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Random rearrangements in functional spaces

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ABSTRACT

We give an operator approach to several inequalities of S. Kwapien and C. Schütt, which allows us to obtain more general results.

Section 0

Let n be an integer, $x = \{x_{ij}\}, 1 \leq i,j \leq n, \Pi = \Pi_n$ be the set of rearrangements of $\{1, 2, \dots, n\}$. Denote by $s_1, s_2, \dots, s_{n^2}$ the rearrangement of $|x_{ij}|$ in the decreasing order. S. Kwapien and C. Schütt proved the following statements.

Theorem A ([3])

The inequalities

$$\frac{1}{2n}\sum_{k=1}^{n} s_k \leq \frac{1}{n!}\sum_{\pi\in\Pi} \max_{1\leq i\leq n} |x_{i\pi(i)}| \leq \frac{1}{n}\sum_{k=1}^{n} s_k$$

are valid.

Theorem B ([5])

If $1 \leq p \leq q < \infty$, *then*

$$\frac{1}{10}\left(\left(\frac{1}{n}\sum_{k=1}^{n}s_k^p\right)^{1/p}+\left(\frac{1}{n}\sum_{k=n+1}^{n^2}s_k^q\right)^{1/q}\right)$$
$$\leq\left(\frac{1}{n!}\sum_{\pi\in\Pi}\left(\sum_{i=1}^{n}|x_{x\pi(i)}|^q\right)^{p/q}\right)^{1/p}$$
$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}s_k^p\right)^{1/p}+\left(\frac{1}{n}\sum_{k=n+1}^{n^2}s_k^q\right)^{1/q}.$$

The operator approach to such problems is presented in this article. It allows to obtain more general results.

Section 1

If $x(t)$ is a measurable function on $[0,1]$, we denote by $x^*(t)$ the decreasing rearrangement of $|x(t)|$. A Banach functional space E on $[0,1]$ with the Lebesgue measure m is said to be rearrangement invariant (r.i.) if it satisfies the following condition: if $y \in E$ and $x^*(t) \leq y^*(t)$ for all $t \in [0,1]$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$. Let $\tau > 0$. The compression operators

$$\sigma_\tau x(t) = \begin{cases} x(\frac{t}{\tau}), & 0 \leq t \leq \min(\tau,1) \\ 0, & \min(\tau,1) < t \leq 1 \end{cases}$$

act in every r.i. space. The numbers

$$\alpha_E = \lim_{\tau\to 0}\frac{\ln\|\sigma_\tau\|_E}{\ln\tau}, \quad \beta_E = \lim_{\tau\to\infty}\frac{\ln\|\sigma_\tau\|_E}{\ln\tau}$$

are named Boyd indexes of the r.i. space E. It's known that $0 \leq \alpha_E \leq \beta_E \leq 1$. Let $x, y \in L_1$. We denote $x \prec y$ if

$$\int_0^\tau x^*(t)dt \leq \int_0^\tau y^*(t)dt$$

for each $\tau \in [0,1]$. If a r.i. space E is separable or isometric to the conjugate of some separable r.i. space, then $x \prec y$ implies $\|x\|_E \leq \|y\|_E$. For simplicity we shall assume that a r.i. space E satisfies this assumption. The Hardy operator

$$Hx(t) = \int_t^1 \frac{x(s)}{s}ds$$

is bounded in a r.i. space E iff $\alpha_E > 0$. Without loss of generality $\|1\|_E = 1$.

Orlicz, Lorentz, Marcinkiewicz spaces are r.i. ones. If a function $M(u)$ is even, convex, increasing on $[0,\infty)$ and

$$\lim_{u\to\infty}\frac{M(u)}{u}=\infty,\quad M(0)=0$$

then

$$\|x\|_{L_M}=\inf\left\{\lambda:\lambda>0,\ \int_0^1 M\Big(\frac{x(t)}{\lambda}\Big)dt\le 1\right\}.$$

Let $\varphi(t)$ be an increasing concave function on $[0,1]$ s.t. $\varphi(0)=0$. Then

$$\|x\|_{\Gamma(\varphi)}=\int_0^1 x^*(t)d\varphi(t).$$

All the above mentioned properties of r.i. spaces can be found in [1,2,4].

