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Besov spaces and function series on Lie groups II

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ABSTRACT

In the paper we investigate the absolute convergence in the sup-norm of two-sided Harish-Chandra's Fourier series of functions belonging to Zygmund-Hölder spaces defined on non-compact connected Lie groups.

Let G be an n-dimensional connected unimodular Lie group countable at infinity and let K be a k-dimensional connected compact subgroup of G. Let $\Sigma(K)$ denote the set of all equivalence classes of finite-dimensional irreducible representation of K. For any $\sigma \in \Sigma(K)$ let χ_δ be a character of the class δ and $d(\delta)$ its degree.

We put

$$\alpha_\delta = d(\delta)\bar{\chi}_\delta. \tag{1}$$

Let $L(x)$ denote the left regular representation of G on $C^\infty(G)$ (or $C_0^\infty(G)$) i.e. $(L(x)f)(y) = f(x^{-1}y)$ and $R(x)$ be the right regular representation of G on the same spaces i.e. $(R(x)f)(y) = f(yx)$. If f is a suitable function on G then

$$(\alpha_\delta * f)(x) = \int_K \alpha_\delta(y) f(y^{-1}x)dy, \quad x \in G, \tag{2}$$

and

$$(f * \alpha_\delta)(x) = \int_K \alpha_\delta(y^{-1}) f(xy)dy, \quad x \in G, \tag{3}$$

are called a *δ-Fourier component* of the function f with respect to the representation $L(x)$ and $R(x)$ respectively, dy being the normalized Haar measure on K. The group G is countable at infinity therefore the space of smooth function $C^\infty(G)$ and the space of smooth functions with compact support $C_0^\infty(G)$ taken with their usual topologies are locally convex complete and metrizable vector topological spaces. Let $D'(G)$ be the continuous dual of $C_0^\infty(G)$. We call the elements of $D'(G)$ distributions on G. Identifying α_δ with an element of the space of Radon measures with compact support on G we can regard (2) and (3) as the convolutions on G. Due to this identification, we can define the *δ-Fourier component* with respect to $L(x)$ and $R(x)$ of every distribution $T \in D'(G)$ by

$$\alpha_\delta * T, \quad T * \alpha_\delta \quad \text{(convolutions of distributions).}$$

Theorem 1 (cf. [2])

Let $f \in C^\infty(G)$ $(f \in C_0^\infty(G))$ then the Fourier series

$$\sum_{\delta \in \Sigma(K)} \alpha_\delta * f \text{ and } \sum_{\delta \in \Sigma(K)} f * \alpha_\delta$$

converge absolutely to f in $C^\infty(G)$ $(C_0^\infty(G))$.

Corollary 1 (cf. [14] §4.4.3)

The Fourier series of the distribution T converges to T in $D'(G)$ equipped with the topology of uniform convergence on bounded subsets.

Note that $L(x)$ and $R(y)$ $(x, y \in G)$ commute and hence

$$\alpha * (f * \beta) = (\alpha * f) * \beta \quad \big(\alpha, \beta \in C(K)\big).$$

We may therefore simply write $\alpha * f * \beta$. In the present paper we will regard also the following series

$$\sum_{\delta_1, \delta_2 \in \Sigma(K)} \alpha_{\delta_1} * f * \alpha_{\delta_2}. \tag{4}$$

Generally the above series does not coincide with the series defined in (2) and (3). If the group G is abelian then the last series coincides with the previous ones because $\alpha_\delta * \alpha_\delta = \alpha_\delta$ and $\alpha_{\delta_1} * \alpha_{\delta_2} = 0$ if $\delta_1 \neq \delta_2$.

Proposition 1 (cf. [2])

Let E denote either one of the spaces $C^\infty(G)$ or $C_0^\infty(G)$. Then for any $f \in E$ the series (4) converges absolutely to f in E.

