \quad (n=1,2,\ldots)\,.$$

Since $\|x_n\|^0=1, \overline{k}=\sup_n k_n<\infty$. Further, for any D,

$$\begin{aligned}\overline{k}&\geq k_n\geq R_M(k_nx_n) \\ &\geq \int_{\{t:|k_nx_n(t)|>D\}} M\big(k_nx_n(t)\big)d\mu\geq M(D)\mu\{t\in G: |k_nx_n(t)|>D\}\,.\end{aligned}$$

Thus $\mu\{t: |kx(t)| > D\} < \sigma/4, \mu\{t: |k_n x_n(t)| > D\} < \sigma/4$ $(n = 1, 2, \ldots)$ for some D large enough.

Denote all left and right extreme points of affine segments of $M(u)$ as $c_1, c_2, \ldots$ Since $kx(t) \neq c_i$ $(i = 1, 2, \ldots)$ there are open segments V_i including c_i with $\mu\{t: kx(t) \in V_i\} < \sigma/(2^i \cdot 4)$ $(i = 1, 2, \ldots)$, then

$$\mu\left\{t: kx(t) \in \bigcup_{i=1}^{\infty} V_i\right\} \leq \frac{\sigma}{4}.$$

Hence

$$\mu G_n = \mu\Big\{t \in G: |k_n x_n(t) - kx(t)| \geq \varepsilon, \\ |k_n x_n(t)|, |kx(t)| \leq D, kx(t) \in S_M \setminus \bigcup_{i=1}^{\infty} V_i\Big\} \geq \frac{\sigma}{4}.$$

For the bounded closed set of three dimension space

$$\left\{(u, v, \lambda): |u - v| \geq \varepsilon, |u|, |v| \leq D, v \in S_M \setminus \bigcup_{i=1}^{\infty} V_i, \lambda \in \left[\frac{1}{1 + \overline{k}}, \frac{\overline{k}}{1 + \overline{k}}\right]\right\}$$

there is a common $\delta, 0 < \delta < 1$ such that for all (u, v, λ) of the above set

$$M\big(\lambda u + (1 - \lambda) v\big) \leq (1 - \delta)\big(\lambda M(u) + (1 - \lambda) M(v)\big),$$

so, for $t \in G_n$,

$$M\Big(\frac{k \cdot k_n}{k + k_n}(x_n(t) + x(t))\Big) \leq (1 - \delta)\Big(\frac{k}{k + k_n} M(k_n x_n(t)) + \frac{k_n}{k + k_n} M(kx(t))\Big).$$

Hence

$$
\begin{aligned}
0 &\leftarrow \|x_n\|^0 + \|x\|^0 - \|x_n + x\|^0 \\
&\geq \frac{1}{k_n}(1 + R_M(k_n x_n)) + \frac{1}{k}(1 + R_M(kx)) \\
&\quad - \frac{k+k_n}{k\cdot k_n}\left(1 + R_M\Big(\frac{k\cdot k_n}{k+k_n}(x_n + x)\Big)\right) \\
&= \frac{k+k_n}{k\cdot k_n}\int_G \Big[\frac{k}{k+k_n}M\big(k_n x_n(t)\big) + \frac{k_n}{k+k_n}M\big(kx(t)\big) - M\Big(\frac{k\cdot k_n}{k+k_n}(x_n(t) \\
&\quad + x(t))\Big)\Big]d\mu \\
&\geq \frac{k+k_n}{k\cdot k_n}\int_{G_n} \Big[\frac{k}{k+k_n}M\big(k_n x_n(t)\big) + \frac{k_n}{k+k_n}M\big(kx(t)\big) - M\Big(\frac{k\cdot k_n}{k+k_n}(x_n(t) \\
&\quad + x(t))\Big)\Big]d\mu \\
&\geq \frac{k+k_n}{k\cdot k_n}\delta\int_{G_n}\Big(\frac{k}{k+k_n}M\big(k_n x_n(t)\big) + \frac{k_n}{k+k_n}M\big(kx(t)\big)\Big)d\mu \\
&\geq \frac{1}{k}\delta M\Big(\frac{\varepsilon}{2}\Big)\frac{\delta}{4} > 0
\end{aligned}
$$

this contradiction shows that $k_n x_n - kx \xrightarrow{\mu} 0$. □

Lemma 8

Let $x \in S(L_M)$ and $k \in \mathbb{K}(x)$. Then for any $\varepsilon > 0$ there is y such that $\|x - y\|^0 < \varepsilon, k \in \mathbb{K}(y)$ and $R_N(p(\frac{ky}{1-\frac{\varepsilon}{2}})) < \infty$.

