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Properties of some bivariate approximants

Paulina Pych-Taberska

Institute of Mathematics, Adam Mickiewicz University,
Matejki 48/49, 60-769 Poznań, Poland

Abstract

The smoothness and approximation properties of certain discrete operators for bivariate functions are examined.

1. Preliminaries

Let I be a finite or infinite interval and let Q be the square $I \times I$. A bivariate (complex-valued) function f defined on Q is said to be Bögel-continuous, in symbols $f \in BC(Q)$, if for every $(x,y) \in Q$ there holds

$$\lim_{(u,v)\to(x,y)} \Delta_{u,v} f(x,y) = 0,$$

where $(u.v) \in Q, \Delta_{u,v}f(x,y) := f(u,v) - f(u,y) - f(x,v) + f(x,y)$. The mixed modulus of continuity $\omega(f;\delta,\eta)_Q$ of a function f on Q is defined for $\delta \geq 0, \eta \geq 0$ as the supremum of $|\Delta_{u,v}f(x,y)|$ extended over all $(x,y) \in Q, (u,v) \in Q$ such that $|u-x| \leq \delta, |v-y| \leq \eta$. As is known, if f is uniformly Bögel-continuous on Q then

$$\lim_{(\delta,\eta)\to(0,0)} \omega(f;\delta,\eta)_Q = 0.$$

In particular, this relation holds if $f \in BC(Q)$ on a compact square Q. Some other properties of the Bögel-continuous functions and their mixed moduli of continuity can be found e.g. in [2] and [3].

Let φ be a positive bivariate function on the square $(0,1]\times(0,1]$, non-decreasing in each variable, with $\varphi(1,1)\le 1$ and $\varphi(s,t)\to 0$ as $(s,t)\to(0,0)$. Take a positive number A and denote by $H_A^{\varphi}(Q)$ the class of all functions $f\in BC(Q)$ for which

$$\omega(f;\delta,\eta)_Q \le A\varphi(\delta,\eta) \quad \text{if} \quad 0<\delta\le 1,\ 0<\eta\le 1.$$

Write $H_A^{\alpha,\beta}(Q)$ instead of $H_A^{\varphi}(Q)$ when $\varphi(s,t)=s^\alpha t^\beta \quad (\alpha\ge 0,\ \beta\ge 0)$.

Given a rectangle P and a bivariate (complex-valued) function f defined on P we introduce the quantities

$$\|f\|_P := \sup|f(x,y)|$$

and

$$\|f\|_{P;\varphi} := \|f\|_P + \sup\left\{\frac{|\Delta_{u,v}f(x,y)|}{\varphi(|u-x|,|v-y|)}\right\},$$

where the first supremum is taken over all $(x,y)\in P$ and the second one is extended over all $(x,y)\in P, (u,v)\in P$ such that $0<|u-x|\le 1, 0<|v-y|\le 1$. Clearly, if $f\in H^{\varphi}(Q) := \bigcup_{A>0} H_A^{\varphi}(Q)$ then $\|f\|_{P;\varphi}$ is finite for every rectangle $P\subseteq Q$ on which f is bounded. This non-negative number is called the Hölder-type norm of f on P.

Consider now a sequence $J_1, J_2, \dots$ of some index sets contained in $\mathbb{Z} := \{0,\pm1,\pm2,\dots\}$, choose real numbers $\xi_{j,k}\in I$ and real-valued functions $p_{j,k}$ continuous on an interval $\widetilde{I}\subseteq I$ and write, formally,

$$L_k g(t) := \sum_{j\in J_k} g(\xi_{j,k})p_{j,k}(t) \quad (t\in\widetilde{I}, k\in\mathbb{N}) \tag{1}$$

for univariate (complex-valued) functions g defined on I. For bivariate (complex-valued) functions f defined on Q we introduce, also formally, the Boolean sums $L_{m,n}$ $(m,n\in\mathbb{N})$ of parametric extensions of operators L_m and L_n, i.e.

$$L_{m,n}f := (L_m^{\cdot} + L_n^{*} - L_m^{\cdot}\circ L_n^{*})f, \tag{2}$$

where

$$L_m^{\cdot}f(x,y) := L_m f^y(x), \quad L_n^{*}f(x,y) := L_n f^x(y)$$

and $f^y(x)=f^x(y)=f(x,y)$ for $(x,y)\in\widetilde{Q} := \widetilde{I}\times\widetilde{I}$.