On the Lie group G we can define a left-invariant Riemannian metric tensor g and a right-invariant Riemannian metric tensor $\tilde{g}$ as well (cf. [3]). The Riemannian manifolds (G, g) and $(G, \tilde{g})$ are both connected complete Riemannian manifolds with a positive injectivity and bounded geometry. Therefore we can define the two scales of *Besov spaces* on G $B^s_{p,q}(G)$ and $\overline{B}^s_{p,q}(G)$, $-\infty < s < \infty$, $0 < p \leq \infty$, $0 < q \leq \infty$. The first scale corresponds to the Riemannian manifold (G, g) the second one to $(G, \tilde{g})$ in the sense of the Triebel definition (cf. [10], [12]). Generally these two scales of function spaces do not coincide.

Let $\mathcal{R}$ be the Lie algebra of K. Since K is compact we can choose a positive-defined quadratic form Q on $\mathcal{R}$ invariant with respect of the adjoint representation Ad_k. Let $X_1, \dots, X_k$ be a basis of $\mathcal{R}$ orthonormal with respect to Q, then the differential operator

$$\Omega = I - (X_1^2 + \cdots + X_k^2) \tag{5}$$

commutes with both left and right translation of K. It is well known that the functions α_δ, $\delta \in \Sigma(K)$, are eigenvectors of Ω with eigenvalues $c(\delta) \geq 1$, and that

$$\sum_{\delta \in \Sigma(K)} d(\delta)^2 c(\delta)^{-m} < \infty$$

for a sufficiently large positive number m (cf. [2]), $d(\delta)$ being the degree of the class δ.

Thus for every r, $0 < r \leq 2$, there is the smallest number m_r such that

$$\sup_{\delta \in \Sigma(K)} d(\delta)^r c(\delta)^{-m} < \infty, \quad \text{for every } m > m_r\,. \tag{6}$$

Let $C(G)$ denote the Banach space of bounded continuous functions on G with the standard norm. In [7] we proved the following theorem

Theorem 2

Let $1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $s > \frac{n}{p} + 2m_1 + \max(0, \frac{k}{2} - \frac{k}{p})$. Let $f \in \overline{B}^s_{p,q}(G)$ ($f \in B^s_{p,q}(G)$). Then the Fourier series

$$\sum_{\delta \in \Sigma(K)} \alpha_\delta * f \quad \Big(\sum_{\delta \in \Sigma(K)} f * \alpha_\delta \Big)$$

converges absolutely in $C(G)$ to the function f. Moreover, there is a constant C such that

$$\sum_{\delta \in \Sigma(K)} \|\alpha_\delta * f\|_\infty \leq C \|f|\overline{B}^s_{p,q}(G)\|, \qquad \sum_{\delta \in \Sigma(K)} \|f * \alpha_\delta\|_\infty \leq C \|f|B^s_{p,q}(G)\|.$$

2. The absolute convergence of the series $\Sigma\alpha_{\delta_1} * f * \alpha_{\delta_2}$

The main result of the paper reads as follows.

Theorem 3

Let $s > 4m_1 + k$. Then for every $f \in B^s_{\infty,\infty}(G) \cap \overline{B}^s_{\infty,\infty}(G)$ the series

$$\sum_{\delta_1,\delta_2 \in \Sigma(K)} \alpha_{\delta_1} * f * \alpha_{\delta_2}$$

converges absolutely to f in $C(G)$ and

$$\sum_{\delta_1,\delta_2 \in \Sigma(K)} \|\alpha_{\delta_1} * f * \alpha_{\delta_2}\|_\infty \le C \max\left(\|f|B^s_{\infty,\infty}(G)\|,\ \|f|\overline{B}^s_{\infty,\infty}(G)\|\right).$$

Proof. We divide the proof into several steps.