Proof. If $R_N(p((1+\eta)kx)) < \infty$ for some $\eta > 0$, we can take $y = x$. Assume that $R_N(p((1+\eta)kx)) = \infty$ for every $\eta > 0$. Let $\varepsilon > 0$. Since $R_N(p((1-\varepsilon)kx)) \leq 1$ and $R_N(p((1+\varepsilon)kx)) = \infty$, we can find $G_c = \{t \in G\colon |x(t)| > c\}$ with

$$
R_N\Big(p((1+\varepsilon)kx)\chi_{G\setminus G_c}\Big) + R_N\Big(p((1-\varepsilon)kx)\chi_{G_c}\Big) = 1 .
$$

Clearly $\mu G_c > 0$. Put $y(t) = (1+\varepsilon)x\chi_{G\setminus G_c} + (1-\varepsilon)x\chi_{G_c}$. Then $\|y - x\|^0 = \varepsilon, k \in \mathbb{K}(y), R_N(p(ky)) = 1$ and

$$
\begin{aligned}
R_N\left(p\Big(\frac{ky}{1-\frac{\varepsilon}{2}}\Big)\right) &= R_N\left(p\Big(\frac{1+\varepsilon}{1-\frac{\varepsilon}{2}}kx\Big)\chi_{G\setminus G_c}\right) + R_N\left(p\Big(\frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}kx\Big)\chi_{G_c}\right) \\
&\leq N\left(p\Big(\frac{1+\varepsilon}{1-\frac{\varepsilon}{2}}c\Big)\right)\mu G + R_N(p_-(kx)) < \infty .
\end{aligned}
$$

Theorem 1

Let $x \in S(L_M)$ and let $k \in \mathbb{K}(x)$. A point x is W^*UR if and only if

(1) $M \in \nabla_2$

(2) $\mu\{t \in G: |kx(t)| \in \mathbb{R} \setminus S_M \cup \{a'\} \cup \{b'\}\} = 0$

(3) if $\mu G_a > 0$ *then* $R_N(p_-(kx)) = 1$;
if $\mu G_b > 0$ *then* $R_N(p(kx)) = 1$ *and* $R_N(p\frac{kx}{1-\tau})) < \infty$, *for some* $0 < \tau < 1$.

(4) If $R_N(p_-(kx)) < 1$, *then for any* $\varepsilon, \varepsilon' > 0$ *there is* $\delta > 0$ *such that for any measurable function* $u(t)$ *and* $e \subset G$ *satisfying* $\mu e < \delta$ *and* $\varepsilon\varepsilon' \leq \varepsilon' u(t) \leq kx(t) \leq u(t)$ *for all* $t \in e$, *if* $M(u(t)) \geq \varepsilon N(p_-(u(t)))$ *and*

$$M\Big(\frac{u(t) + kx(t)}{2}\Big) > (1-\delta)\frac{M(u(t)) + M(kx(t))}{2},$$

then $R_M(u\chi_e) < \varepsilon$.

Proof. Necessity, see Lemmas 3-6.
Sufficiency. We shall discuss three cases.

Let $\|x\|^0 = \frac{1}{k_n}(1 + R_M(k_n x_n)) = 1, \|x_n + x\|^0 \to 2$. Without loss of generality we can assume that $x_n(t) \geq 0, x(t) \geq 0$ and, by Lemma 8, $R_N(p(\xi_n k_n x_n)) < \infty$ for some $\xi_n > 1$.

(I) $\mu G_a = \mu G_b = 0$

In this case, by (2) and Lemma 7, it follows that $k_n x_n - kx \xrightarrow{\mu} 0$. To prove that $x_n - x \xrightarrow{\mu} 0$, it is enough to show $k_n \to k$. Note that $k_n - k = R_M(k_n x_n) - R_M(kx)$, so by the Egoroff theorem, it suffices to prove $\lim\limits_{\mu e \to 0} \sup\limits_n R_M(k_n x_n \chi_e) = 0$. This will be split also into two cases.

