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Boundary spaces for inclusion map between rearrangement invariant spaces

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ABSTRACT

Let $E([0,1];m)$ be a rearrangement invariant space (RIS) on $[0,1]$ with Lebesgue measure m. That is, E is a Banach lattice and if $m(t: |x(t)| > \tau) = m(t: |y(t)| > \tau) \forall \tau$, then $\|x\|_E = \|y\|_E$. For each of this kind of spaces we have inclusions $C \subset L_\infty \subset E \subset L_1$ and canonical inclusion maps $I(C,E)$ or $I(E_1,E_2)$. The aim of this paper is to represent a number of RIS, which are boundary for various properties of canonical inclusion maps. There are still some unsolved problem in this area.

1. Strict singularity

An operator $T \in \mathcal{L}(X,Y)$ between two Banach spaces (BS) X and Y is called strictly singular if there is no infinite dimensional subspace Z of X such that the restriction $T|Z$ is an isomorphism. The set of this kind of operators will be denoted $\sigma(X,Y)$. It is an ideal in the Pietsch sense.

According to a well-known Grothendieck's theorem $I(L_\infty, L_p) \in \sigma, 1 \leq p < \infty$ (see, for example, the text book of W. Rudin). A more general fact seems to be true:

Theorem 1

Let E be a RIS and $E \neq L_\infty$. Then $I(L_\infty, E)$ is strictly singular.

Proof. The function $\varphi_E(t) := \|\mathbf{1}_{[0,t]}\|_E$, the so-called fundamental function of the space E is a quasi-concave function. We may define Lorentz space $\Lambda(\varphi_E) := \{f: \int_0^1 f^* d\varphi_E < \infty\}$, where f^* is the decreasing rearrangement of $|f|$, besides we have another inclusion: $E \supset \Lambda(\varphi_E), E \neq L_\infty$. It's known, that if $E \neq L_\infty$, then the function φ_E is continuous at zero and the space $\Lambda(\varphi_E)$ is weakly sequential complete. From this we deduce that the p-convexification $\Lambda_p(\varphi_E) := \{f: |f|^p \in \Lambda(\varphi_E)\}$ is reflexive for $1 < p < \infty$. So, we have: $E \supset \Lambda(\varphi_E) \supset \Lambda_p(\varphi_E) \supset L_\infty$. As L_∞ has the Dunford-Pettis property (i.e., $\forall Y, \forall$ weakly compact $T \in \mathcal{L}(L_\infty, Y), \forall$ convex weakly compact $K \subset L_\infty, T(K)$ is compact in Y), we have that the unit ball B_H of each subspace $H \subset E$, such that $H \subset L_\infty$, is compact in E. □

In spite of the fact that Theorem 1 solves the problem of strict singularity of the inclusion map $I(L_\infty, E)$, there are still left a lot of problems concerning inclusion maps between general RIS $E_1 \subset E_2$. For example, there is no full description of the set of such RIS E, for which $I(E, L_1) \in \sigma$. In this direction we know only a partial answer:

Theorem 2

If RIS $E \subset L_2$, then $I(E, L_1) \in \sigma$ iff $E \not\supset G$, where G is the closure of $C[0,1]$ in the Orlicz space $L_N, N(u) = e^{u^2} - 1$.

Proof. If $E \supset G$, then according to the classical result of Rodin-Semenov ([6], [2]), E contains an infinite dimensional subspace R closed in L_1.