Step 1. Let $\widetilde{G} = G \times G$ and $\widetilde{K} = K \times K$, where $\times$ denotes the cartesian product of groups and manifolds as well. The group $\widetilde{G}$ is a $2n$-dimensional connected Lie group, and $\widetilde{K}$ is its compact subgroup. The Lie algebra of $\widetilde{G}$ is isomorphic to the direct sum $\mathcal{G} \ominus \mathcal{G}$ of the Lie algebra $\mathcal{G}$ of G. In this step we describe a Riemannian structure on $\widetilde{G}$ needed later on.

Let π_i, $i = 1, 2$, denote a projection of $\widetilde{G}$ onto the corresponding factor of the product. We define the Riemannian metric $\widetilde{g}$ on $\widetilde{G}$ as a cartesian product $\widetilde{g} = \overline{g} \times g$ of the Riemannian metric $\overline{g}$ and g i.e.

$$\widetilde{g}_{(x,y)}(X, Y) = \overline{g}_x(d_{(x,y)}\pi_1 X, d_{(x,y)}\pi_1 Y) + g_y(d_{(x,y)}\pi_2 X, d_{(x,y)}\pi_2 Y),$$

$(x, y) \in \widetilde{G}$, $X, Y \in T_{(x,y)}\widetilde{G}$. The manifold $(\widetilde{G}, \tilde{g})$ is a connected homogeneous Riemannian manifold. The mappings

$$\Phi_{(a,b)}: \widetilde{G} \ni (x, y) \to (xa, b^{-1}y) \in \widetilde{G}, \quad a, b \in G$$

form a group of isometries acting transitively on $\widetilde{G}$. The transitivity of the action is obvious. Since $\Phi_{(a,b)} = \Phi_{(a,e)} \circ \Phi_{(e,b)}$, it is sufficient to prove that $\Phi_{(a,e)}$ and $\Phi_{(e,b)}$ are isometries. To prove that $\Phi_{(a,e)}$ is an isometry we ought to show that

$$\widetilde{g}_{(x,y)}(X, Y) = \overline{g}_{(xa,y)}(d_{(x,y)}\Phi_{(a,e)} X, d_{(x,y)}\Phi_{(a,e)} Y), \tag{7}$$

for every $(x,y) \in \widetilde{G}$, $X,Y \in T_{(x,y)}\widetilde{G}$. Using the product structure of $\widetilde{G}$ it is not difficult to see that $d_{(xa,y)}\pi_1 \circ d_{(x,y)}\Phi_{(a,e)} = d_y r_a \circ d_{(x,y)}\pi_1$, and $d_{(xa,y)}\pi_2 \circ d_{(x,y)}\Phi_{(a,e)} = d_{(x,y)}\pi_2$, where $r_a: G \ni x \to xa \in G$. These identities and the fact that r_a is an isometry of $(G,\overline{g})$ imply (7). The proof for $\Phi_{(e,b)}$ is the same. Thus $(\widetilde{G},\widetilde{g})$ is a Riemannian manifold with positive injectivity radius and bounded geometry and the spaces $B^s_{p,p}(\widetilde{G})$ are well defined on $\widetilde{G}$.

The following relation between the covariant differentiation $\widetilde{\nabla}$ of $(\widetilde{G},\widetilde{g})$, ∇ of (G,g) and $\overline{\nabla}$ of $(G,\overline{g})$ is well known:
$\nabla_{(X_1,X_2)}(Y_1,Y_2) = (\overline{\nabla}_{X_1}Y_1, \nabla_{X_2}Y_1)$, where X_1,X_2,Y_1,Y_2 are vector fields on G. The last identity makes it obvious that

$$\exp_{(x,y)} X = (\overline{\exp}_x d_{(x,y)}\pi_1 X, \exp_y d_{(x,y)}\pi_2 X), \quad X \in T_{(x,y)}\widetilde{G}.$$