(I1) $R_N(p_-(kx)) = 1$.

Assume that $k_n x_n(t) \to kx(t)$, $t \in G, \mu$ - a.e., passing to a subsequence if necessary. Since $p_-(u)$ is left-continuous and nondecreasing, we get $\liminf\limits_{n\to\infty} N(p_-(k_n x_n(t))) \geq N(p_-(kx(t)))$, $t \in G, \mu$ - a.e. Since $R_N(p_-(k_n x_n)) \leq 1$ it follows that $1 \geq \liminf\limits_{n\to\infty} R_N(p_-(k_n x_n)) \geq R_N(p_-(kx)) = 1$. Since $\mu(G) < \infty$ we get $\lim\limits_{\mu e\to 0} \sup\limits_n R_N(p_-(k_n x_n \chi_e)) = 0$. From $M \in \nabla_2, M(u) \leq up_-(u) \leq dN(p_-(u))$ we deduce

$$\lim_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_e) = 0$$

(I2) $R_N(p_-(kx)) < 1$.

Let $\overline{k} = \sup\limits_n k_n$ and $\varepsilon > 0$. Then there exists $0 < \delta' < 1$ such that $M(\frac{k+k_n/2}{k+k_n}u) \leq (1-\delta')\frac{k+k_n/2}{k+k_n}M(u)$, for every $u \geq \varepsilon$. Let $0 < \varepsilon' < \delta'/\overline{k}$ and let $\delta > 0$ be taken from (4) for those ε and ε'.

For $e \subset G$ and each n, denote

$$A_n = \Big\{t \in e : k_n x_n(t) < \varepsilon \quad \text{or} \quad k_n x_n(t) < kx(t) \\ \text{or} \quad M\big(k_n x_n(t)\big) < \varepsilon N\big(p_-(k_n x_n(t))\big)\Big\}$$

$$B_n = \Big\{t \in e \setminus A_n : M\Big(\frac{k \cdot k_n}{k+k_n}(x_n(t)+x(t))\Big) \le (1-\delta)\Big(\frac{k}{k+k_n} M\big(k_n x_n(t)\big) \\ + \frac{k_n}{k+k_n} M\big(kx(t)\big)\Big) \quad \text{or} \quad kx(t) < \varepsilon' k_n x_n(t)\Big\}$$

$$C_n = e \setminus A_n \setminus B_n = \Big\{t \in e : \varepsilon\varepsilon' \le \varepsilon' k_n x_n(t) \le kx(t) \le k_n x_n(t),$$

$$M\big(k_n x_n(t)\big) \ge \varepsilon N\big(p_-(k_n x_n(t))\big), \quad \text{and}$$

$$M\Big(\frac{k \cdot k_n}{k+k_n}(x_n(t)+x(t))\Big) > (1-\delta)\Big(\frac{k}{k+k_n} M\big(k_n x_n(t)\big) + \frac{k_n}{k+k_n} M\big(kx(t)\big)\Big)\Big\}.$$

From $R_M(k_n x_n \chi_{A_n}) \le M(\varepsilon)\mu e + R_M(kx\chi_e) + \varepsilon R_N(p_-(k_n x_n))$, combining with $R_M(kx\chi_e) \to 0 \;\; (\mu e \to 0)$ and $R_N(p_-(k_n x_n)) \le 1$ we deduce

$$\lim_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_{A_n}) = \mathrm{O}(\varepsilon)$$

If $kx(t) < \varepsilon' k_n x_n(t)$ then

$$\begin{aligned} M\Big(\frac{k \cdot k_n}{k+k_n}(x_n(t)+x(t))\Big) &\le M\Big(\frac{k+\varepsilon' k_n}{k+k_n} k_n x_n(t)\Big) \\ &\le (1-\delta)\frac{k+\varepsilon' k_n}{k+k_n} M(k_n x_n(t)) \\ &= (1-\delta')\frac{k+\varepsilon' k_n}{k}\frac{k}{k+k_n} M\big(k_n x_n(t)\big) \\ &\le (1-\delta')(1+\varepsilon'\overline{k})\frac{k}{k+k_n} M\big(k_n x_n(t)\big) \\ &\le (1-\delta'^2)\Big(\frac{k}{k+k_n} M\big(k_n x_n(t)\big) + \frac{k_n}{k+k_n} M\big(kx(t)\big)\Big). \end{aligned}$$