Now suppose that $I(E, L_1)$ is not strictly singular. It means that E contains an infinite dimensional subspace H, closed in L_1. This subspace is closed in L_2 also (cf. condition). Let $\{f_i\}$ be a sequence of elements of H, equivalent to the unit basis of l_2 and $\|f_i\|_{L_2} = 1, i = 1, 2, \ldots$. We can assume that $f_i \to 0$ weakly in L_2 and $\liminf \|f_i\|_{L_1} > 0$; this may be done by choosing subsequences. The last inequality ensures the existence of a function $0 \le g \in L_1$ with $m(\text{supp } g) > 0$ such that $f_i^2 \to g$ weakly in L_1. Now we will use the following theorem of V. Gaposhkin ([1], Th. 1.5.1):

If $\{f_k\}$ is a sequence of functions such that:

1) $\|f_k\|_{L_2} = 1 \ \forall k$;
2) $f_k \to 0$ weakly in L_2;
3) $\exists g \in L_1^+, \|g\|_{L_1} = 1$ such that $f_k \to g$ weakly in L_1;

then it's possible to choose a subsequence $\{f_{k_i}\}$ such that the next equality, like in central limit theorem, takes place

$$\lim_{m\to\infty} m\left\{t: \frac{1}{\sqrt{m}} \sum_{i=1}^{m} f_{k_i}(t) \ge s\right\} = \frac{1}{2\pi} \int_0^1 dt \int_{s/\sqrt{g(t)}}^{\infty} \exp\left(\frac{-u^2}{2}\right) du.$$

Using this fact it's not difficult to see that the function $(\ln \frac{1}{t})^{\frac{1}{2}} \in E''$, where E'' is the Köthe dual of E. The last condition is known to be equivalent to the inclusion $E \supset G$. □

2. Absolutely summing properties

DEFINITION. An operator T is called (q,p) - absolutely summing ($T \in \Pi_{q,p}(X,Y)$ if $\exists C > 0 : \forall \{x_1, x_2, \dots, x_n\} \in X$

$$\Big(\sum_i \|Tx_i\|^q\Big)^{1/q} \le C \sup\Big\{\Big(\sum_i |F(x_i)|^p\Big)^{1/p} : \|F\|_{X^*} \le 1\Big\}.$$

This definition makes sense only if $0 < p \le q < \infty$; if $p > q$ then only 0 - operator is (q,p) - absolutely summing. For $p = q$ we use the notations Π_p and "p – absolutely summing".

Theorem 3

Let $E_1 \subset E_2$ and $p \ge 1$. The inclusion map $I(E_1, E_2) \in \Pi_p$ iff $E_1 = L_\infty, E_2 \supset L_p$.

Proof. Sufficiency is obvious. Now assume that $I(E_1, E_2)$ is p - absolutely summing. Then each weak convergent sequence in E_1 is convergent in norm in E_2. Repeating the proof of Theorem 1 we deduce that $E_1 = L_\infty$. From classical factorization theorem of Pietsch we have: $\exists$ probability measure ν on $[0,1]$ such that

$$\|f\|_{E_2} \le \pi_p(I)\Big(\int |f(s)|^p d\nu(s)\Big)^{1/p}, \ f \in C[0,1].$$

Now let $t \in [0,1]$ and $f_t(s) := f(t+s)$, addition by mod 1. We have:

$$\|f_t\|_{E_2}^p \le \pi_p(I)\Big(\int |f_t(s)|^p d\nu(s)\Big), \ t \in [0,1].$$

Integrating this inequality by Lebesgue measure, we have

$$\|f\|_{E_2} \le \pi_p(I)\|f\|_{L_p}, \ f \in C[0,1]. \ \square$$

In order to give the analogous fact for (q,p) – absolutely summing operators, we again return to Lorentz spaces $L_{q,1} := \Lambda(\varphi_q)$, where $\varphi_q(t) = t^{\frac{1}{q}}$. Another description of its norm is as following: $\|f\| = \int_0^\infty (m(|f| > t))^{\frac{1}{q}} dt$.

Theorem 4

Let $1 \le p < q < \infty$. The following assertions are equivalent:
1) $I(C[0,1];\ E[0,1]) \in \Pi_{q,p}$;
2) $\exists K > 0: \varphi_E(t) \le K t^{\frac{1}{q}},\ 0 \le t \le 1$;
3) $E \supset L_{q,1}$.