Let $i(G), i(\overline{G})$ and $i(\widetilde{G})$ be the injectivity radius of $(G,g),(G,g)$ and $(\widetilde{G},\widetilde{g})$, respectively. Let $\varepsilon < \min \frac{i(G),i(\overline{G}),i(\widetilde{G})}{8}$. Then there are positive numbers α and β, $0<\alpha<\beta<\varepsilon$, and sequences of points $\{x_i\},\{y_i\} \subset G$ such that the family of geodesic balls $\{B(x_i,\beta)\}$, $(\{B(y_i,\beta)\})$ forms a uniformly locally finite covering of (G,g) (and $(G,\overline{g})$, respectively), and the balls $B(x_i,\alpha)$ $(B(y_i,\alpha))$ are pairwise disjoint. The sets $B(y_i,\beta)\times B(x_j,\beta)$ are also pairwise disjoint. The geodesic balls $B_{ij} = B((y_i,x_j),\sqrt{2}\beta)$ form a covering of $\widetilde{G}$. It is not difficult to see that this covering is uniformly locally finite. In fact, the manifold $\widetilde{G}$ has bounded geometry therefore there are constants $C_1,C_2>0$ such that $\mathrm{vol}(B(x,3\sqrt{2}\beta)) < C_1$ and $\mathrm{vol}(B(x,\alpha)) < C_2$ for every $x \in \widetilde{G}$. Let $J_{ij} = \{(k,l): B((y_k,x_l),\sqrt{2}\beta)\cap B((y_i,x_j),\sqrt{2}\beta) \neq \emptyset\}$. Then

$$C_1 > \mathrm{vol}\big(B(y_i,x_j),3\sqrt{2}\beta\big) > \sum_{k,l\in J_{ij}} \mathrm{vol}\big(B(y_k,x_l),\alpha\big) > C_2|J_{ij}|$$

(cf. [1] Lemma 2.25 and 2.26, [8]).

Step 2. Let $\widetilde{f}(y,x) = f(yx)$, $x,y \in G$. We prove that $\widetilde{f} \in \overline{B}^s_{\infty,\infty}(\widetilde{G})$ if $f \in B^s_{\infty,\infty}(G) \cap \overline{B}^s_{\infty,\infty}(G)$. We will need the following lemma, which is a direct consequence of Theorem 2.5.13 in [13].

Lemma 1

Let $1 \le p \le \infty$ and $s > 0$. Then

$$\|f|B^s_{p,p}(\mathbb{R}^{n+m})\| \sim \Big\| \, \|f(\cdot,y)|B^s_{p,p}(\mathbb{R}^n)\| L_p(\mathbb{R}^m)\Big\| + \Big\| \, \|f(x,\cdot)|B^s_{p,p}(\mathbb{R}^m)\| L_p(\mathbb{R}^n)\Big\| .$$

Lemma 2

Let $-\infty < s < \infty, 0 < p \le \infty, 0 < q \le \infty$. Then $L(x)(R(x))$ is an isomorphism of $B^s_{p,q}(G)$ $(\overline{B}^s_{p,q}(G))$, and there is a constant C such that $\|L(x)\| \le C$, $(\|R(x)\| \le C)$ for every $x \in G$.

Proof of Lemma 2. Let κ and κ_0 be a rotation invariant C^∞ functions in $\mathbb{R}^n$ such that $\operatorname{supp}\kappa \subseteq B(0,1)$, and $\kappa(0) \neq 0$, $\hat{\kappa}_0(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, where $\hat{}$ denotes the Fourier transform. Let $k_{0,t}(x) = \kappa_0(t^{-1}\exp_e^{-1} x)$, and $k_{N,t}(x) = t^{-n}\kappa_N(t^{-1}\exp_e^{-1} x)$, where $\kappa_N = \left(\sum_{j=1}^{n} \frac{\partial}{\partial x_j^2}\right)^N \kappa$, $N = 1, 2, \dots$ Then for sufficiently small $\varepsilon > 0$ and $r > 0$ and $N > \max(s, 5 + 2\frac{n}{p}) + \max(o, n(\frac{1}{p} - 1))$ the expression