Putting $\delta'' = \min\{\delta, \delta'^2\}$, we obtain

$$M\Big(\frac{k \cdot k_n}{k + k_n}\big(x_n(t) + x(t)\big)\Big) \le (1 - \delta'')\Big(\frac{k}{k + k_n} M\big(k_n x_n(t)\big) + \frac{k_n}{k + k_n} M\big(kx(t)\big)\Big)$$

for $t \in B_n$. So

$$\begin{aligned} 0 &\leftarrow \|x_n\|^0 + \|x\|^0 - \|x_n + x\|^0 \\ &\ge \frac{k + k_n}{k \cdot k_n} \int_{B_n} \delta'' \Big[\frac{k}{k + k_n} M\big(k_n x_n(t)\big) + \frac{k_n}{k + k_n} M\big(kx(t)\big)\Big] d\mu. \end{aligned}$$

Thus $R_M(k_n x_n \chi_{B_n}) < \varepsilon$ for n large enough. Hence we have

$$\lim_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_{B_n}) = O(\varepsilon).$$

Finally from (4), we get $\lim\limits_{\mu e \to 0} \sup\limits_n R_M(k_n x_n \chi_{C_n}) \le \varepsilon$. By the arbitrariness of ε, combining with the above, we obtain $\lim\limits_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_e) = 0$.

(II) $\mu G_a > 0$.

From $1 = R_N(p_-(kx)) < R_N(p(kx))$, combining with (3), it follows $\mu G_b = 0$. Now, analogously as in Lemma 7, we deduce $k_n x_n - kx \xrightarrow{\mu} 0$ on $G \setminus G_a$, so without loss of generality, we can assume that $k_n x_n(t) \to kx(t)$ $t \in G \setminus G_a$ μ-a.e. Since $p_-(u)$ is left-continuous and nondecreasing, we get $\lim\limits_{n \to \infty} \inf N(p_-(k_n x_n(t))) \ge N(p_-(kx(t)))$ $t \in G \setminus G_a$ μ-a.e. Hence $\lim\limits_{n \to \infty} \inf R_N(p_-(k_n x_n \chi_{G \setminus G_a})) \ge R_N(p_-(kx\chi_{G \setminus G_a}))$.

On the other hand, $R_N(p_-(k_n x_n)) \le 1 = R_N(p_-(kx))$, so we have

$$\limsup_{n \to \infty} R_N\big(p_-(k_n x_n \chi_{G_a})\big) \le R_N\big(p_-(kx\chi_{G_a})\big).$$

Denote $G_n = \{t \in G_a : k_n x_n(t) \le kx(t)\}$. Applying the fact that a' is a left extreme point of an affine segment of $M(u)$, we obtain, like in Lemma 7,

$$\mu\{t \in G_a : k_n x_n(t) \le kx(t) - \varepsilon\} \to 0$$

for any $\varepsilon > 0$. Since $\lim\limits_{\mu e \to 0} \sup\limits_n R_N(p_-(k_n x_n \chi_e)) = 0$ for all $e \subset G_n$, we deduce

$$\lim_{n \to \infty} \sup R_N\big(p_-(k_n x_n \chi_{G_n})\big) = \lim_{n \to \infty} \sup R_N\big(p_-(kx\chi_{G_n})\big),$$

so

$$\lim_{n \to \infty} \sup R_N\big(p_-(k_n x_n \chi_{G_a \setminus G_n})\big) \le \lim_{n \to \infty} \sup R_N\big(p_-(kx\chi_{G_a \setminus G_n})\big).$$

From $k_n x_n(t) > kx(t)$ $(t \in G_a \setminus G_n)$ we derive

$$\lim_{n \to \infty} \sup \int_{G_a \setminus G_n} \big[N\big(p_-(k_n x_n(t))\big) - N\big(p_-(kx(t))\big)\big] d\mu = 0.$$

By $p_-(a) < p(a)$ $(a \in \{a\})$,

$$\mu\{t \in G_a \setminus G_n : k_n x_n(t) \ge kx(t) + \varepsilon\} \longrightarrow 0.$$

Combining the above we deduce $k_n x_n - kx \xrightarrow{\mu} 0$ on G, like (I1), which follows that $\lim_{\mu e \to 0} \sup_n R_N(p_-(k_n x_n \chi_e)) = 0$ and $\lim_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_e) = 0$, hence $x_n - x \xrightarrow{\mu} 0$.