Proof. This theorem may be easily deduced from the recent factorization theorem of G. Pisier [5], but we prefer the direct way from the rather old paper of I. Novikov [3].

1) $\Rightarrow$ 2). Let $I(C,E) \in \Pi_{q,p}$. Then, as is known from the results of B. Maurey $I \in \Pi_{q,1}$, that is

$$\exists K > 0: \forall \{x_1, \ldots, x_n\} \in C[0,1], \Big(\sum \|x_i\|^q\Big)^{1/q} \le K \Big\|\sum |x_i|\Big\|.$$

This inequality may be continued on $\{x_1, \ldots, x_n\} \subset L_\infty$. If we set $x_i = \mathbf{1}_{[\frac{i-1}{n}, \frac{i}{n}]}$, then $n\varphi^q(\frac{1}{n}) \le K$, $n = 1, 2, \ldots$; that is equivalent to 2).

2)$\Rightarrow$3) is well-known ([7]).

3)$\Rightarrow$1). Simple calculations (cf. [8] for $q = 2$) show that $I(C, L_{q,1})$ is $(q,1)$ – absolutely summing. □

There are some open problems in this area. As far as I know, there is not a single result concerning the (q,p) – absolutely summing property of inclusion map $I(E_1, E_2)$ for another RIS besides L_p-spaces.

3. Another ideal properties

DEFINITION. An operator $T \in \mathcal{L}(X,Y)$ is of gaussian cotype q if for some $C > 0$ and all sequences (x_i) of X, we have $(\sum \|Tx_i\|^q)^{\frac{1}{q}} \le CE\|g_i x_i\|$, where (g_i) denotes a sequence of independent normalized $N(0,1)$ –gaussian random variables. The set of all operators of such kind forms an ideal and will be denoted by $\mathcal{C}_q^{(g)}$. Not long ago M. Talagrand (preprint) and S. Montgomery-Smith (dissertation) found boundary spaces for the gaussian cotype 2 – property of inclusion map. Their result is the following

Theorem 5

$I(C,E)$ is of gaussian cotype 2 iff $E \supset L_{\Phi,2}$, where $\Phi(t) = t^2 \log t$. The space $L_{\Phi,2}$ is defined by the following norm:

$$\|f\| = \Big(\int \theta\big(m(|f| \ge t)\big) dt^2\Big)^{1/2}, \quad \text{where} \quad \theta(t) = t \ln \frac{2}{t}.$$

It's not difficult to show that $L_{\Phi,2} \supseteq L_{2,1}$ and so $I(C; L_{2,1}) \in \Pi_{2,1} \setminus C_2^{(g)}$, i.e. we have a nice counterexample to the conjecture $C_2^{(g)} = \Pi_{2,1}$. Thus, the space $L_{2,1}$ is still a rich source of counterexamples. Another example of this statement is the following. Let $E \subseteq L_2$. The following conjecture was made by M. Braverman, N. Carothers and others. If $(f_1) \subset E$ and (f_i) are independent, identically distributed random variables such that $Ef_i = 0$, then [span $(f_i)]_E$ is isomorphic to l_2. But this conjecture is not true. As shown in [4] the following equality is valid: $A(L_{2,1}) := \{(a_i) \in R^\infty : \sum a_i f_i$ converges for each sequence of i.i.d. $\{f_i\} : \int f_i = 0, f_1 \in L_{2,1}\} = l_{2,1}$. If the conjecture were true, we would have to have that $A(L_{2,1}) = l_2$. The Theorems 1–5 give the basis for the following

CONJECTURE. For each ideal $\mathcal{U}$ of operators there exists a boundary RIS $E_{\mathcal{U}}$ such that $I(C, E) \in \mathcal{U}$ iff $E \supset E_{\mathcal{U}}$, where the inclusion in the right hand may be strict or unstrict in dependence of the ideal $\mathcal{U}$. As far as I know there is no answer to the question about the boundary space for the ideal of Rademacher cotype q – property.

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