Clearly, if

$$\sum_{j\in J_k} |p_{j,k}(t)| \le c_1 \quad \text{for all} \quad t\in\widetilde{I},\ k\in\mathbb{N}, \tag{3}$$

with a positive constant c_1, then all $L_{m,n}f$ are well-defined for every function f bounded on Q. In the case of unbounded Q, under the additional assumption

$$|\mu_{2,k}|(t) := \sum_{j\in J_k} (\xi_{j,k} - t)^2 |p_{j,k}(t)| < \infty \quad \text{for all} \quad t \in \widetilde{I},\ k \in \mathbb{N},$$

$L_{m,n}f$ are meaningful also for unbounded functions f such that $f(x, y) = \mathrm{O}((1 + x^2)(1 + y^2))$ uniformly in $(x, y) \in Q$.

In Section 2 of this paper we examine the relations between the mixed moduli of continuity of functions f and $L_{m,n}f$ satisfying some appropriate conditions. With the help of the results obtained here we estimate, in Section 3, the degree of approximation of f by $L_{m,n}f$ in the supremum norm and in the Hölder-type one. Analogous problems concerning the rate of convergence of univariate operators (1) were discussed in [5].

2. Smoothness properties

Let $\{J_k, p_{j,k}; j \in J_k, k \in \mathbb{N}\}$ be a system satisfying (3) and let

$$\sum_{j\in J_k} p_{j,k}(t) = s_k \quad \text{for all} \quad t \in \widetilde{I}, k \in \mathbb{N}, \tag{4}$$

where s_k are real numbers independent of $t \in \widetilde{I}$. Suppose, moreover, that $\xi_{j,k} \in I$ and that $p_{j,k}$ have continuous derivatives $p'_{j,k}$ such that

$$\sum_{j\in J_k} |(\xi_{j,k} - t)p'_{j,k}(t)| \le c_2 \quad \text{for all} \quad t \in \operatorname{Int} \widetilde{I}, k \in \mathbb{N}, \tag{5}$$

c_2 being a positive constant. Then the ordinary moduli of continuity of univariate functions g on I and $L_k g$ on $\widetilde{I}$ satisfy

$$\omega(L_k g; \delta)_{\widetilde{I}} \le 2(c_1 + c_2)\omega(g; \delta)_I$$

for all $\delta \ge 0, k \in \mathbb{N}$. This fact, when $I = \widetilde{I}$, was proved recently by W. Kratz and U. Stadtmüller [4]. Under some additional assumptions, the same inequality with an improved constant was derived in [1]. Corresponding result for the mixed moduli of continuity of bivariate functions f and $L_{m,n}f$ can be stated as follows.

Theorem 1

Suppose that $f \in BC(Q) \cap \mathrm{Dom}\,(L_{m,n})$ $(m, n \in \mathbb{N})$ *and that conditions (3)-(5) are fulfilled. Then, for all positive numbers* δ, η,

$$\omega(L_{m,n}f;\delta,\eta)_{\widetilde{Q}} \le c_3\omega(f;\delta,\eta)_Q,$$

with $c_3 = 4(c_1 + c_2)(1 + c_1 + c_2)$.

Proof. Let $(x, y) \in \widetilde{Q}, (u, v) \in \widetilde{Q}, 0 < u - x \le \delta, 0 < v - y \le \eta$ and let $x_0 := (x + u)/2, y_0 := (y + v)/2$.