(III) $\mu G_b > 0$.

In this case, $1 = R_N(p(kx)) > R_N(p_-(kx))$, so $\mu G_a = 0$. From $R_N(p(\frac{kx}{1-\tau})) < \infty$, so $R_M(\frac{kx}{1-\tau}) < \infty$, it yields that $\lim_{\mu e \to 0} \sup_n R_M(k_n x_n \chi_e) = 0$. Indeed if we suppose that for some $\varepsilon_0 > 0$, there exist $e_n \subset G$ and x_n, if necessary pass to a subsequence, such that $R_M(k_n x_n \chi_{e_n}) \geq \varepsilon_0$, then

$$
\begin{aligned}
2 \leftarrow & \frac{k+k_n}{k \cdot k_n}\left(1 + R_M\Big(\frac{k \cdot k_n}{k+k_n}(x_n + x)\Big)\right) \\
= & \frac{k+k_n}{k \cdot k_n}\left(1 + R_M\Big(\frac{k \cdot k_n}{k+k_n}(x_n + x)\chi_{G \setminus e_n}\Big)\right. \\
& \left. + R_M\Big(\Big(\frac{k(1+\tau_n)}{k+k_n}\frac{k_n x_n}{1+\tau_n} + \frac{k_n(1-\tau)}{k+k_n}\frac{kx}{1-\tau}\Big)\chi_{e_n}\Big)\right) \\
\leq & \frac{k+k_n}{k \cdot k_n}\left(1 + R_M\Big(\frac{k \cdot k_n}{k+k_n}(x_n + x)\chi_{G \setminus e_n}\Big)\right. \\
& \left. + \frac{k(1+\tau_n)}{k+k_n} R_M\Big(\frac{k_n x_n}{1+\tau_n}\chi_{e_n}\Big) + \frac{k_n(1-\tau)}{k+k_n} R_M\Big(\frac{kx}{1-\tau}\chi_{e_n}\Big)\right) \\
\leq & \frac{k+k_n}{k \cdot k_n}\left(1 + R_M\Big(\frac{k \cdot k_n}{k+k_n}(x_n + x)\chi_{G \setminus e_n}\Big)\right. \\
& \left. + \frac{k(1+\tau_n)}{k+k_n}(1-\delta)\frac{R_M(k_n x_n \chi_e)}{1+\tau_n} + \frac{k_n(1-\tau)}{k+k_n} R_M\Big(\frac{kx}{1-\tau}\chi_{e_n}\Big)\right) \\
\leq & \frac{k+k_n}{k \cdot k_n}\left(1 + \frac{k}{k+k_n} R_M(k_n x_n \chi_{G \setminus e_n}) + \frac{k_n}{k+k_n} R_M(kx \chi_{G \setminus e_n})\right. \\
& + \frac{k}{k+k_n} R_M(k_n x_n \chi_{e_n}) + \frac{k_n}{k+k_n} R_M(kx \chi_{e_n}) - \frac{k_n}{k+k_n} R_M(kx \chi_{e_n}) \\
& \left. - \frac{k\delta}{k+k_n} R_M(k_n x_n \chi_{e_n}) + \frac{k_n(1-\tau)}{k+k_n} R_M\Big(\frac{kx}{1-\tau}\chi_{e_n}\Big)\right) \\
\leq & \, 2 - \frac{\delta}{\overline{k}} R_M(k_n x_n \chi_{e_n}) - \frac{1}{k} R_M(kx \chi_{e_n}) + \frac{1-\tau}{k} R_M\Big(\frac{kx}{1-\tau}\chi_{e_n}\Big) \\
\longrightarrow & \, 2 - \frac{\delta \varepsilon_0}{\overline{k}}
\end{aligned}
$$

(where $\tau_n > 0, k\tau_n = k_n\tau,\ \tau, \delta > 0$ satisfying $M(\frac{u}{1+\tau/k}) \le (1-\delta)\frac{1}{1+\tau/k}M(u)\ u \ge \varepsilon_0$) - a contradiction.