$$\|f|B^s_{p,q}(G)\|_1 = \|f * k_{0,\varepsilon}|L_p(G)\| + \left(\int_0^r t^{-sq}\|f * k_{N,t}|L_p(G)\|^q \frac{dt}{t}\right)^{1/q}$$

is an equivalent norm in $B^s_{p,q}(G)$ (cf. [11]). But $(L(x)f) * k_{N,t} = L(x)(f * k_{N,t})$. Thus $\|f|B^s_{p,q}(G)\|_1 = \|L(x)f|B^s_{p,q}(G)\|_1$. For right translations the proof is similar. □

Let $\{\varphi_i\}$ be the resolution of unity corresponding to the covering $\{B(x_i, \beta)\}$ and $\{\psi_j\}$ the resolution of unity corresponding to the covering $\{B(y_i, B)\}$. If these resolutions of unity satisfy the assumptions needed to define the scale of Besov spaces (cf. [10], [12]) then $\chi_{ij}(y, x) = \psi_i(y)\varphi_j(x)$ is the resolution of unity corresponding to the covering B_{ij} and satisfying the same assumptions. We have

$$\begin{aligned}
\|\widetilde{f}|B^s_{\infty,\infty}(\widetilde{G})\| &= \sup_{i,j} \|\chi_{ij}\widetilde{f} \cdot \exp_{(y_i,x_j)} |B^s_{\infty,\infty}(\mathbb{R}^{2n})\| \\
&\le \sup_{i,j} \|\psi_i(\exp_{y_i}\xi)\|\varphi_j(\cdot)\widetilde{f}(\overline{\exp})_{y_i}\xi, \exp_{x_j}\cdot)|B^s_{\infty,\infty}(\mathbb{R}^n)\||L_\infty(\mathbb{R}^n)\| \\
&\quad + \sup_{i,j} \|\varphi_j(\exp_{x_j}\xi)\|\psi_i(\cdot)\widetilde{f}(\overline{\exp}_{y_i}\cdot, \exp_{x_j}\xi)|B^s_{\infty,\infty}(\mathbb{R}^n)\||L_\infty(\mathbb{R}^n)\| \\
&\le \sup_j \sup_{x\in G} \|\varphi_j(\cdot)\widetilde{f}(x, \exp_{x_j}\cdot)|B^s_{\infty,\infty}(\mathbb{R}^n)\| \\
&\quad + \sup_i \sup_{y\in G} \|\psi_i(\cdot)\widetilde{f}(\overline{\exp}_{y_i}\cdot, y)|B^s_{\infty,\infty}(\mathbb{R}^n)\| \\
&\le \sup_{x\in G} \|f(x\cdot)|B^s_{\infty,\infty}(G)\| + \sup_{y\in G} \|f(\cdot y)|\overline{B}^s_{\infty,\infty}(G)\| \\
&\le \|f|B^s_{\infty,\infty}(G)\| + \|f|\overline{B}^s_{\infty,\infty}(G)\|.
\end{aligned}$$

The last inequality follows from Lemma 2. Thus

$$\|\widetilde{f}|B^s_{\infty,\infty}(\widetilde{G})\| \le C \max\left(\|f|B^s_{\infty,\infty}(G)\|, \|f|\overline{B}^s_{\infty,\infty}(G)\|\right). \tag{8}$$