Section 2

Let us fix some one-to-one correspondence ℓ of Π into $\{1,2,\ldots,n!\}$. Let $1\le q\le\infty$ and x is an n-square matrix. We define the quasi-linear operator

$$T_qx(t)=\Big(\sum_{i=1}^{n}|x_{i\pi(i)}|^q\Big)^{1/q},\quad t\in\Big(\frac{\ell(\pi)-1}{n!},\frac{\ell(\pi)}{n!}\Big)$$

with usual modification for $q=\infty$. It's evident that $\|T_qx\|_E$ does not depend on ℓ if E is a r.i. space. Define the operator

$$Sx(t)=s_k,\quad t\in\Big(\frac{k-1}{n},\frac{k}{n}\Big),\quad 1\le k\le n.$$

The following statement generalizes Theorem A.

Theorem 1

Let E be a r.i. space. Then

$$\frac{1}{2}\|Sx\|_E\le\|T_\infty x\|_E\le\|Sx\|_E.$$

Proof. For simplicity we assume that $s_1>s_2>\ldots>s_n>0$. Then $m\{t:Sx(t)\ge s_k\}=k/n$ for each $k=1,2,\ldots,n$. As

$$m\{t:T_\infty x(t)=s_k\}\le\frac{(n-1)!}{n!}=\frac{1}{n},\ k=1,2,\ldots,n$$

then

$$m\{t: T_\infty x(t) \geq s_k\} \leq \frac{k}{n}.$$

Hence $(T_\infty x)^*(t) \leq Sx(t)$ and $\|T_\infty x\|_E \leq \|Sx\|_E$.

To prove the left side of the inequality, we fix $k \in \{1, 2, \ldots, n\}$ and construct the matrix

$$y_{ij} = \begin{cases} |x_{ij}|, & |x_{ij}| \geq s_k \\ 0, & |x_{ij}| < s_k \end{cases}.$$

Applying Theorem A to matrix $\underline{Y} = \{y_{ij}\}$ we have

$$\frac{1}{2n}\sum_{1}^{k} s_i \leq \frac{1}{n!}\sum_{\pi\in\Pi} \max_{1\leq i\leq n} y_{i\pi(i)}. \tag{1}$$

Denote $e_k = \{t: T_\infty y(t) \neq 0\}$. Then $me_k \leq k/n$ and

$$\begin{aligned}\frac{1}{2n}\sum_{\pi\in\Pi} \max_{1\leq i\leq n} y_{i\pi(i)} &= \int_{e_k} T_\infty y(t)dt \\ &\leq \int_{e_k} T_\infty x(t)dt \leq \int_0^{k/n} (T_\infty x)^*(t)dt\,. \end{aligned} \tag{2}$$

As

$$\frac{1}{n}\sum_{i=1}^{n} s_i = \int_0^{k/n} Sx(t)dt$$

then (1) and (2) imply the inequalities

$$\frac{1}{2}\int_0^{k/n} Sx(t)dt \leq \int_0^{k/n} (T_\infty x)^*(t)dt.$$

The function

$$\int_0^{\tau} (T_\infty x)^*(t)dt$$

is concave on $[0, 1]$ and the function

$$\frac{1}{2}\int_0^{\tau} Sx(t)dt$$

is linear on each interval $[\frac{k-1}{n}, \frac{k}{n}]$, $1 \leq k \leq n$. Therefore the inequality

$$\frac{1}{2}\int_0^{\tau} Sx(t)dt \leq \int_0^{\tau} (T_\infty x)^*(t)dt$$

is valid for each $\tau \in [0, 1]$. Hence $\frac{1}{2}Sx \prec T_\infty x$ and

$$\frac{1}{2}\|Sx\|_E \leq \|T_\infty x\|_E. \ \square$$

Usually the Orlicz space L_M where $M(u) = e^{|u|} - 1$ is denoted by $\exp L$.