Since

$$M(u) > \int_{(1-\theta)u}^{u} p(t)d\mu \ge p\big((1-\theta)u\big)\theta u$$
$$= \frac{\theta}{1-\theta}(1-\theta)u \cdot p\big((1-\theta)u\big) \ge \frac{\theta}{1-\theta}N\big(p((1-\theta)u)\big)$$

we have that for $\theta, 0 < \theta < 1$,

$$\lim_{\mu e \to 0} \sup_n R_N\big(p((1-\theta)k_n x_n \chi_e)\big) = 0\,. \qquad (*)$$

For θ small enough, if $k_n x_n(t) \le \frac{kx(t)}{1-\tau/2}$ then

$$(1+\theta)\frac{k \cdot k_n}{k+k_n}\big(x_n(t) + x(t)\big) \le (1+\theta)\frac{kx(t)}{1-\tau/2} \le \frac{kx(t)}{1-\tau}$$

and if $k_n x_n(t) > \frac{kx(t)}{1-\tau/2}$ then

$$(1+\theta)\frac{k \cdot k_n}{k+k_n}\big(x_n(t) + x(t)\big) \le (1+\theta)\Big(1 - \frac{\tau}{2(1+\bar{k})}k_n x_n(t)\Big)$$
$$= (1-\theta')k_n x_n(t)\,.$$

Combining $R_N(p(\frac{kx}{1-\tau})) < \infty$ and $(*)$, we have

$$\lim_{\mu e \to 0} \sup_n R_N\Big(p\big((1+\theta)\frac{k \cdot k_n}{k+k_n}(x_n + x)\chi_e\big)\Big) = 0\,. \qquad (**)$$

In the following we shall show that for any $\eta > 0$

$$\lim_{n \to \infty} \sup_n R_N\Big(p\big((1+\eta)\frac{k \cdot k_n}{k+k_n}(x_n + x)\big)\Big) \ge 1\,. \qquad (***)$$

Indeed if we suppose that for some $\eta_0 > 0, \theta_0 > 0$ and for all n (if necessary passing to a subsequence)

$$R_N\Big(p\big((1+\eta_0)\frac{k \cdot k_n}{k+k_n}(x_n + x)\big)\Big) \le 1 - \theta_0\,.$$

For $h_n = \frac{k \cdot k_n}{k+k_n}$, we have shown $\frac{1}{h_n}(1 + R_M(h_n(x_n + x))) - \|x_n + x\|^0 \to 0$, thus

$$\begin{aligned}
0 \leftarrow & \frac{1}{h_n}\Big(1 + R_M\big(h_n(x_n + x)\big) - \inf_{h>0} \frac{1}{h}\Big(1 + R_M\big(h(x_n + x)\big)\Big)\Big) \\
& \geq \frac{1}{h_n}\Big(1 + R_M\big(h_n(x_n + x)\big) - \frac{1}{(1+\eta_0)h_n}\Big[1 + R_M\big((1+\eta_0)h_n(x_n + x)\big)\Big]\Big) \\
& = \frac{\eta_0}{(1+\eta_0)h_n}\Big\{1 - \frac{1+\eta_0}{\eta_0}\big[R_M\big((1+\eta_0)h_n(x_n + x)\big) - R_M\big(h_n(x_n + x)\big)\big] \\
& \quad + R_M\big((1+\eta_0)h_n(x_n + x)\big)\Big\} \\
& \geq \frac{\eta_0}{(1+\eta_0)h_n} \\
& \quad \times \left\{1 - \Big[\frac{1+\eta_0}{\eta_0} \int_G \eta_0 h_n\big(x_n(t) + x(t)\big)p\big[(1+\eta_0)h_n\big(x_n(t) + x(t)\big)\big]d\mu\Big]\right. \\
& \quad \left. + R_M\big((1+\eta_0)h_n(x_n + x)\big)\right\} \\
& = \frac{\eta_0}{(1+\eta_0)h_n}\Big(1 - R_N(p((1+\eta_0)h_n(x_n + x)))\Big) \geq \frac{\eta_0\theta_0}{(1+\eta_0)h_n} \\
& \geq \frac{2\eta_0\theta_0}{(1+\eta_0)\overline{k}}.
\end{aligned}$$

This contradiction shows that $(***)$ holds.