By the definition,

$$(L_m^{\cdot} \circ L_n^*)f(x,y) = \sum_{i\in J_m}\sum_{j\in J_n} f(\xi_{i,m},\xi_{j,n})p_{i,m}(x)p_{j,n}(y).$$

Hence, in view of (4),

$$\begin{aligned}
&\Delta_{u,v}(L_m^{\cdot} \circ L_n^*)f(x,y) \\
&\quad= \sum_{i\in J_m}\sum_{j\in J_n} f(\xi_{i,m},\xi_{j,n})\{p_{i,m}(u) - p_{i,m}(x)\}\{p_{j,n}(v) - p_{j,n}(y)\} \\
&\quad= \sum_{i\in J_m}\sum_{j\in J_n} \Delta_{x_0,y_0} f(\xi_{i,m},\xi_{j,n})\{p_{i,m}(u) - p_{i,m}(x)\}\{p_{j,n}(v) - p_{j,n}(y)\}.
\end{aligned}$$

Applying the known property of the mixed modulus of continuity ([2], Lemma 2.1) we get

$$|\Delta_{u,v}(L_m^{\cdot} \circ L_n^*)f(x,y)| \le A_m(x,u;\delta)A_n(y,v;\eta)\omega(f;\delta,\eta)_Q,$$

where

$$\begin{aligned}
A_m(x,u;\delta) &:= \sum_{i\in J_m}\Big(1 + \Big[\frac{|\xi_{i,m} - x_0|}{\delta}\Big]\Big)|p_{i,m}(u) - p_{i,m}(x)| \\
&\le \sum_{i\in J_m} |p_{i,m}(u) - p_{i,m}(x)| + \frac{1}{\delta}\int_x^u \sum_{|\xi_{i,m}-x_0|\ge\delta} |\xi_{i,m} - x_0||p'_{i,m}(t)|dt.
\end{aligned}$$

Observing that $|\xi_{i,m} - x_0| \le 2|\xi_{i,m} - t|$ if $x < t < u$, $|\xi_{i,m} - x_0| \ge u - x$ and using (3), (5) we easily verify that

$$A_m(x,u;\delta) \le 2(c_1 + c_2)$$

for all $m \in \mathbb{N}, u, x \in \widetilde{I}, \delta > 0$ (see [4], p. 330). Consequently,

$$\omega\big((L_m^{\cdot} \circ L_n^*)f;\delta,\eta\big)_{\widetilde{Q}} \le 4(c_1+c_2)^2\omega(f;\delta,\eta)_Q.$$

Analogously, one can get inequalities for the mixed moduli of continuity of the remaining terms of the Boolean sum (2). Namely, we have

$$\begin{aligned}|\Delta_{u,v}L_m^{\cdot} f(x,y)| &= \Big|\sum_{i\in J_m}\Delta_{x_0,v}f(\xi_{i,m},y)\{p_{i,m}(u)-p_{i,m}(x)\}\Big| \\ &\le A_m(x,u;\delta)\omega(f;\delta,\eta)_Q,\end{aligned}$$

which implies

$$\omega(L_m^{\cdot} f;\delta,\eta)_{\widetilde{Q}} \le 2(c_1+c_2)\omega(f;\delta,\eta)_Q.$$

By symmetry, the same inequality remains also valid for the mixed modulus of continuity of $L_n^* f$.

These results together with (2) lead to the desired inequality. □

For many well-known operators the "weights" $p_{j,k}$ ($j \in J_k, k \in \mathbb{N}$) satisfy the assumptions

$$p_{j,k}(t) \ge 0, \quad \sum_{j\in J_k} p_{j,k}(t) = 1 \quad \text{for all} \quad t \in \widetilde{I}$$

and

$$|\mu_{2,k}|(t) > 0, \quad |\mu_{2,k}|(t)p'_{j,k}(t) = (\xi_{j,k}-t)p_{j,k}(t) \quad \text{for all} \quad t \in \operatorname{Int}\widetilde{I}.$$

Hence, in these cases, $c_1 = c_2 = 1$ and the constant c_3 in Theorem 1 equals 24.