Step 3. In the third step we deal with expansions of functions from the spaces $B^s_{\infty,\infty}(\widetilde{K})$ needed later on. On the Lie algebra $\widetilde{\mathfrak{K}}$ of $\widetilde{K}$ we define a positive-defined quadratic form $\widetilde{Q}$ by

$$\widetilde{Q}(X, Y) = Q(d_{(e,e)}\pi_1 X, d_{(e,e)}\pi_1 Y) + Q(d_{(e,e)}\pi_2 X, d_{(e,e)}\pi_2 Y), \quad X, Y \in \widetilde{\mathfrak{K}},$$

where Q is the form on $\mathfrak{R}$ described in §1. The form $\widetilde{Q}$ is invariant with respect to $Ad_{\widetilde{K}}$. Let $X_1, \ldots, X_k$ be the base in $\mathfrak{R}$ orthonormal with respect to Q. Then the vectors

$$\widetilde{X}_1 = (X_1, 0), \ldots, \widetilde{X}_k = (X_k, 0), \widetilde{X}_{k+1} = (0, X_1), \ldots, \widetilde{X}_{2k} = (0, X_k)$$

form a basis of $\widetilde{\mathfrak{R}}$ orthonormal with respect to $\widetilde{Q}$, and therefore the differential operator

$$\widetilde{\Omega} = I - \sum_{i=1}^{2k} \widetilde{X}_i^2$$

commutes with both left and right translations of $\widetilde{K}$. The operator $\widetilde{\Omega}$ is a positive-defined self-adjoint operator in $L_2(\widetilde{K})$ so we can define the abstract Besov spaces $B_q^s(\widetilde{\Omega}), s > 0, 1 \leq q \leq \infty$, connected with this operator (cf. [6], §6.2). The abstract Besov space $B_q^s(\widetilde{\Omega})$ coincides with the space $B_{2,q}^{2s}(\widetilde{K})$ defined on $\widetilde{K}$ by the Riemannian approach (cf. [10], [12]).

The functions $\beta_{\delta_1,\delta_2}(x,y) = \alpha_{\delta_1}(x)\alpha_{\delta_2}(y)$ as well as the functions $\chi_{\delta_1,\delta_2}(x,y) = \chi_{\delta_1}(x)\chi_{\delta_2}(y)$ are the eigenfunctions of $\widetilde{\Omega}$ with eigenvalues $c(\delta_1,\delta_2) = c(\delta_1) + c(\delta_2) - 1$, $\delta_1, \delta_2 \in \Sigma(K)$. The operator $\widetilde{\Omega}$ has a pure point spectrum and the functions χ_{δ_1,δ_2} form the orthonormal system of eigenvectors of $\widetilde{\Omega}$ therefore for every $r, w \in \mathbb{R}$ such that $w + k(1 - \frac{r}{2}) > 0$, there is a positive constant C such that

$$\sum_{\delta_1,\delta_2 \in \Sigma(K)} c(\delta_1,\delta_2)^w | < \chi_{\delta_1,\delta_2}, f > |^r \leq C \| f | B_{2,r}^s(\widetilde{K}) \|^r$$

holds for all $f \in B_{2,r}^s(\widetilde{K}), s = 2\frac{w}{r} + 2k(\frac{1}{r} - \frac{1}{2})$ (cf. Theorem 6.4.3 in [6]). Here $< \cdot, \cdot >$ denotes the scalar product in $L_2(\widetilde{K})$. Thus

$$\begin{aligned} &\sum_{\delta_1,\delta_2 \in \Sigma(K)} c(\delta_1,\delta_2)^w | < \beta_{\delta_1,\delta_2}, f > |^r \\ &\qquad \leq C \sup_{\delta_1,\delta_2} \left(c(\delta_1,\delta_2)^{-m} d(\delta_1)^r d(\delta_2)^r \right) \| f | B_{2,r}^s(\widetilde{K}) \|^r \end{aligned}$$

holds for $s = 2\frac{w+m}{r} + 2k(\frac{1}{r} - \frac{1}{2})$. But $c(\delta_1,\delta_2) \geq \max(c(\delta_1), c(\delta_2))$ therefore the last inequality and (6) imply

$$\sum_{\delta_1,\delta_2 \in \Sigma(K)} c(\delta_1,\delta_2)^w | < \beta_{\delta_1,\delta_2}, f > |^r \leq C \| f | B_{2,r}^s(\widetilde{K}) \|^r, \tag{9}$$

for $s > \frac{2}{r}(w + 2m_r) + 2k(\frac{1}{r} - \frac{1}{2})$.