Denote $G_n = \{t \in G_b : k_n x_n(t) < kx(t)\}$, like in Lemma 7, we can derive that $k_n x_n \chi_{G_b \setminus G_n} - kx \xrightarrow{\mu} 0$ on $(G \setminus G_b)$, i.e. for any $\varepsilon > 0$,

$$\mu\{t \in (G \setminus G_b) \cup (G_b \setminus G_n) : |k_n x_n(t) - kx(t)| \geq \varepsilon\} \to 0 \quad \text{as} \quad n \to \infty\,.$$

Finally we show that $k_n x_n - kx \xrightarrow{\mu} 0$ on G. Indeed if suppose that for some $\varepsilon > 0, \sigma > 0$, and all n, if necessary passing to a subsequence, there are $\widetilde{G}_n \subset G_n$

$$\mu\widetilde{G}_n = \mu\{t \in G_n : k_n x_n(t) < kx(t) - \varepsilon\} \geq \sigma\,.$$

Taking into account that $p(u)$ is right-continuous and nondecreasing, $k_n x_n(t) < kx(t)$ $(t \in G_n)$ and $(***)$, we derive

$$
\begin{aligned}
1 &\le \lim_{n\to\infty} R_N\Big(p\big((1+\eta)\frac{k\cdot k_n}{k+k_n}(x_n+x)\big)\Big)\\
&= \lim_{n\to\infty}\Big\{R_N\big(p((1+\eta)\frac{k\cdot k_n}{k+k_n}(x_n+x)\chi_{G\setminus\widetilde{G}_n})\big)\\
&\quad + R_N\big(p((1+\eta)\frac{k\cdot k_n}{k+k_n}(x_n+x)\chi_{\widetilde{G}_n})\big)\Big\}\\
&\le \lim_{n\to\infty}\Big\{R_N\big(p((1+\eta)kx\chi_{G\setminus\widetilde{G}_n})\big)\\
&\quad + \int_{\widetilde{G}_n} N\big(p((1+\eta)(kx(t))-\frac{\varepsilon}{1+\overline{k}})\big)d\mu\Big\}\\
&= \lim_{n\to\infty}\big\{R_N\big(p((1+\eta)kx\chi_{G\setminus\widetilde{G}_n})\big) + N(p(a))\mu\widetilde{G}_n\big\}\\
&\le \lim_{n\to\infty}\big\{R_N\big(p((1+\eta)kx\chi_{G\setminus\widetilde{G}_n})\big)\\
&\quad - \big(N(p(b))-N(p(a))\big)\mu\widetilde{G}_n + N(p(b))\mu\widetilde{G}_n\big\}\\
&\le R_N(p((1+\eta)kx)) - \big(N(p(b))-N(p(a))\big)\sigma\,.
\end{aligned}
$$

By $R_N(p(\frac{kx}{1-\tau}))<\infty$, let $\eta\to 0$, we get a contradiction:

$$1\le 1-\big(N(p(b))-N(p(a))\big)\sigma\,.$$

So $k_nx_n-kx\xrightarrow{\mu}0$ on G. From $\lim_{\mu e\to 0}\sup_n R_M(k_nx_n\chi_e)=0$, we have that $x_n-x\xrightarrow{\mu}0$. □

By Theorem 1, we easily deduce the correct criteria of UR and WUR points.

Theorem 2

Let $x\in S(L_M)$ and $k\in\mathbb{K}(x)$. A point x is a UR (WUR) if and only if

(i) $M\in\Delta_2\cap\nabla_2$

(ii) $\mu\{t\in G\colon |kx(t)|\in\mathbb{R}\setminus S_M\cup\{a'\}\cup\{b'\}\}=0$

(iii) if $\mu G_a>0$ *then* $R_N(p_-(kx))=1$

if $\mu G_b>0$ *then* $R_N(p(kx))=1$.

Proof. Necessity. Since a WUR point is a W^*UR point, it follows that $M\in\nabla_2$, (ii) and (iii) hold. On the other hand, suppose $M\notin\Delta_2$.

If $x\in L_M^0\setminus E_M^0$, take the transversal function $x_n\in B(L_M)$, $\|x_n+x\|^0\ge 2\|x_n\|^0\to 2\|x\|^0=2$ (where $x_n(t)=x(t)$ if $|x(t)|\le n;=0$ if $|x(t)|>n$). By Hahn-Banach theorem there is a singular functional $\varphi,\varphi(x)\ne 0$, and $\varphi(x_n-x)=-\varphi(x)$, so $x_n-x\overset{w}{\not\to}0$.