Theorem 2

There exists a constant $C > 0$ such that

$$\|T_1 x\|_{\exp L} \le C\left(\max_{1\le i,j\le n} |x_{ij}| + \frac{1}{n}\sum_{i,j=1}^{n} |x_{ij}| \right). \tag{3}$$

Proof. It is well known that the extremal points of the convex set

$$\max_{1\le i,j\le n} |x_{ij}| \le 1 \quad \text{and} \quad \sum_{i,j=1}^{n} |x_{ij}| \le n$$

are matrices such that some n elements x_{ij} $(1 \le i,j \le n)$ are equal to ± 1 and $n^2 - n$ elements are equal to 0. It is sufficient to prove inequality (3) only for such matrices.

Let matrix z belong to this set and $z \ge 0$, $1 \le j \le n$. We have

$$m\left\{t : T_1\big(z(t)\big) = j\right\} \le C_n^j \frac{(n-j)!}{n!} = \frac{1}{j!}. \tag{4}$$

Therefore

$$\int_0^1 (e^{\frac{T_1 z(t)}{\lambda}} - 1)dt \le \sum_{j=1}^{\infty} \frac{e^{j/\lambda}}{j!} - 1 = e^{e^{1/\lambda}} - 2.$$

This means that C in (3) may be chosen as $\frac{1}{\ln\ln 3}$. □

Lemma 3

If x is an $n \times n$ matrix and

$$\left|\{(i,j) : x_{ij} \ne 0\}\right| \le n$$

then

$$T_1 x \prec 8HSx.$$

Proof. First we consider the case:

$$s_i = \begin{cases} 1, & 1 \le i \le k \\ 0, & k < i \le n \end{cases}$$

for some $k \le n$. Given $1 \le j \le k$ we denote

$$R_j = \left\{\pi : \pi \in \Pi, \sum_{i=1}^{n} x_{i\pi(i)} = j\right\}, \quad Q_j = \bigcup_{m=j}^{k} R_m$$

and $\tau_j = \frac{|Q_j|}{n!}$. It is clear that

$$\tau_j \le \frac{2C_k^j(n-j)!}{n!} = \frac{2k!(n-j)!}{j!(k-j)!n!} = \frac{2(k-j+1)\dots k}{j!(n-j+1)\dots n} \le \frac{2k}{j!n}.$$

As

$$HSx(t) = \begin{cases} \ln \frac{k}{nt}, & 0 < t \le \frac{k}{n} \\ 0, & \frac{k}{n} \le t \le 1 \end{cases}$$

then

$$m\{t\colon HSx(t) \ge j\} = \frac{k}{n}e^{-j}.$$

Therefore

$$m\{t\colon T_1x(t) \ge j\} = \tau_j \le \frac{2e^2k}{2n}e^{-j} \le 8m\{t\colon HSx(t) \ge j\}.$$

So

$$T_1x \prec 8HSx.$$

Let us consider the general case. There exist $a_k \ge 0$, n-square matrices z_k $(1 \le k \le n)$ such that some k elements of z_k are equal to 1 and the other $n^2 - k$ elements are equal to 0,

$$\{(i,j)\colon (z_k)_{ij} = 1\} \subset \{(i,j)\colon (z_{k+1})_{ij} = 1\}$$

for each $k = 1, 2, \dots, n-1$ and

$$x = \sum_{k=1}^{n} a_k z_k.$$

Then

$$\int_0^\tau (T_1x)^*(t)dt \le \sum_{k=1}^{n} a_k \int_0^\tau (T_1z_k)^*(t)dt$$
$$\le 8\sum_{k=1}^{n} a_k \int_0^\tau HSz_k(t)dt = 8\int_0^\tau HSx(t)dt. \ \square$$

Theorem 4

Let $1 \le q < \infty$, E be a r.i. space, $\alpha_E > 0$. Then

$$\|T_qx\|_E \le C\left(\|Sx\|_E + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}\right) \tag{5}$$

where C depends only on E.