Inequalities obtained in the proof of Theorem 1 yield the implication

$$\begin{aligned}&f \in H_A^{\varphi}(Q) \cap \operatorname{Dom}(L_{m,n}) \Rightarrow \\ &\qquad \Rightarrow \big(L_m^{\cdot} f \in H_B^{\varphi}(\widetilde{Q}), L_n^* f \in H_B^{\varphi}(\widetilde{Q}),\ (L_m^{\cdot} \circ L_n^*)f \in H_M^{\varphi}(\widetilde{Q})\big),\end{aligned}$$

where $B = 2(c_1+c_2)A$, $M = 4(c_1+c_2)^2A$. This means that, under assumptions of Theorem 1, the terms of the Boolean sum (2) have the property of preserving the Hölder class with the same order that f but with the different constants. In the case $\varphi(s,t) = s^\alpha t^\beta$ $(0 < \alpha \le 1,\ 0 < \beta \le 1)$ we will indicate a wide class of operators for which the order (α,β) as well as the Hölder constant A are retained.

To this end, let us introduce a sequence $(\psi_k)_1^\infty$ of continuous functions on $I_0 = [0,\infty)$, with values $\psi_k(0) = 1$, satisfying for some positive numbers $q = q(k)$ and a certain interval $\widetilde{I}_0 \subseteq I_0$ (such that $0 \in \widetilde{I}_0$) the following conditions

1^0 $(-1)^j D_q^j \psi_k(t) \geq 0$ whenever $t \in \widetilde{I}_0,\ j \in \mathbb{N}_0 := \{0, 1, \ldots\}$, where $D_q^0 \psi_k := \psi_k,\ D_q^1 \psi_k(t) := (\psi_k(t+q) - \psi_k(t))/q$ and

$$D_q^j \psi_k(t) := D_q^1(D_q^{j-1}\psi_k)(t) \quad \text{if} \quad j > 1;$$

2^0 under the restriction $t, x \in \widetilde{I}_o$,

$$\psi_k(t) = \sum_{j=0}^{\infty} \frac{(t-x)^{(j,q)}}{j!} D_q^j \psi_k(x),$$

where

$$h^{(0,\rho)} := 1,\ h^{(j,\rho)} := h(h-\rho)\ldots(h-(j-1)\rho) \quad \text{if} \quad j \geq 1 \quad (h, \rho \in \mathbb{R}).$$

Consider, as in [6], the class of linear operators $V_k = L_k$ defined by (1) for univariate functions g on I_0, with $J_k = \mathbb{N}_0, \xi_{j,k} = j/k$ and

$$p_{j,k}(t) = (-1)^j \frac{t^{(j,-q)}}{j!} D_q^j \psi_k(t) \quad (t \in \widetilde{I}_0).$$

We note occasionally that from some operators of this class (with q independent of k) the classical Bernstein polynomials, the Szász-Mirakyan operators or the Baskakov operators can be obtained by letting $q \to 0+$.

Theorem 2

Suppose that $V_k e_1(t) = \gamma_k t$ for all $t \in \widetilde{I}_0,\ k \in \mathbb{N}$, where $e_1(\tau) = \tau$ $(\tau \geq 0)$ and γ_k are some constants from $[0,1]$. Denote by $V_{m,n}$ $(m, n \in \mathbb{N})$ the Boolean sum of parametric extensions of univariate operators V_m and V_n. Put $Q_0 = I_0 \times I_0$, $\widetilde{Q}_0 = \widetilde{I}_0 \times \widetilde{I}_0$. Then if $f \in H_A^{\alpha,\beta}(Q_0) \cap$ Dom $(V_{m,n})$, with $0 < \alpha, \beta \leq 1$, the functions $V_m^{\cdot} f$, $V_n^* f$ and $(V_m^{\cdot} \circ V_n^*) f$ are in the class $H_A^{\alpha,\beta}(\widetilde{Q}_0)$.

Proof. Let $f \in H_A^{\alpha,\beta} Q_0 \cap$ Dom $(V_{m,n})$ and let δ, η be arbitrary positive numbers. Consider $(x,y) \in \widetilde{Q}_0, (u,v) \in \widetilde{Q}_0$ such that $0 < u - x \leq \delta, 0 < v - y \leq \eta$.