If $x \in E_M^0$, take $z \in L_M^0 \setminus E_M^0, R_M(z) < \infty$. Choose a singular function $\varphi(z) \neq 0$. Denote $G_n = \{t \in G: |z(t)| \leq n\}$, then $\mu(G \setminus G_n) \to 0$. Define

$$x_n(t) = x\chi_{G_n}(t) + \frac{1}{k} z\chi_{G\setminus G_n}(t).$$

Then $\|x_n\|^0 \leq \frac{1}{k}(1 + R_M(kx\chi_{G_n}) + R_M(z\chi_{G\setminus G_n})) \to \frac{1}{k}(1 + R_M(kx)) = \|x\|^0 = 1$ and $\|x_n + x\|^0 \geq \|2x\chi_{G_n}\|^0 \to 2$. But $\varphi(x_n - x) = \varphi(\frac{1}{k}z\chi_{G\setminus G_n}) = \frac{1}{k}\varphi(z) \neq 0$, and $x_n - x \overset{w}{\not\to} 0$.

Sufficiency. In this case, (1) and (2) of Theorem 1 hold. By $M \in \Delta_2$, it yields that $R_N(p((1+\tau)kx)) < \infty$, i.e. (3) of Theorem 1 holds. Also by $M \in \Delta_2$, we derive $R_M(u\chi_e) \leq R_M(\frac{kx}{\varepsilon}\chi_e) \leq DR_M(kx\chi_e) \to 0$ (as $\mu e \to 0$) i.e., (4) of Theorem 1 holds.

Therefore, $\|x_n + x\|^0 \to 2$ implies $x_n - x \xrightarrow{\mu} 0$. By $M \in \Delta_2$ and [5], it yields that $\|x_n - x\|^0 \to 0$, hence x is a UR point. □

For Orlicz sequence spaces, we obtain the same results as in function spaces, and omit the proof.

Theorem 3

Let $x \in S(l_M)$ and $k \in \mathbb{K}(x)$. A point x is W^*UR if and only if

(1) $M \in \nabla_2$,

(2) $\{j: kx(j) \neq 0\}$ is a single element set, or (i), (ii) and (iii) hold

(i) $\{j: |kx(j)| \in \mathbb{R} \setminus S_M \cup \{a'\} \cup \{b'\}\} = \emptyset$

(ii) if $|kx(i)| = a \in \{a\}, \sum_{j\neq i} N(p_-(kx(j))) + N(p(kx(i))) > 1$
if $|kx(i)| = b \in \{b\}, \sum_{j\neq i} N(p(kx(j))) + N(p_-(kx(i))) < 1$ and $R_N(p(\frac{kx}{1-\tau})) < \infty$ for some $0 < \tau < 1$

(iii) for any $\varepsilon, \varepsilon' > 0$, there is n_0 such that for all summable sequences $\{u(i)\}$ and subsequences e of natural numbers $\mathbb{N}$ with $\min\{i: i \in e\} > n_0$ and for all $i \in e$

$$\varepsilon'\varepsilon \leq \varepsilon'|u(i)| \leq k|x(i)| \leq |u(i)|, M(u(i)) \geq \varepsilon N(p_-(u(i)))$$

$$M\Big(\frac{u(i) + kx(i)}{2}\Big) > \Big(1 - \frac{1}{n_0}\Big)\frac{M(u(i)) + M(kx(i))}{2}$$

we have $\sum_{i\in e} M(u(i)) < \varepsilon$.

Theorem 4

Let $x \in S(l_M)$ and $k \in \mathbb{K}(x)$. A point x is UR (WUR) if and only if

(1) $M \in \Delta_2 \cap \nabla_2$

(2) $\{j : kx(j) \neq 0\}$ *is a singleton set, or (i) and (ii) hold*

(i) $\{i : |kx(i)| \in \mathbb{R} \setminus S_M \cup \{a'\} \cup \{b'\}\} = \emptyset$

(ii) if $|kx(i)| = a \in \{a\}, \sum_{j \neq i} N(p_-(kx(j))) + N(p(kx(i))) > 1$

if $|kx(i)| = b \in \{b\}, \sum_{j \neq i} N(p(kx(j))) + N(p_-(kx(i))) < 1\,.$

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