Proof. Let

$$\|Sx\|_E \le 1, \quad \frac{1}{n} \sum_{k=n+1}^{n^2} s_k^q \le 1. \tag{6}$$

We find n-square matrices y and z such that their supports are disjoint, $x = y + z$, $|\operatorname{supp} y| \le n$ and $Sx = Sy$. By Lemma 3, we have

$$\begin{aligned} \|T_q y\|_E &\le 8\|HSy\|_E \le 8\|H\|_E\|Sy\|_E \\ &= 8\|H\|_E\|Sx\|_E. \end{aligned}$$

Denote $|z_{ij}|^q = u_{ij}$, $1 \le i, j \le n$. Then

$$\|T_q z\|_E = \|(T_1 u)^{1/q}\|_E \le \|T_1 u\|_E^{1/q}.$$

Assumptions (6) imply that

$$0 \le u_{ij} \le 1, \ 1 \le i, \ j \le n, \ \sum_{i,j=1}^{n} u_{ij} \le n.$$

Applying Theorem 2 we have

$$\|T_1 u\|_{\exp L} \le \frac{1}{\ln\ln 3}.$$

It is well known that the assumption $\alpha_E > 0$ implies $E \supset L_\tau$ for some $\tau < \infty$. So $E \supset \exp L$ and

$$\|x\|_E \le C_1\|x\|_{\exp L}$$

for some $C_1 > 0$ and every $x \in \exp L$. Therefore

$$\|T_q z\|_E \le \left(\frac{C_1}{\ln\ln 3}\right)^{1/q}$$

and

$$\begin{aligned} \|T_q x\|_E &\le \|T_q y\|_E + \|T_q z\|_E \\ &\le 8\|H\|_E + \left(\frac{C_1}{\ln\ln 3}\right)^{1/q}. \ \square \end{aligned}$$

The assumption $\alpha_E > 0$ in Theorem 4 is essential, however it is not necessary. In fact, the function $T_q I_n(t)$ takes the value $n^{1/q}$ on some interval of length $1/n!$. Hence

$$\lim_{n\to\infty} \|T_q I_n\|_{L_\infty} = \infty.$$

On the other hand, $SI_n(t) = 1$ and $s_k = 0$ for $n < k \le n^2$.

The inequality inverse to (5) is true without any restrictions.

Theorem 5

Let E be a r.i. space and $1 \leq q < \infty$. Then

$$\frac{1}{12}\left(\|Sx\|_E + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}\right) \leq \|T_q x\|_E.$$

Proof. By Theorem 3,

$$\|Sx\|_E \leq 2\|T_\infty x\|_E \leq 2\|T_q x\|_E.$$

A space E is embedded into L_1 with constant 1 ([2], II.4.1). Applying Theorem B with $p = 1$ we have

$$\left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q} \leq 10\|T_q x\|_{L_1} \leq 10\|T_q x\|_E\,.$$

From the above given inequality we obtain the needed one. □

Corollary 6

If $M \in \Delta_2$, $1 \leq q < \infty$, then

$$\|T_q x\|_{L_M} \approx \|Sx\|_{L_M} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}.$$

Corollary 7

$1 \leq p,\ q < \infty$ *then*

$$\|T_q x\|_{L_p} \approx \left(\frac{1}{n}\sum_{k=1}^{n} s_k^p\right)^{1/p} + \left(\frac{1}{n}\sum_{k=n+1}^{n^2} s_k^q\right)^{1/q}.$$

Corollary 7 states that the restriction $p \leq q$ in Theorem B is superfluous.

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