The argumentation similar to that of the proof of Theorem 2.1 of [6] yields the identities

$$V_m^{\cdot} f(u,t) = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{i+l} \frac{x^{(i,-q)}(u-x)^{(l,-q)}}{i!l!} D_q^{i+l} \psi_m(u) f\Big(\frac{i+l}{m}, t\Big),$$

$$V_m^{\cdot} f(x,t) = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{i+l} \frac{x^{(i,-q)}(u-x)^{(l,-q)}}{i!l!} D_q^{i+l} \psi_m(u) f\Big(\frac{i}{m}, t\Big),$$

for every $t \in \widetilde{I}_0$. Consequently,

$$
\begin{aligned}
&|\Delta_{u,v} V^{\cdot}_m f(x,y)| \\
&= \Big|\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{i+l}\frac{x^{(i,-q)}(u-x)^{(l,-q)}}{i!l!}D_q^{i+l}\psi_m(u)\Delta_{\frac{i+l}{m},v}f\Big(\frac{i}{m},y\Big)\Big| \\
&\le A\eta^{\beta}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{i+l}\frac{x^{(i,-q)}(u-x)^{(l,-q)}}{i!l!}D_q^{i+l}\psi_m(u)\Big(\frac{l}{m}\Big)^{\alpha} \\
&= A\eta^{\beta}V_m g_{\alpha}(u-x),
\end{aligned}
$$

where $g_{\alpha}(\tau) = \tau^{\alpha}$ $(\tau \ge 0)$. Since

$$
V_k g_{\alpha}(t) \le t^{\alpha} \quad \text{for all} \quad t \in \widetilde{I}_0,\ k \in \mathbb{N} \tag{6}
$$

(see [6], p. 128), we have

$$
|\Delta_{u,v} V^{\cdot}_m f(x,y)| \le A\eta^{\beta}(u-x)^{\alpha}.
$$

This implies the inequality

$$
\omega(V^{\cdot}_m f;\delta,\eta)_{\widetilde{Q}_0} \le A\delta^{\alpha}\eta^{\beta}.
$$

Analogously,

$$
\omega(V^{*}_n f;\delta,\eta)_{\widetilde{Q}_0} \le A\delta^{\alpha}\eta^{\beta}.
$$

Considering the superposition $V^{\cdot}_m \circ V^{*}_n$ we easily observe that

$$
\begin{aligned}
&|\Delta_{u,v}(V^{\cdot}_m \circ V^{*}_n)f(x,y)| \\
&= \Big|\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{i+l+j+r}\frac{x^{(i,-q)}(u-x)^{(l,-q)}}{i!l!} \\
&\quad\times \frac{y^{(j,-q)}(v-y)^{(r,-q)}}{j!r!}D_q^{i+l}\psi_m(u)D_q^{j+r}\psi_n(v)\Delta_{(i+l)/m,(j+r)/n}f\Big(\frac{i}{m},\frac{j}{n}\Big)\Big| \\
&\le A V_m g_{\alpha}(u-x)V_n g_{\beta}(v-y).
\end{aligned}
$$

Applying inequality (6) we get

$$
\omega\big((V^{\cdot}_m \circ V^{*}_n)f;\delta,\eta\big)_{\widetilde{Q}_0} \le A\delta^{\alpha}\eta^{\beta},
$$

and this completes the proof. □

3. Approximation properties

Let us return to the general operators $L_{m,n}$ given by (2) for functions f defined on Q. Make the standing assumption

$$\sum_{j\in J_k} p_{j,k}(t) = 1 \quad \text{for all} \quad t \in \widetilde{I},\ k \in \mathbb{N}.$$

In this case, for any f ∈Dom $(L_{m,n})$ and all $(x,y) \in \widetilde{Q}$,

$$f(x,y) - L_{m,n}f(x,y) = \sum_{i\in J_m}\sum_{j\in J_n} \Delta_{x,y} f(\xi_{i,m},\xi_{j,n}) p_{i,m}(x) p_{j,n}(y).$$

By a small modification of the proof of Theorem 2.2 in [2] one can get

Theorem 3

Let condition (3) be satisfied and let for a certain interval $Y \subseteq \widetilde{I}$,

$$\sup_{t\in Y} |\mu_{2,k}|(t) \le \lambda d_k^2, \tag{7}$$

where λ is a positive constant and $(d_k)_1^\infty$ is a sequence of positive numbers not greater than 1. Suppose that f belongs to the class $BC(Q) \cap$ Dom $(L_{m,n})$ and that $P := Y \times Y$. Then

$$\|f - L_{m,n}f\|_P \le (c_1 + \lambda)^2 \omega(f; d_m, d_n)_Q.$$

In order to estimate the deviation $L_{m,n}f$ from f in Hölder-type norm it is convenient to apply the following