The following embedding is a consequence of the compactness of the manifold $\widetilde{K}$:

$$B^s_{\infty,\infty}(\widetilde{K}) \subset B^{s_0}_{2,r}(\widetilde{K}),\ 1 \leq r \leq \infty,\ -\infty < s_0 < s < \infty,\ \text{cf. [7]}.$$

Now if $w = 0$ and $r = 1$ then (9) implies that there is a positive constant $c > 0$ such that

$$\sum_{\delta_1,\delta_2\in\Sigma(K)} | < \beta_{\delta_1\delta_1}, f > | \leq C\|f|B^s_{\infty,\infty}(\widetilde{K})\| \tag{10}$$

holds for every $f \in B^s_{\infty,\infty}(\widetilde{K}),\ s > 4m_1 + k$.

Step 4. Let f be a suitable function on G. Then

$$(\alpha_{\delta_1} * f * \alpha_{\delta_2})(x) = \int_K \int_K \alpha_{\delta_1}(y)\alpha_{\delta_2}(z^{-1})f(y^{-1}xz)dzdy$$
$$= \int_K \int_K \overline{\alpha_{\delta_1}(y)\alpha_{\delta_2}(z)}f(yxz)dzdy = \int_{\widetilde{K}} \overline{\beta_{\delta_1\delta_2}(y,z)}f(yxz)dydz.$$

We put $\widetilde{f}_x(y,z) = f(yxz),\ x,y,z \in G$. Then $f_x = f_e \circ \Phi_{(e,x^{-1})}$ and

$$(\alpha_{\delta_1} * f * \alpha_{\delta_2})(x) = \int_{\widetilde{K}} \overline{\beta_{\delta_1,\delta_2}(y,z)}\widetilde{f}_x(y,z)dydz =< \widetilde{f}_x, \beta_{\delta_1,\delta_2} > . \tag{11}$$

Let $\widetilde{K}_x = \{(y,z) \in \widetilde{G}: (y, x^{-1}z) \in K\} = \Phi_{e,x^{-1}}(\widetilde{K}), x \in G$. Then $\widetilde{K}_x$ is a compact submanifold of G. Let $\mathcal{R}_x: B^s_{\infty,\infty}(\widetilde{G}) \to B^s_{\infty,\infty}(\widetilde{K}_x)$ be the restriction operator (cf. [8]). We recall that it is a continuous surjective linear operator. It was proved in Lemma 1 of [7] that the norms in the spaces $B^s_{\infty,\infty}(\widetilde{K}_x)$ can be defined in such a way that

$$\|\mathcal{R}_e(\widetilde{f}_x)|B^s_{\infty,\infty}(\widetilde{K})\| = \|\mathcal{R}_x(\widetilde{f}_e)|B^s_{\infty,\infty}(\widetilde{K}_x)\| \quad \text{and} \quad \|\mathcal{R}_x\| \leq C, \tag{12}$$

where C is a constant independent of x.

If $f \in B^s_{\infty,\infty}(G) \cap \overline{B}^s_{\infty,\infty}(G), s > 4m_1 + k$, then $\widetilde{f}_e \in B^s_{\infty,\infty}(\widetilde{G})$ (cf. Step 2). Now it follows from (8) and (10)-(12) that

$$\sum_{\delta_1,\delta_2\in\Sigma(K)} \|\alpha_{\delta_1} * f * \alpha_{\delta_2}\|_\infty \leq C \max\left(\|f|B^s_{\infty,\infty}(G)\|, \|f|\overline{B}^s_{\infty,\infty}(G)\|\right).$$

Thus the series converges absolutely in $C(G)$. But it converges to f in the sense of the strong topology of $\mathcal{D}'(G)$ so it converges to f also in $C(G)$. □

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