Lemma

Suppose that $f \in$ Dom $(L_{m,n})$ and that $0 < \delta_m \le 1$, $0 < \eta_n \le 1$. Then, for every rectangle $P \subseteq \widetilde{Q}$,

$$\|f - L_{m,n}f\|_{P;\varphi} \le \Big(1 + \frac{4}{\varphi(\delta_m,\eta_n)}\Big)\|f - L_{m,n}f\|_P + \sup \frac{1}{\varphi(\delta,\eta)}\{\omega(f;\delta,\eta)_P + \omega(L_{m,n}f;\delta,\eta)_P\},$$

the supremum being taken over all pairs (δ,η) belonging to the set $R(\delta_m,\eta_n) := (0,1] \times (0,1] \setminus (\delta_m, 1] \times (\eta_n, 1]$.

The above inequality follows at once from the two obvious facts:
(i) if $(x,y) \in P$, $(u,v) \in P$, $\delta_m \le |u-x| \le 1$, $\eta_n \le |v-y| \le 1$, then

$$\frac{|\Delta_{u,v}(f - L_{m,n}f)(x,y)|}{\varphi(|u-x|,|v-y|)} \le \frac{4}{\varphi(\delta_m,\eta_n)} \|f - L_{m,n}f\|_P;$$

(ii) if $0 < |u-x| \le \delta_m, 0 < |v-y| \le 1$ or if $\delta_m \le |u-x| \le 1, 0 < |v-y| \le \eta_n$, then

$$\frac{|\Delta_{u,v}(f - L_{m,n}f)(x,y)|}{\varphi(|u-x|,|v-y|)} \le \frac{\omega(f;|u-x|,|v-y|)_P + \omega(L_{m,n}f;|u-x|,|v-y|)_P}{\varphi(|u-x|,|v-y|)}.$$

Combining Theorems 1, 3 and Lemma we obtain

Theorem 4

Suppose that conditions (3), (5) and (7) are fulfilled. Then if $f \in H^\varphi(Q) \cap$ Dom $(L_{m,n})$ and $P := Y \times Y$, we have

$$\|f - L_{m,n}f\|_{P;\varphi} \le c_4 \sup \left\{ \frac{\omega(f;\delta,\eta)_Q}{\varphi(\delta,\eta)} \right\},$$

where $c_4 = 5(c_1+\lambda)^2 + c_3 + 1$ and the supremum is taken over all $(\delta,\eta) \in R(d_m,d_n)$.

Corollary

Let $f \in H_A^{\alpha,\beta}(Q) \cap$ Dom $(L_{m,n})$ and let $0 < a < \alpha \le 1, 0 < b < \beta \le 1$. Then, in case $\varphi(s,t) = s^a t^b$ $(0 < s,t \le 1)$,

$$\|f - L_{m,n}f\|_{P;\varphi} \le Ac_4(d_m^{\alpha-a} + d_n^{\beta-b})$$

whenever assumptions (3), (5), (7) hold.

Remark. For operators $V_{m,n}$ considered in Theorem 2 and functions $f \in H_A^{\alpha,\beta}(Q_0)$ $(0 < \alpha,\beta \le 1)$ satisfying the condition $f(x,y) = O((1+x^2)(1+y^2))$ uniformly in $(x,y) \in Q_0$, the relation $V_{m,n}f \in H_{3A}^{\alpha,\beta}(\tilde{Q}_0)$ is valid. Further, by Theorem 3,

$$\|f - V_{m,n}f\|_P \le (1+\lambda)^2 \omega(f;d_m,d_n)_{Q_0},$$

where $P = Y \times Y$, λ and d_k are determined via condition (7). Applying Lemma we get, for all $m,n \in \mathbb{N}$, the estimate of $\|f - V_{m,n}f\|_{P;\varphi}$ as in Corollary, with c_4 replaced by $5(1+\lambda)^2 + 4$.

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